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Fredholm-type integral equation in controlled metric-like spaces

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Abstract
In this article we make an improvement in the Banach contraction using a controlled function in controlled metric like spaces, which generalizes many results in the literature. Moreover, we present an application on Fredholm type integral equation.

Keywords: Fixed point; Controlled metric-like spaces; Fredholm-type integral equations

1 Introduction

One of the most interesting applications of fixed point theory is solving integral and differential equations; see, for example, [1]. The Banach contraction principle was generalized many times to extend its application. As an example of these generalizations, b-spaces (see [2]) are an extension of the regular metric spaces; see [3–15]. Lately, Kamran [16] introduced what the so-called extended b-metric spaces by adding a control function $\theta(\mathfrak{p},\mathfrak{q})$ in the triangle inequality. For more on b-metric spaces and its extensions, we refer the reader to [17–23]. First, we start by reminding the reader the definition of extended b-metric spaces.

Definition 1.1 ([16]) Consider the set $X \neq \emptyset$ and $\theta : X \times X \to [1,\infty)$. Let $d_e : X \times X \to [0,\infty)$ be such that for all $\mathfrak{p},\mathfrak{q},z \in X$,

(1) $d_e(\mathfrak{p},\mathfrak{q}) = 0$ if and only if $\mathfrak{p} = \mathfrak{q}$;
(2) $d_e(\mathfrak{p},\mathfrak{q}) = d_e(\mathfrak{q},\mathfrak{p})$;
(3) $d_e(\mathfrak{p},\mathfrak{q}) \leq \theta(\mathfrak{p},\mathfrak{q})[d_e(\mathfrak{p},z) + d_e(z,\mathfrak{q})]$.

Then (X,d_e) is called an extended b-metric space.

Mlaiki et al. [24] gave an extension to this type of metric spaces as follows.

Definition 1.2 ([24]) Consider the set $X \neq \emptyset$ and $\varrho : X \times X \to [1,\infty)$. Suppose that a function $d_c : X \times X \to [0,\infty)$ satisfies the following:

(1) $d_c(\mathfrak{p},\mathfrak{q}) = 0$ if and only if $\mathfrak{p} = \mathfrak{q}$;
(2) $d_c(\mathfrak{p},\mathfrak{q}) = d_c(\mathfrak{q},\mathfrak{p})$;
(3) $d_c(\mathfrak{p},\mathfrak{q}) \leq \varrho(\mathfrak{p},z)d_c(\mathfrak{p},z) + \varrho(z,\mathfrak{q})d_c(z,\mathfrak{q})$ for all $\mathfrak{p},\mathfrak{q},z \in X$.

Then (X,d_c) is called a controlled metric-type space.

In 2021, a new generalization of the b-metric spaces introduced in [25], the so-called controlled metric-like spaces.

Definition 1.3 ([25]) Consider the set $X \neq \emptyset$ and $\varrho : X \times X \to [1,\infty)$. Suppose that a function $d_c : X \times X \to [0,\infty)$ satisfies the following:

(CML1) $d_c(s,r) = 0 \Rightarrow s = r$;

(CML2) $d_c(s,r) = d_c(r,s)$;

(CML3) $d_c(s,r) \leq \varrho(s,z)d_c(s,z) + \varrho(z,r)d_c(z,r)$ for all $s,r,z \in X$.

Then (X,d_c) is called a controlled metric-like space.

Example 1.4 ([25]) Let $X = \{0,1,2\}$. Define the function d_c by

$$d_c(0,0) = d_c(1,1) = 0, \qquad d_c(2,2) = \frac{1}{10}$$

and

$$d_c(0,1) = d_c(1,0) = 1, \qquad d_c(0,2) = d_c(2,0) = \frac{1}{2}, \qquad d_c(1,2) = d_c(2,1) = \frac{2}{5}.$$

Let $\varrho : X \times X \to [1,\infty)$ a symmetric function defined by

$$\varrho(0,0) = \varrho(1,1) = \varrho(2,2) = \varrho(0,2) = 1, \qquad \varrho(1,2) = \frac{5}{4}, \qquad \varrho(0,1) = \frac{11}{10}.$$

Here d_c is a controlled metric-like on X.

We have $d_c(2,2) = \frac{1}{10} \neq 0$, which implies that (X,d_c) is not a controlled metric-type space.

Definition 1.5 ([25]) Let (X,d_c) be a controlled metric-like space, and let $\{s_n\}_{n\geq 0}$ be a sequence in X.

(1) $\{s_n\}$ converges to s in X if and only if

$$\lim_{n\to\infty} d_c(s_n,s) = d_c(s,s).$$

Then we write $\lim_{n\to\infty} s_n = s$.

(2) $\{s_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d_c(s_n,s_m)$ exists and is finite.

(3) We say that (X,d_c) is complete if for every Cauchy sequence $\{s_n\}$, there is $s \in \chi$ such that

$$\lim_{n\to\infty} d_c(s_n,s) = d_c(s,s) = \lim_{n,m\to\infty} d_c(s_n,s_m).$$

Definition 1.6 ([26]) Let (X,d_c) be a controlled metric-like space. Let $s \in X$ and $\tau > 0$.

(i) The open ball $B(s,\tau)$ is

$$B(s,\tau) = \bigl\{y \in X, \bigl|d_c(s,r) - d_c(s,s)\bigr| < \tau\bigr\}.$$

We denote controlled metric-like spaces by *CMLS*.

Note that if $\mathfrak{T}$ is continuous at $\mathfrak{p}$ in the *CMLS* (X,d_c), then $\mathfrak{p}_n \to \mathfrak{p}$ implies that $\mathfrak{T}\mathfrak{p}_n \to \mathfrak{T}\mathfrak{p}$ as $n \to \infty$.

Now let (X, d_c) be a controlled metric-like space, and let $\mathfrak{T} : X \to X$. The following control functions were introduced by Sintunavarat et al. [27] (in this paper, we will exclude zero from their range):

$$\mathrm{A} = \{\vartheta : X \to (0,1), \vartheta(\mathfrak{T}\mathfrak{p}) \leq \vartheta(\mathfrak{p}) \text{ for all } \mathfrak{p} \in X\}.$$

and

$$\mathrm{B} = \{\vartheta : X \to (0,1/2), \vartheta(\mathfrak{T}\mathfrak{p}) \leq \vartheta(\mathfrak{p}) \text{ for all } \mathfrak{p} \in X\}.$$

2 Main results

Our first main result corresponds to a nonlinear Banach-type result on *CMLS*, which is also an extension of the results in [28].

Theorem 2.1 *Let (X, d_c) be a complete CMLS. Consider the mapping $\mathfrak{T}: X \to X$ such that*

$$d_c(\mathfrak{T}\mathfrak{p}, \mathfrak{T}\mathfrak{q}) \leq \vartheta(\mathfrak{p}) d_c(\mathfrak{p}, \mathfrak{q}) \tag{2.1}$$

for all $\mathfrak{p}, \mathfrak{q} \in X$, where $\vartheta \in \mathrm{A}$. For $\mathfrak{p}_0 \in X$, take $\mathfrak{p}_n = \mathfrak{T}^n \mathfrak{p}_0$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_{i+2})}{\varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1})} \varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_m) < \frac{1}{\vartheta(\mathfrak{p}_0)}. \tag{2.2}$$

Also, assume that for every $\mathfrak{p} \in X$, we have

$$\lim_{n \to \infty} \varrho(\mathfrak{p}_n, \mathfrak{p}) \quad \textit{and} \quad \lim_{n \to \infty} \varrho(\mathfrak{p}, \mathfrak{p}_n) \quad \textit{exist and are finite.} \tag{2.3}$$

Then $\mathfrak{T}$ has a unique fixed point.

Proof Consider the sequence $\{\mathfrak{p}_n = \mathfrak{T}^n \mathfrak{p}_0\}$. By (2.1) we get

$$d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \leq \vartheta(\mathfrak{p}_{n-1}) d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_n) \quad \text{for all } n \geq 1.$$

Since $\vartheta \in \mathrm{A}$, we have

$$d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \leq \vartheta(\mathfrak{p}_0) d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_n) \quad \text{for all } n \geq 1.$$

By induction,

$$d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \leq [\vartheta(\mathfrak{p}_0)]^n d_c(\mathfrak{p}_0, \mathfrak{p}_1) \quad \text{for all } n \geq 0. \tag{2.4}$$

Choose $k =: \vartheta(\mathfrak{p}_0) \in (0,1)$. For all natural numbers $n < m$, as in [24], we have

$$\begin{aligned} d_c(\mathfrak{p}_n, \mathfrak{p}_m) &\leq \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1}, \mathfrak{p}_m) d_c(\mathfrak{p}_{n+1}, \mathfrak{p}_m) \\ &\leq \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) k^n d_c(\mathfrak{p}_0, \mathfrak{p}_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \varrho(\mathfrak{p}_j, \mathfrak{p}_m) \right) \varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1}) k^i d_c(\mathfrak{p}_0, \mathfrak{p}_1) \\ &\leq \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) k^n d_c(\mathfrak{p}_0, \mathfrak{p}_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \varrho(\mathfrak{p}_j, \mathfrak{p}_m) \right) \varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1}) k^i d_c(\mathfrak{p}_0, \mathfrak{p}_1). \end{aligned}$$

Let

$$S_p = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \varrho(\mathfrak{p}_j, \mathfrak{p}_m) \right) \varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1}) k^i.$$

Hence we have

$$d_c(\mathfrak{p}_n, \mathfrak{p}_m) \le d_c(\mathfrak{p}_0, \mathfrak{p}_1) \big[k^n \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) + (S_{m-1} - S_n) \big]. \tag{2.5}$$

Now by condition (2.2) and the ratio test, we deduce that $\lim_{n\to\infty} S_n$ exists, and therefore $\{S_n\}$ is a Cauchy sequence. Taking the limit in (2.5), we obtain

$$\lim_{n,m\to\infty} d_c(\mathfrak{p}_n, \mathfrak{p}_m) = 0. \tag{2.6}$$

Hence $\{\mathfrak{p}_n\}$ is a Cauchy sequence. Since (X, d_c) is complete, we deduce that $\{\mathfrak{p}_n\}$ converges to some $u \in X$. We claim that u is a fixed point of $\mathfrak{T}$. To prove this claim, we start by applying the triangle inequality of the *CMLS* as follows:

$$d_c(u, \mathfrak{p}_{n+1}) \le \varrho(u, \mathfrak{p}_n) d_c(u, \mathfrak{p}_n) + \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}).$$

By (2.2), (2.3), and (2.6) we conclude that

$$\lim_{n\to\infty} d_c(u, \mathfrak{p}_{n+1}) = 0. \tag{2.7}$$

Thus

$$\begin{aligned} d_c(u, \mathfrak{T}u) &\le \varrho(u, \mathfrak{p}_{n+1}) d_c(u, \mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1}, \mathfrak{T}u) d_c(\mathfrak{p}_{n+1}, \mathfrak{T}u) \\ &\le \varrho(u, \mathfrak{p}_{n+1}) d_c(u, \mathfrak{p}_{n+1}) + \vartheta(\mathfrak{p}_n) \varrho(\mathfrak{p}_{n+1}, \mathfrak{T}u) d_c(\mathfrak{p}_n, u) \\ &\le \varrho(u, \mathfrak{p}_{n+1}) d_c(u, \mathfrak{p}_{n+1}) + \vartheta(\mathfrak{p}_0) \varrho(\mathfrak{p}_{n+1}, \mathfrak{T}u) d_c(\mathfrak{p}_n, u). \end{aligned}$$

Note that, as $n \to \infty$ in (2.3) and (2.7), we have $d_c(u, \mathfrak{T}u) = 0$, that is, $\mathfrak{T}u = u$. Now we may assume that $\mathfrak{T}$ has fixed points u and v. Hence

$$d_c(u, v) = d_c(\mathfrak{T}u, \mathfrak{T}v) \le \vartheta(u) d_c(u, v) < d_c(u, v),$$

which leads us to a contradiction. Thereby $d_c(u, v) = 0$, which implies $u = v$, as desired. □

Next, we present the following example.

Example 2.2 Let $X = [0, 1]$. Consider the *CMLS* (X, d_c) defined by

$$d_c(\mathfrak{p}, \mathfrak{q}) = |\mathfrak{p} - \mathfrak{q}|^2,$$

where $\varrho(\mathfrak{p},\mathfrak{q}) = \mathfrak{p}\mathfrak{q} + 1$ for $\mathfrak{p},\mathfrak{q} \in X$. Take $\mathfrak{T}\mathfrak{p} = \frac{\mathfrak{p}^2}{4}$. Choose $\vartheta : X \to [0,1)$ as $\vartheta(\mathfrak{p}) = \frac{\mathfrak{p}+1}{4}$. Then $\vartheta \in \mathrm{A}$. Take $\mathfrak{p}_0 = 0$, so (2.2) is satisfied. Let $\mathfrak{p},\mathfrak{q} \in X$. Then

$$\begin{aligned} d_c(\mathfrak{T}\mathfrak{p},\mathfrak{T}\mathfrak{q}) &= \frac{(\mathfrak{p}^2-\mathfrak{q}^2)^2}{16} = \frac{1}{16}|\mathfrak{p}-\mathfrak{q}|^2(\mathfrak{p}+\mathfrak{q})^2 \\ &\leq \frac{1}{4}|\mathfrak{p}-\mathfrak{q}|^2 \\ &\leq \frac{\mathfrak{p}+1}{4}|\mathfrak{p}-\mathfrak{q}|^2 \\ &= \vartheta(\mathfrak{p})d_c(\mathfrak{p},\mathfrak{q}). \end{aligned}$$

Note that all the hypotheses of Theorem 2.1 are satisfied. Thus there exists an element $u \in X$ such that $\mathfrak{T}u = u$, which is $u = 0$.

In the following theorem, we propose a fixed point result using the nonlinear Kannan-type contraction via the auxiliary function $\vartheta \in \mathrm{B}$.

Theorem 2.3 *Let (X,d_c) be a complete CMLS by the function $\varrho : X \times X \to [1,\infty)$. Let $\mathfrak{T}: X \to X$ where*

$$d_c(\mathfrak{T}\mathfrak{p},\mathfrak{T}\mathfrak{q}) \leq \vartheta(\mathfrak{p})\big[d_c(\mathfrak{p},\mathfrak{T}\mathfrak{p}) + d_c(\mathfrak{q},\mathfrak{T}\mathfrak{q})\big] \tag{2.8}$$

for all $\mathfrak{p},\mathfrak{q} \in X$, where $\vartheta \in \mathrm{B}$. For $\mathfrak{p}_0 \in X$, take $\mathfrak{p}_n = \mathfrak{T}^n\mathfrak{p}_0$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\varrho(\mathfrak{p}_{i+1},\mathfrak{p}_{i+2})}{\varrho(\mathfrak{p}_i,\mathfrak{p}_{i+1})}\varrho(\mathfrak{p}_{i+1},\mathfrak{p}_m) < \frac{1-\vartheta(\mathfrak{p}_0)}{\vartheta(\mathfrak{p}_0)}. \tag{2.9}$$

Also, assume that for every $\mathfrak{p} \in X$, we have

$$\lim_{n\to\infty}\varrho(\mathfrak{p},\mathfrak{p}_n) \quad \textit{exists, is finite and} \quad \lim_{n\to\infty}\varrho(\mathfrak{p}_n,\mathfrak{p}) < \frac{1}{\vartheta(\mathfrak{p}_0)}. \tag{2.10}$$

Then there exists a unique fixed point of $\mathfrak{T}$.

Proof Let $\{\mathfrak{p}_n = \mathfrak{T}\mathfrak{p}_{n-1}\}$ be a sequence in X satisfying hypotheses (2.9) and (2.10). From (2.8) we obtain

$$\begin{aligned} d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) &= d_c(\mathfrak{T}\mathfrak{p}_{n-1},\mathfrak{T}\mathfrak{p}_n) \\ &\leq \vartheta(\mathfrak{p}_{n-1})\big[d_c(\mathfrak{p}_{n-1},\mathfrak{T}\mathfrak{p}_{n-1}) + d_c(\mathfrak{p}_n,\mathfrak{T}\mathfrak{p}_n)\big] \\ &\leq \vartheta(\mathfrak{p}_0)\big[d_c(\mathfrak{p}_{n-1},\mathfrak{p}_n) + d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1})\big]. \end{aligned}$$

Consider $a = \vartheta(\mathfrak{p}_0)$. Then $d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) \leq \frac{a}{1-a}d_c(\mathfrak{p}_{n-1},\mathfrak{p}_n)$. By induction we get

$$d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) \leq \left(\frac{a}{1-a}\right)^n d_c(\mathfrak{p}_1,\mathfrak{p}_0), \quad \forall n \geq 0. \tag{2.11}$$

For all natural numbers n, m, we have

$$d_c(\mathfrak{p}_n,\mathfrak{p}_m) \leq \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{p}_m)d_c(\mathfrak{p}_{n+1},\mathfrak{p}_m).$$

Following the steps of the proof of Theorem 2.1, we deduce

$$\begin{aligned} d_c(\mathfrak{p}_n,\mathfrak{p}_m) &\le \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\varrho(\mathfrak{p}_j,\mathfrak{p}_m)\right)\varrho(\mathfrak{p}_i,\mathfrak{p}_{i+1})d_c(\mathfrak{p}_i,\mathfrak{p}_{i+1}) \\ &\quad + \prod_{k=n+1}^{m-1}\varrho(\mathfrak{p}_k,\mathfrak{p}_m)d_c(\mathfrak{p}_{m-1},\mathfrak{p}_m) \\ &\le \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})\left(\frac{a}{1-a}\right)^n d_c(\mathfrak{p}_0,\mathfrak{p}_1) \\ &\quad + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\varrho(\mathfrak{p}_j,\mathfrak{p}_m)\right)\varrho(\mathfrak{p}_i,\mathfrak{p}_{i+1})\left(\frac{a}{1-a}\right)^i d_c(\mathfrak{p}_0,\mathfrak{p}_1) \\ &\quad + \prod_{i=n+1}^{m-1}\varrho(\mathfrak{p}_i,\mathfrak{p}_m)\left(\frac{a}{1-a}\right)^{m-1} d_c(\mathfrak{p}_0,\mathfrak{p}_1). \end{aligned}$$

Since $0 \le a < \frac{1}{2}$, we have $\frac{a}{1-a} \in (0,1)$. Therefore $\{\mathfrak{p}_n\}$ is a Cauchy sequence, and since (X,d_c) is a complete *CMLS*, $\{\mathfrak{p}_n\}$ converges to some $u \in X$. Suppose that $\mathfrak{T}u \neq u$. Then

$$\begin{aligned} 0 &< d_c(u,\mathfrak{T}u) \le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)d_c(\mathfrak{p}_{n+1},\mathfrak{T}u) \\ &\le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)\vartheta(\mathfrak{p}_n)\big[d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + d_c(u,\mathfrak{T}u)\big] \\ &\le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)\vartheta(\mathfrak{p}_0)\big[d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + d_c(u,\mathfrak{T}u)\big]. \end{aligned} \tag{2.12}$$

As $n \to \infty$ in (2.12) and by (2.10), we conclude that $0 < d_c(u,\mathfrak{T}u) < d_c(u,\mathfrak{T}u)$, which leads us to a contradiction. Thereby $\mathfrak{T}u = u$. Now we may assume that $\mathfrak{T}$ has fixed points u and v. Thus

$$\begin{aligned} d_c(u,v) &= d_c(\mathfrak{T}u,\mathfrak{T}v) \le \vartheta(u)\big[d_c(u,\mathfrak{T}u) + d_c(v,\mathfrak{T}v)\big] \\ &= \vartheta(u)\big[d_c(u,u) + d_c(v,v)\big] = 0. \end{aligned}$$

Hence $u = v$. Therefore the fixed point is unique, as required. □

Example 2.4 Consider $X = \{0,1,2\}$. Take the controlled metric-like d_c defined as

$$d_c(0,1) = \frac{1}{2}, \qquad d_c(0,2) = \frac{11}{20}, \qquad d_c(1,2) = \frac{3}{20}.$$

Let $\varrho : X \times X \to [1,\infty)$ be defined by

$$\begin{aligned} &\varrho(0,0) = \varrho(1,1) = \varrho(2,2) = \varrho(1,2) = \varrho(2,1) = 1, \\ &\varrho(0,2) = \varrho(2,0) = 2, \qquad \varrho(0,1) = \varrho(1,0) = \frac{3}{2}. \end{aligned}$$

Let $\mathfrak{T} : X \to X$ be given by

$$\mathfrak{T}0 = 2 \quad \text{and} \quad \mathfrak{T}1 = \mathfrak{T}2 = 1.$$

Let $\vartheta : X \to [0, \frac{1}{2})$ be given by $\vartheta(0) = \frac{99}{200}$, $\vartheta(1) = \frac{3}{10}$, and $\vartheta(2) = \frac{49}{100}$. Then $\vartheta \in \mathrm{B}$. Take $\mathfrak{p}_0 = 0$, so that (2.9) is satisfied.

Also, it is easy to see that (2.8) holds. By Theorem 2.3 there exists a unique u such that $\mathfrak{T}u = u$, that is, $u = 1$.

Now,we again give a response to an open question in [24], which is a study of a nonlinear Chatterjea-type contraction via an auxiliary function $\vartheta \in \mathrm{B}$.

Theorem 2.5 *Let (X, d_c) be a complete CMLS by the function*

$$\varrho : X \times X \to [1, \infty).$$

$$\textit{Let } \mathfrak{T}: X \to X \textit{ be such that } d_c(\mathfrak{T}\mathfrak{p}, \mathfrak{T}\mathfrak{q}) \le \vartheta(\mathfrak{p})\bigl[d_c(\mathfrak{p}, \mathfrak{T}\mathfrak{q}) + d_c(\mathfrak{q}, \mathfrak{T}\mathfrak{p})\bigr] \tag{2.13}$$

for all $\mathfrak{p}, \mathfrak{q} \in X$, where $\vartheta \in \mathrm{B}$. For $\mathfrak{p}_0 \in X$, take $\mathfrak{p}_n = \mathfrak{T}^n \mathfrak{p}_0$. Suppose that

$$\sup_{i \ge 1} \varrho(\mathfrak{p}_{i-1}, \mathfrak{p}_i) = \beta \quad \text{(exists and is finite)}, \tag{2.14}$$

$$0 < \vartheta(\mathfrak{p}_0) < \frac{1}{2\beta}, \tag{2.15}$$

and

$$\sup_{m \ge 1} \lim_{i \to \infty} \frac{\varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_{i+2})}{\varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1})} \varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_m) < \frac{\beta\vartheta(\mathfrak{p}_0)}{1 - \beta\vartheta(\mathfrak{p}_0)}. \tag{2.16}$$

Also, assume that d_c is continuous with respect to the first variable and that for every $\mathfrak{p} \in X$,

$$\lim_{n \to \infty} \varrho(\mathfrak{p}, \mathfrak{p}_n) \quad \textit{exists, is finite, and} \quad \lim_{n \to \infty} \varrho(\mathfrak{p}_n, \mathfrak{p}) < \frac{1}{\vartheta(\mathfrak{p}_0)}. \tag{2.17}$$

Then $\mathfrak{T}$ possesses a unique fixed point in X.

Proof Consider the sequence $\{\mathfrak{p}_n = \mathfrak{T}\mathfrak{p}_{n-1}\}$ in X satisfying hypotheses (2.14), (2.15), (2.16), and (2.17). From (2.13) and (2.14) we obtain

$$\begin{aligned} d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) &= d_c(\mathfrak{T}\mathfrak{p}_{n-1}, \mathfrak{T}\mathfrak{p}_n) \\ &\le \vartheta(\mathfrak{p}_{n-1})\bigl[d_c(\mathfrak{p}_{n-1}, \mathfrak{T}\mathfrak{p}_n) + d_c(\mathfrak{p}_n, \mathfrak{T}\mathfrak{p}_{n-1})\bigr] \\ &= \vartheta(\mathfrak{p}_{n-1}) d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_{n+1}) \\ &\le \vartheta(\mathfrak{p}_0)\bigl[\varrho(\mathfrak{p}_{n-1}, \mathfrak{p}_n) d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_n) + \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1})\bigr] \\ &\le \beta\vartheta(\mathfrak{p}_0)\bigl[d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_n) + d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1})\bigr]. \end{aligned}$$

Let $b = \frac{\beta\vartheta(\mathfrak{p}_0)}{1-\beta\vartheta(\mathfrak{p}_0)}$. By (2.15) we have $b \in (0, 1)$. Then $d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \le b d_c(\mathfrak{p}_{n-1}, \mathfrak{p}_n)$. By induction we get

$$d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \le b^n d_c(\mathfrak{p}_0, \mathfrak{p}_1), \quad \forall n \ge 0. \tag{2.18}$$

For all natural numbers n, m, we have

$$d_c(\mathfrak{p}_n,\mathfrak{p}_m) \le \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{p}_m)d_c(\mathfrak{p}_{n+1},\mathfrak{p}_m).$$

Following the steps of the proof of Theorem 2.1, we get

$$\begin{aligned} d_c(\mathfrak{p}_n,\mathfrak{p}_m) &\le \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})d_c(\mathfrak{p}_n,\mathfrak{p}_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\varrho(\mathfrak{p}_j,\mathfrak{p}_m)\right)\varrho(\mathfrak{p}_i,\mathfrak{p}_{i+1})d_c(\mathfrak{p}_i,\mathfrak{p}_{i+1}) \\ &\quad + \prod_{k=n+1}^{m-1}\varrho(\mathfrak{p}_k,\mathfrak{p}_m)d_c(\mathfrak{p}_{m-1},\mathfrak{p}_m) \\ &\le \varrho(\mathfrak{p}_n,\mathfrak{p}_{n+1})(b^n d_c(\mathfrak{p}_0,\mathfrak{p}_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\varrho(\mathfrak{p}_j,\mathfrak{p}_m)\right)\varrho(\mathfrak{p}_i,\mathfrak{p}_{i+1})b^i d_c(\mathfrak{p}_0,\mathfrak{p}_1) \\ &\quad + \prod_{i=n+1}^{m-1}\varrho(\mathfrak{p}_i,\mathfrak{p}_m)b^{m-1}d_c(\mathfrak{p}_0,\mathfrak{p}_1). \end{aligned}$$

This implies that $\{\mathfrak{p}_n\}$ is a Cauchy sequence *CMLS* (X,d_c). Since the space is complete, the sequence $\{\mathfrak{p}_n\}$ converges to some $u \in X$. Now suppose that $\mathfrak{T}u \neq u$. Then

$$\begin{aligned} 0 < d_c(u,\mathfrak{T}u) &\le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)d_c(\mathfrak{p}_{n+1},\mathfrak{T}u) \\ &\le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)\vartheta(\mathfrak{p}_n)\big[d_c(\mathfrak{p}_n,\mathfrak{T}u) + d_c(u,\mathfrak{p}_{n+1})\big] \\ &\le \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + \varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)\vartheta(\mathfrak{p}_0)\big[d_c(\mathfrak{p}_n,\mathfrak{T}u) + d_c(u,\mathfrak{p}_{n+1})\big]. \end{aligned} \tag{2.19}$$

As $n \to \infty$ in (2.19), by (2.17) and using the continuity of d_c with respect to its first variable, we deduce that $0 < d_c(u,\mathfrak{T}u) < d_c(u,\mathfrak{T}u)$, which leads us to a contradiction. Thus $\mathfrak{T}u = u$.

Now let us assume that $\mathfrak{T}$ has fixed points u and v. Then

$$\begin{aligned} d_c(u,v) = d_c(\mathfrak{T}u,\mathfrak{T}v) &\le \vartheta(u)\big[d_c(u,\mathfrak{T}v) + d_c(v,\mathfrak{T}u)\big] \\ &= \vartheta(u)\big[d_c(u,u) + d_c(v,v)\big] = 0. \end{aligned}$$

Therefore $u = v$, and thus the fixed point of $\mathfrak{T}$ is unique. □

Now we introduce cyclical orbital contractions in the class of *CMLS*.

Definition 2.6 Let U and V be two nonempty subsets of a *CMLS* (X,d_c). Let $\mathfrak{T}: U \cup V \to U \cup V$ be a cyclic mapping (i.e., $\mathfrak{T}(U) \subseteq V$ and $\mathfrak{T}V \subseteq U$) such that for some $\mathfrak{p} \in U$, there exists $k_{\mathfrak{p}} \in (0,1)$ such that

$$d_c\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}\mathfrak{q}\big) \le k_{\mathfrak{p}} d_c\big(\mathfrak{T}^{2n-1}\mathfrak{p},\mathfrak{q}\big), \tag{2.20}$$

where $n = 1,2,\ldots$ and $\mathfrak{q} \in U$. Then $\mathfrak{T}$ is called a controlled cyclic orbital contraction mapping.

Finally, we prove the following result.

Theorem 2.7 *Let U and V be two nonempty closed subsets of a complete CMLS (X, d_c). Let $\mathfrak{T}: X \to X$ be a controlled cyclic orbital contraction mapping. For $\mathfrak{p}_0 \in U$, take $\mathfrak{p}_n = \mathfrak{T}^n \mathfrak{p}_0$. Suppose that*

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_{i+2})}{\varrho(\mathfrak{p}_i, \mathfrak{p}_{i+1})} \varrho(\mathfrak{p}_{i+1}, \mathfrak{p}_m) < \frac{1}{k_{\mathfrak{p}_0}}. \tag{2.21}$$

Also, assume that for every $\mathfrak{p} \in X$,

$$\lim_{n\to\infty} \varrho(\mathfrak{p}_n, \mathfrak{p}) \quad \text{and} \quad \lim_{n\to\infty} \varrho(\mathfrak{p}, \mathfrak{p}_n) \quad \text{exist and are finite.} \tag{2.22}$$

Then $U \cap V$ is nonempty, and $\mathfrak{T}$ has a unique fixed point.

Proof Suppose there exists $\mathfrak{p}$ (say $\mathfrak{p}_0$) in U satisfying (2.20). Define the iterative sequence $\{\mathfrak{p}_n = \mathfrak{T}^n \mathfrak{p}_0\}$. Since $\mathfrak{p}_0 \in U$ and $\mathfrak{T}$ is cyclic, we have

$$\mathfrak{p}_{2n} \in U \quad \text{and} \quad \mathfrak{p}_{2n+1} \in V \quad \text{for all } n \geq 0. \tag{2.23}$$

By (2.20) we get

$$d_c(\mathfrak{T}^2\mathfrak{p}, \mathfrak{T}\mathfrak{p}) \leq k_{\mathfrak{p}} d_c(\mathfrak{T}\mathfrak{p}, \mathfrak{p}).$$

Again,

$$d_c(\mathfrak{T}^3\mathfrak{p}, \mathfrak{T}^2\mathfrak{p}) = d_c(\mathfrak{T}^2\mathfrak{p}, \mathfrak{T}(\mathfrak{T}^2\mathfrak{p})) \leq k_{\mathfrak{p}} d_c(\mathfrak{T}\mathfrak{p}, \mathfrak{T}^2\mathfrak{p}) \leq (k_{\mathfrak{p}})^2 d_c(\mathfrak{T}\mathfrak{p}, \mathfrak{p}).$$

By induction we obtain that

$$d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \leq [k_{\mathfrak{p}}]^n d_c(\mathfrak{p}_0, \mathfrak{p}_1) \quad \text{for all } n \geq 0. \tag{2.24}$$

Similarly to the proof of Theorem 2.1, we can easily deduce that

$$\lim_{n,m\to\infty} d_c(\mathfrak{p}_n, \mathfrak{p}_m) = 0, \tag{2.25}$$

that is, $\{\mathfrak{p}_n\}$ is a Cauchy sequence in the complete *CMLS* (X, d_c), so $\{\mathfrak{p}_n\}$ converges to some $u \in X$. Since $\{\mathfrak{T}^{2n}\mathfrak{p}\}$ is in U and U is closed, the limit u is in S_1. Similarly, $\{\mathfrak{T}^{2n-1}\mathfrak{p}\}$ is in the closed subset V, so $u \in V$, that is, $u \in U \cap V$, and hence $U \cap V$ is not empty. Let us prove that u is a fixed point of $\mathfrak{T}$. We have

$$d_c(u, \mathfrak{p}_{n+1}) \leq \varrho(u, \mathfrak{p}_n) d_c(u, \mathfrak{p}_n) + \varrho(\mathfrak{p}_n, \mathfrak{p}_{n+1}) d_c(\mathfrak{p}_n, \mathfrak{p}_{n+1}).$$

Using (2.21), (2.22), and (2.25), we get that

$$\lim_{n\to\infty} d_c(u, \mathfrak{p}_{n+1}) = 0. \tag{2.26}$$

By (2.20) we deduce

$$\begin{aligned} d_c(u,\mathfrak{T}u) &\le \varrho\big(u,\mathfrak{T}^{2n}\mathfrak{p}\big)d_c\big(u,\mathfrak{T}^{2n}\mathfrak{p}\big) + \varrho\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}u\big)d_c\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}u\big) \\ &\le \varrho\big(u,\mathfrak{T}^{2n}\mathfrak{p}\big)d_c\big(u,\mathfrak{T}^{2n}\mathfrak{p}\big) + k_{\mathfrak{p}}\varrho\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}u\big)d_c\big(\mathfrak{T}^{2n-1}\mathfrak{p},u\big) \\ &= \varrho(u,\mathfrak{p}_{n+1})d_c(u,\mathfrak{p}_{n+1}) + k_{\mathfrak{p}}\varrho(\mathfrak{p}_{n+1},\mathfrak{T}u)d_c(\mathfrak{p}_{2n-1},u). \end{aligned}$$

Taking the limit as $n\to\infty$ and using (2.22) and (2.26), we deduce that $d_c(u,\mathfrak{T}u)=0$, that is, $\mathfrak{T}u=u$. Finally, assume that $\mathfrak{T}$ has two fixed points, say u and v (they are in U). Then

$$d_c(u,v) = d_c(\mathfrak{T}u,\mathfrak{T}v) = d_c\big(\mathfrak{T}^{2n}u,\mathfrak{T}v\big) \le k_u d_c\big(\mathfrak{T}^{2n-1}u,v\big) = k_u d_c(u,v),$$

which holds unless $d_c(u,v)=0$, so $u=v$. Hence $\mathfrak{T}$ has a unique fixed point. □

The following example illustrates Theorem 2.7.

Example 2.8 Let $X = U\cup V$, where $U=[\frac{1}{4},\frac{1}{2}]andV=[\frac{1}{2},1]$. Consider the controlled metric-like d_c defined as

$$d_c(\mathfrak{p},\mathfrak{q}) = |\mathfrak{p}-\mathfrak{q}|^2,$$

where $\varrho(\mathfrak{p},\mathfrak{q}) = \mathfrak{p}\mathfrak{q}+1$ for $\mathfrak{p},\mathfrak{q}\in X$. Take $\mathfrak{T}\mathfrak{p}=\frac{1}{2}$ if $\mathfrak{p}\in U$ and $\mathfrak{T}\mathfrak{p}=\frac{\mathfrak{p}}{2}$ if $\mathfrak{p}\in V\setminus\{\frac{1}{2}\}$. Now let $k_{\mathfrak{p}}: X\to[0,1]$ be defined as $k_{\mathfrak{p}}=\frac{\mathfrak{p}+1}{2}$. Note that for all $\mathfrak{p}\in U$, we have

$$\mathfrak{T}\mathfrak{p}=\frac{1}{2},\qquad \mathfrak{T}^2\mathfrak{p}=\frac{1}{2},\qquad \ldots,\qquad \mathfrak{T}^{2n-1}\mathfrak{p}=\frac{1}{2},\qquad \mathfrak{T}^{2n}\mathfrak{p}=\frac{1}{2},\qquad \ldots.$$

For all $\mathfrak{q}\in U$, using the fact that

$$d_c\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}\mathfrak{q}\big) = d_c\left(\frac{1}{2},\frac{1}{2}\right) = 0,$$

we deduce that

$$d_c\big(\mathfrak{T}^{2n}\mathfrak{p},\mathfrak{T}\mathfrak{q}\big) \le k_{\mathfrak{p}} d_c\big(\mathfrak{T}^{2n-1}\mathfrak{p},\mathfrak{q}\big).$$

It is not difficult to see that $\mathfrak{T}$ satisfies all the hypotheses of Theorem 2.7. Therefore $\mathfrak{T}$ has a unique fixed point $u=\frac{1}{2}$.

3 Fredholm-type integral equation

Consider the set $X = C([0,1],(-\infty,\infty))$ and the following Fredholm-type integral equation:

$$\mathfrak{p}'(t) = \int_0^1 \mathbb{S}\big(t,s,\mathfrak{p}'(t)\big)\,ds \quad \text{for } t\in[0,1], \tag{3.1}$$

where $\mathbb{S}(t,s,\mathfrak{p}'(t))$ is a continuous function from $[0,1]^2$ into $\mathbb{R}$. Now define

$$\begin{aligned} &d_c: X\times X \longrightarrow \mathbb{R}^+ \\ &\quad(\mathfrak{p},\mathfrak{q}) \mapsto \sup_{t\in[0,1]}\left(\frac{|\mathfrak{p}'(t)|+|\mathfrak{q}(t)|}{2}\right). \end{aligned}$$

Note that (X, d_c) is a complete *CMLS*, where

$$\varrho(\mathfrak{p}, \mathfrak{q}) = 2.$$

Theorem 3.1 *Assume that for all* $\mathfrak{p}, \mathfrak{q} \in X$,

(1) $|\mathbb{S}(t,s,\mathfrak{p}'(t))| + |\mathbb{S}(t,s,\mathfrak{q}(t))| \leq \vartheta(\sup_{t\in[0,1]}(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|))(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|)$ *for some* $\vartheta \in \mathrm{B}$.

(2) $\mathbb{S}(t,s,\int_0^1 \mathbb{S}(t,s,\mathfrak{p}'(t))\,ds) < \mathbb{S}(t,s,\mathfrak{p}'(t))$ *for all* t, s.

Then the integral equation (3.1) *has a unique solution.*

Proof Let $\mho : X \longrightarrow X$ be defined by $\mho\mathfrak{p}'(t) = \int_0^1 \mathbb{S}(t,s,\mathfrak{p}'(t))\,ds$. Then

$$d_c(\mho\mathfrak{p}', \mho\mathfrak{q}) = \sup_{t\in[0,1]}\left(\frac{|\mho\mathfrak{p}'(t)| + |\mho\mathfrak{q}(t)|}{2}\right).$$

Now we have

$$\begin{aligned}
d_c\bigl(\mho\mathfrak{p}'(t), \mho\mathfrak{q}(t)\bigr) &= \frac{|\mho\mathfrak{p}'(t)| + |\mho\mathfrak{q}(t)|}{2} \\
&= \frac{|\int_0^1 \mathbb{S}(t,s,\mathfrak{p}'(t))\,ds| + |\int_0^1 \mathbb{S}(t,s,\mathfrak{q}(t))\,ds|}{2} \\
&\leq \frac{\int_0^1 |\mathbb{S}(t,s,\mathfrak{p}'(t))|\,ds + \int_0^1 |\mathbb{S}(t,s,\mathfrak{q}(t))|\,ds}{2} \\
&= \frac{\int_0^1 (|\mathbb{S}(t,s,\mathfrak{p}'(t))| + |\mathbb{S}(t,s,\mathfrak{q}(t))|)\,ds}{2} \\
&\leq \frac{\int_0^1 \vartheta(\sup_{t\in[0,1]}(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|))(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|)\,ds}{2} \\
&\leq \vartheta\Bigl(\sup_{t\in[0,1]}\bigl(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|\bigr)\Bigr) d_c\bigl(\mathfrak{p}'(t), \mathfrak{q}(t)\bigr).
\end{aligned}$$

Thus $d_c(\mho\mathfrak{p}', \mho\mathfrak{q}) \leq \vartheta(\sup_{t\in[0,1]}(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|)) d_c(\mathfrak{p}', \mathfrak{q})$. Also, notice that

$$\varrho(\mathfrak{p}, \mathfrak{q}) < \frac{1}{\vartheta(\sup_{t\in[0,1]}(|\mathfrak{p}'(t)| + |\mathfrak{q}(t)|))}.$$

Therefore all the hypotheses of Theorem 2.1 are satisfied, and hence equation (3.1) has a unique solution. □

4 Conclusion

We have proved the existence and uniqueness of a fixed point for a self-mapping in controlled metric-like spaces under different nonlinear contractions with a control function. Also, we present an application of our results to Fredholm-type integral equations. Moreover, we would like to bring the reader's attention to the following question.

Question 4.1 Under what conditions we can obtain the same results for a self-mapping in double controlled metric-like spaces [26]?

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