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Application of some new contractions for existence and uniqueness of differential equations involving Caputo–Fabrizio derivative

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Abstract

In this paper we study fractional initial value problems with Caputo–Fabrizio derivative which involves nonsingular kernel. First we apply α-ℓ-contraction and α-type F-contraction mappings to study the existence and uniqueness of solutions for such problems. Finally, we use some contraction mappings in complete $\mathfrak{F}$-metric spaces for this purpose.

Keywords: Caputo–Fabrizio fractional derivative; $\mathfrak{F}$-metric space; α-ℓ-contraction mapping; Fixed point

1 Introduction

Fractional calculus is a part of mathematical analysis that studies the performance of derivative and integral operations on non-integer orders. In the past years, early work in fractional calculus was limited to mathematics. But, in the last few years, extensive studies on the applications of fractional operators in the other disciplines have been conducted. Recently, this field had found many applications in various directions such as applied mathematics, electrochemistry, tracer in fluid flows, fractional-order multi-poles in electromagnetism, finance, signal processing, bio-engineering, viscoelasticity, fluid mechanics, and fluid dynamics [10, 24]. These wide applications have led researchers to provide different definitions of fractional derivatives. The main difference between these definitions is related to possessing different kernels. Two famous fractional derivatives, namely the Riemann–Liouville and the Caputo derivatives, have received a lot of attention and so differential and integral equations containing these derivatives by several methods containing numerical and analytical methods ([22, 29, 30], but these definitions included a singular kernel. Thus, recently Caputo and Fabrizio provided a definition with a nonsingular kernel which the properties of this new definition can be found in [25]. Various methods have been used by researchers to solve differential equations including Caputo–Fabrizio fractional derivative and multi-singular point-wise defined equations (see [16, 18, 32] and the references therein). One of the efficient methods in investigating the existence and

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uniqueness of the solutions of differential equations is using of fixed point theory and for this reason there is a long history of presenting various fixed point theorems (see [1–7, 9, 11, 13, 17, 19, 22, 26–28, 34]).

In 2012, Samet et al. [33] proposed the concept of α-ψ contractive mappings and investigated the existence of fixed points for such mappings, then some researchers improved them to large classes of the contractive type mappings (see [8, 12, 14, 15, 20, 22, 23]).

Wardowski et al. [35] proposed and investigated the F-contraction, then Abbas et al. [17, 21] further generalized the concept of F-contraction and proved some fixed point results.

Wasfi Shatanawi and Erdal Karapınar in [34] introduced F_S−contracfions in the sense of Wardowski and Seghal and F_J−contractions in the sense of Wardowski and Jachymski. Then they ensured some existence and uniqueness fixed point results. Throughout the article $\mathfrak{J}$ denote $[0, 1]$.

In this work, we consider the following differential equation with Caputo–Fabrizio derivatives via fixed point theorems:

$$\left(_0^{CF}D^\varsigma \hbar\right)(p) = f\left(p, \hbar(p)\right), \quad p \in \mathfrak{J}, 0 < \varsigma < 1, \tag{1}$$

$$\hbar(0) = \hbar_0,$$

where D^ς is the Caputo–Fabrizio derivative of order ς and f is continuous with $f(0, \hbar(0)) = 0$.

In what we will have below it is supposed that (M, d) be a complete b-metric space and p_1 is its constant, also the elements of Ω are increasing and continuous functions $\ell : [0, \infty) \to [0, \infty)$ satisfying $\ell(qx) \le q\ell(x) \le qx$, $q > 1$; moreover, Λ denotes the family of nondecreasing functions such that, for $p_1 \ge 1$, we have $\varrho : [0, \infty) \to [0, \frac{1}{p_1{}^2})$.

Definition 1.1 ([5]) Let $\psi : M \to M$ and suppose that there exists $\alpha : M \times M \to [0, \infty)$ with

$$\alpha(x, y)\ell\left(p_1{}^3 d(\psi x, \psi y)\right) \le \varrho\left(\ell\left(d(x, y)\right)\right)\ell\left(d(x, y)\right), \tag{2}$$

for $x, y \in M$, $\varrho \in \Lambda$ and $\ell \in \Omega$. Then ψ is called a generalized α-ℓ-Geraghty contraction mapping.

Definition 1.2 ([33]) Let for $\psi : M \to M$ where $M \ne \emptyset$ and $\alpha : M \times M \to [0, \infty)$ we have

$$\alpha(x, y) \ge 1 \quad \Longrightarrow \quad \alpha(\psi x, \psi y) \ge 1, \quad \forall x, y \in M. \tag{3}$$

Then ψ is called an α-admissible mapping.

Now, we have the following fixed point theorem.

Theorem 1.3 ([5]) *Let $\psi : M \to M$ be a generalized α-ℓ-Geraghty and*

(n1) *ψ is α-admissible;*

(n2) *there exist $J_0 \in M$ with $\alpha(J_0, \psi J_0) \ge 1$;*

(n3) $\{J_n\} \subseteq M$, $\lim_{n \to \infty} J_n = u$ where $u \in M$ and $\alpha(J_n, J_{n+1}) \geq 1$
implies $\alpha(J_n, J) \geq 1$.

Then there exists a fixed point for the mapping ψ.

Definition 1.4 Let $0 < \varsigma < 1$, $J \in C^1[0, b]$, $b > 0$. The Caputo–Fabrizio derivative for J of order ς is defined by

$$^{CF}D^\varsigma J(p) = \frac{M(\varsigma)(2-\varsigma)}{2(1-\varsigma)} \int_0^p \exp\left(-\frac{\varsigma}{1-\varsigma}(p-x)\right) J'(x)\, dx, \quad s \geq 0 \tag{4}$$

where $M(\varsigma)$ is a normalization constant depending on ς with $M(0) = M(1) = 1$. Note that $(^{CF}D^\varsigma)(J) = 0$ if and only if J is a constant.

Definition 1.5 Let $0 < \varsigma < 1$. The Caputo–Fabrizio integral for a function J of order ς is defined by

$$^{CF}I^\varsigma(J)(p) = \frac{2(1-\varsigma)}{M(\varsigma)(2-\varsigma)} J(p) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^p J(x)\, dx, \quad p \geq 0. \tag{5}$$

Take as $d: M \times M \to [0, \infty)$ given by

$$d(\hbar, \wp) = \left\| (\hbar - \wp)^2 \right\|_\infty = \sup_{p \in \mathfrak{J}} (\hbar(p) - \wp(p))^2,$$

where $p_1 = 2$ is the constant of (M, d) and $M = \mathcal{C}(\mathfrak{J}, R)$.

In this paper we consider

$$\left(^{CF}_0 D^\varsigma \hbar\right)(p) = f(p, \hbar(p)), \quad p \in \mathfrak{J}, 0 < \varsigma < 1, \tag{6}$$

$$\hbar(0) = \hbar_0,$$

where D^ς is the Caputo–Fabrizio derivative of order ς, also it is supposed that $f: \mathfrak{J} \times M \to M$ satisfies in $f(0, \hbar(0)) = 0$ and is continuous.

It is easy to prove the following lemma.

Lemma 1.6 *If* $0 < \varsigma < 1$, *then*

$$\left(^{CF}I_b^\varsigma {}^{CF}D^\varsigma \hbar\right)(p) = \hbar(p) - \hbar(b). \tag{7}$$

2 Main results

In this section, for existence and uniqueness of a solution for the problem be defined in (6) firs we apply an α-ℓ-contraction, then we continue by using an α-type F-contraction and another contraction in complete $\mathfrak{F}$-metric space to examine the existence and uniqueness of solutions of the mentioned problem.

Theorem 2.1 *Suppose*
(n1) *there exist* $J: R^2 \to R$ *such that*

$$\left| f(p, \hbar(p)) - f(p, \wp(p)) \right| \leq \frac{(2-\varsigma)M(\varsigma)}{4\sqrt{2}} \sqrt{\varrho\left(\ell\left(|\hbar(p) - x(p)|^2\right)\right) \ell\left(|\hbar(p) - \wp(p)|^2\right)}$$

for $p \in \mathfrak{J}$, $\ell \in \Omega$ *and* $\hbar, \wp \in \mathcal{C}(\mathfrak{J}, R)$ *with* $\jmath(\hbar, \wp) \geq 0$;

($n2$) *there exist* $\hbar_1 \in \mathcal{C}(\mathfrak{J})$ *with* $\jmath(\hbar_1(p), \top\hbar_1(p)) \geq 0$, $p \in \mathfrak{J}$, *where* $\top : \mathcal{C}(\mathfrak{J}) \to \mathcal{C}(\mathfrak{J})$ *is defined by*

$$(\top\hbar)(p) = \hbar_0 + {}_0^{CF}I^{\varsigma}f(p, \hbar(p));$$

($n3$) *for* $p \in \mathfrak{J}$ *and* $\hbar, \wp \in \mathcal{C}(\mathfrak{J})$, $\jmath(\hbar(p), \wp(p)) \geq 0$ *implies* $\jmath(\top\hbar(p), \top\wp(p)) \geq 0$;

($n4$) $\{\hbar_n\} \subseteq \mathcal{C}(\mathfrak{J})$, $\hbar_n \to \hbar$ *where* $\hbar \in \mathcal{C}(\mathfrak{J})$ *and* $\jmath(\hbar_n, \hbar_{n+1}) \geq 0$ *implies* $\jmath(\hbar_n, \hbar) \geq 0$, *for* $n \in \mathrm{N}$.

Then there exist at least one solution for the problem (6).

Proof Applying the Caputo–Fabrizio integral and using Proposition 1.6, from (6) we have

$$\hbar(p) = \hbar_0 + {}_0^{CF}I^{\varsigma}f(p, \hbar(p)) = \top\hbar(p).$$

We prove that $\top$ has a fixed point. Thus,

$$
\begin{aligned}
&\left|\top\hbar(p) - \top\wp(p)\right|^2 \\
&= \left|{}_0^{CF}I^k\big[f(p, \hbar(p)) - f(p, \wp(p))\big]\right|^2 \\
&\leq \Bigg\{\left|\left|\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\big[f(p, \hbar(p)) - f(p, \wp(p))\big]\right.\right. \\
&\qquad \left.\left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^p \big[f(p, \hbar(p)) - f(p, \wp(p))\big]\,dp\right|\right|\Bigg\}^2 \\
&\leq \Bigg\{\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\left|f(p, \hbar(p)) - f(p, \wp(p))\right| \\
&\qquad + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^p \left|f(p, \hbar(p)) - f(p, \wp(p))\right|\,dp\Bigg\}^2 \\
&\leq \left\{\frac{(2-\varsigma)M(\varsigma)}{4\sqrt{2}} \cdot \frac{2}{(2-\varsigma)M(\varsigma)}\right\}^2 \Bigg\{(1-\varsigma)\sqrt{\varrho\big(\ell(|\hbar(p) - \wp(p)|^2)\big)\ell(|\hbar(p) - \wp(p)|^2)} \\
&\qquad + \varsigma\int_0^p \sqrt{\varrho\big(\ell(|\hbar(p) - \wp(p)|^2)\big)\ell(|\hbar(p) - \theta(p)|^2)}\,dp\Bigg\}^2 \\
&= \left\{\left(\frac{(1-\varsigma)}{2\sqrt{2}} + \frac{\varsigma}{2\sqrt{2}}\right)\left(\sqrt{\varrho\Big(\ell\Big(\sup_{p\in\mathfrak{J}}|\hbar(p) - \wp(p)|^2\Big)\Big)\ell\Big(\sup_{p\in\mathfrak{J}}|\hbar(p) - \wp(p)|^2\Big)}\right)\right\}^2 \\
&\leq \left\{\frac{1}{2\sqrt{2}}\sqrt{\varrho\Big(\ell\Big(\sup_{p\in\mathfrak{J}}|\hbar(p) - \wp(p)|^2\Big)\Big)\ell\Big(\sup_{p\in\mathfrak{J}}|\hbar(p) - \wp(p)|^2\Big)}\right\}^2 \\
&= \left\{\frac{1}{2\sqrt{2}}\sqrt{\varrho\big(\ell(d(\hbar, \wp))\big)\ell(d(\hbar, \wp))}\right\}^2 \\
&= \frac{1}{8}\varrho\big(\ell(d(\hbar, \wp))\big)\ell(d(\hbar, \wp)).
\end{aligned}
$$

Hence for $\hbar, \wp \in \mathcal{C}(\mathfrak{J})$, $p \in \mathfrak{J}$ with $\jmath(\hbar(p), \wp(p)) \geq 0$, we have

$$8\left\|(\top\hbar - \top\wp)^2\right\|_{\infty} \leq \varrho\big(\ell(d(\hbar, \wp))\big)\ell(d(\hbar, \wp)).$$

Put $\alpha : \mathcal{C}(\mathfrak{J}) \times \mathcal{C}(\mathfrak{J}) \to [0, \infty)$ by

$$\alpha(\hbar, \wp) = \begin{cases} 1 & J(\hbar(p), \wp(p)) \geq 0, p \in \mathfrak{J}, \\ 0 & \text{else,} \end{cases}$$

and

$$\alpha(\hbar, \wp)\ell\big(8d(\top\hbar, \top\wp)\big) \leq 8d(\top\hbar, \top\wp)$$
$$\leq \varrho\big(\ell\big(d(\hbar, \wp)\big)\big)\ell\big(d(\hbar, \wp)\big).$$

So, $\top$ is an α-ℓ-contraction. To show that $\top$ is α-admissible, we have from (n3)

$$\alpha(\hbar, \wp) \geq 1 \quad \Rightarrow \quad J\big(\hbar(p), \wp(p)\big) \geq 0 \quad \Rightarrow \quad J\big(\top(\hbar), \top(\wp)\big) \geq 0$$
$$\Rightarrow \quad \alpha\big(\top(\hbar), \top(\wp)\big) \geq 1,$$

for $\hbar, \wp \in \mathcal{C}(\mathfrak{J})$. By (n2), we have $\hbar_0 \in \mathcal{C}(\mathfrak{J})$ such that $\alpha(\hbar_0, \top\hbar_0) \geq 1$. From (n4) and Theorem 1.3, there exists $\hbar^* \in \mathcal{C}(\mathfrak{J})$ such that $\hbar^* = \top\hbar^*$.   □

Now to define an α-type F-contraction mapping, let $\mathfrak{F}$ be the family of strictly increasing functions $F : \mathfrak{R}_+ \to R$ such that there exists $k \in (0, 1)$ for which $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$ and also $\lim_{n \to \infty} F(\alpha_n) = -\infty$ if and only if $\lim_{n \to \infty} \alpha_n = 0$ for each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers.

Definition 2.2 Suppose that there exist $y > 0$, $F \in \mathfrak{F}$ and $\alpha : M \times M \to \{-\infty\} \cup (0, \infty)$ such that for $\hbar, \wp \in M$ we have $d(\psi\hbar, \psi\wp) > 0$ and

$$y + \alpha(\hbar, \wp)F\big(d(\psi\hbar, \psi\wp)\big) \leq F\big(d(\hbar, \wp)\big).$$

Then $\psi : M \to M$ is called an α-type F-contraction on M where (M, d) is a metric space.

We present the following theorem.

Theorem 2.3 ([17]) *Let (M, d) be a metric space and $\psi : M \to M$ be an α-type F-contraction such that*:
  (n1) *there exist $\hbar_0 \in M$ with $\alpha(\hbar_0, \psi\hbar_0) \geq 1$,*
  (n2) *ψ is α-admissible,*
  (n3) *if $\{\hbar_n\} \subseteq M$ with $\alpha(\hbar_n, \hbar_{n+1}) \geq 1$ and $\hbar_n \to \hbar$, then $\alpha(\hbar_n, \hbar) \geq 1$, $n \in \mathbb{N}$,*
  (n4) *F is continuous.*
Then ψ has a fixed point $\hbar^* \in M$ and for $\hbar_0 \in M$ the sequence $\{\psi^n \hbar_0\}_{n \in \mathbb{N}}$ is convergent to $\hbar^*$.

If we consider the metric

$$d(\hbar, \wp) = \sup_{t \in \mathfrak{J}}\big\{\big|\hbar(t) - \wp(t)\big|\big\},$$

then we can prove the following theorem in $X = C(\mathfrak{J}, R)$.

Theorem 2.4 *There exist $J : R^2 \to R$ such that*

(H1) $|f(p, \hbar(p)) - f(p, \wp(p))| \leq \frac{(2-\varsigma)M(\varsigma)}{2} e^{-y} |\hbar(p) - \wp(p)|$ *for $p \in \mathfrak{J}$ and $\hbar, \wp \in R$;*

(H2) $\exists \hbar_1 \in \mathcal{C}(\mathfrak{J})$ *such that* $J(\hbar_1(p), \top \hbar_1(p)) \geq 0$ *for $p \in \mathfrak{J}$, where $\top : \mathcal{C}(\mathfrak{J}) \to \mathcal{C}(\mathfrak{J})$ defined by the following*

$$\top \hbar(p) = \hbar_0 + {}_0^{CF} I^\varsigma f(p, \hbar(p));$$

(H3) *for $p \in \mathfrak{J}$ and $\hbar, \wp \in \mathcal{C}(\mathfrak{J})$, $J(\hbar(p), \wp(p)) \geq 0$ implies $J(\top \hbar(p), \top \wp(p)) \geq 0$:*

(H4) $\{\hbar_n\} \subseteq \mathcal{C}(\mathfrak{J})$, $\lim_{n \to \infty} \hbar_n = \hbar$ *where $\hbar \in \mathcal{C}(\mathfrak{J})$ and $J(\hbar_n, \hbar_{n+1}) \geq 0$ implies $J(\hbar_n, \hbar) \geq 0$, for $n \in \mathrm{N}$.*

Then there exists at least one fixed point for $\top$ which is the solution of problem (6).

Proof We show that $\top$ has a fixed point. Thus,

$$\begin{aligned}
&\left| \top \hbar(p) - \top \wp(p) \right| \left| \top \hbar(p) - \top \wp(p) + 1 \right| \\
&= \left| {}^{CF} I \left[f(p, \hbar(p)) - f(p, \wp(p)) \right] \right| \left| {}^{CF} I \left[f(p, \hbar(p)) - f(p, \wp(p)) \right] + 1 \right| \\
&\leq \left(\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \left| f(p, \hbar(p)) - f(p, \wp(p)) \right| \right. \\
&\quad \left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^p \left| f(p, \hbar(p)) - f(p, \wp(p)) \right| dp \right) \\
&\quad \times \left(\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \left| f(p, \hbar(p)) - f(p, \wp(p)) \right| \right. \\
&\quad \left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^p \left| f(p, \hbar(p)) - f(p, \wp(p)) \right| dp + 1 \right) \\
&\leq \left(\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} e^{-y} \left| \hbar(p) - \wp(p) \right| \right. \\
&\quad \left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} \int_0^p e^{-y} \left| \hbar(p) - \wp(p) \right| dp \right) \\
&\quad \times \left(\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} e^{-y} \left| \hbar(p) - \wp(p) \right| \right. \\
&\quad \left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} \int_0^p e^{-y} \left| \hbar(p) - \wp(p) \right| dp + 1 \right) \\
&\leq \left(e^{-y} \sup_{p \in \mathfrak{J}} \left| \hbar(p) - \wp(p) \right| \right) \left(e^{-y} \sup_{p \in \mathfrak{J}} \left| \hbar(p) - \wp(p) \right| + 1 \right) \\
&= \left(e^{-y} d(\hbar, \wp) \right) \left(e^{-y} d(\hbar, \wp) + 1 \right) \\
&= \left(e^{-y} d(\hbar, \wp) \right)^2 + \left(e^{-y} d(\hbar, \wp) \right) \\
&= e^{-y} \left[e^{-y} \left(d(\hbar, \wp) \right)^2 + d(\hbar, \wp) \right] \\
&\leq e^{-y} \left[\left(d(\hbar, \wp) \right)^2 + d(\hbar, \wp) \right].
\end{aligned}$$

Hence for $\hbar, \wp \in \mathcal{C}(\mathfrak{J})$, $p \in \mathfrak{J}$ with $J(\hbar(p), \wp(p)) \geq 0$, we have

$$\begin{aligned}
&\left(d(\top \hbar(p), \top \wp(p)) \right)^2 + d(\top \hbar(p), \top \wp(p)) \\
&\leq e^{-y} \left[\left(d(\hbar, \wp) \right)^2 + d(\hbar, \wp) \right].
\end{aligned}$$

So

$$\mathrm{Ln}\big[\big(d\big(\top\hbar(p),\top\wp(p)\big)\big)^2 + d\big(\top\hbar(p),\top\wp(p)\big)\big]$$
$$\leq \mathrm{Ln}\, e^{-y} + \mathrm{Ln}\big[\big(d(\hbar,\wp)\big)^2 + d(\hbar,\wp)\big],$$

therefore

$$y + \mathrm{Ln}\big[\big(d\big(\top\hbar(p),\top\wp(p)\big)\big)^2 + d\big(\top\hbar(p),\top\wp(p)\big)\big]$$
$$\leq \mathrm{Ln}\big[\big(d(\hbar,\wp)\big)^2 + d(\hbar,\wp)\big].$$

Now, suppose $F : [0,\infty) \to R$ is defined by $F(u) = \mathrm{Ln}(u^2 + u)$, $u > 0$, then it is straightforward to show that $F \in \mathfrak{F}$.

Set $\alpha : \mathcal{C}(\mathfrak{J}) \times \mathcal{C}(\mathfrak{J}) \to \{-\infty\} \cup [0,\infty)$ by

$$\alpha(\hbar,\wp) = \begin{cases} 1 & J(\hbar(p),\wp(p)) \geq 0, \text{ for all } p \in \mathfrak{J}, \\ -\infty & \text{else}, \end{cases}$$

then we have $y + \alpha(\hbar,\wp)F(d(\top\hbar,\top\wp)) \leq F(d(\hbar,\wp))$ for $\hbar,\wp \in M$ with $d(\top\hbar,\top\wp) > 0$. Therefore we conclude that $\top$ satisfies all conditions of definition of α-type F-contraction. By (H3),

$$\alpha(\hbar,\wp) \geq 1 \quad \Rightarrow \quad J\big(\hbar(p),\wp(p)\big) \geq 0 \quad \Rightarrow \quad J\big(\top(\hbar),\top(\wp)\big) \geq 0$$
$$\Rightarrow \quad \alpha\big(\top(\hbar),\top(\wp)\big) \geq 1, \quad \hbar,\wp \in \mathcal{C}(\mathfrak{J}),$$

which shows that $\top$ is α-admissible. From (H2) we have $\hbar_0 \in \mathcal{C}(\mathfrak{J})$ such that $\alpha(\hbar_0,\top\hbar_0) \geq 1$. According to (H4) and Theorem 2.3, we can obtain $\hbar^* \in \mathcal{C}(\mathfrak{J})$ where $\hbar^* = \top\hbar^*$ which is a fixed point of $\top$ and therefore a solution of the problem. □

Now to use in the next definition let us define $\mathfrak{F}$ involved the functions $\psi : (0,\infty) \to R$ such that:

($\mathfrak{F}_1$) $0 < p < t$ implies $\psi(p) \leq \psi(t)$;
($\mathfrak{F}_2$) $s_n \to 0$ if and only if $\psi(s_n) \to -\infty$,
where $\{s_n\} \subset (0,+\infty)$.

Definition 2.5 ([20]) Let $\psi \in \mathfrak{F}, a \in [0,+\infty)$ and $d : M \times M \to [0,+\infty)$ with the following conditions:

(d_1) $(\hbar,\wp) \in M \times M, d(\hbar,\wp) = 0 \Leftrightarrow \hbar = \wp$;
(d_2) $d(\hbar,\wp) = d(\wp,\hbar)$, for $(\hbar,\wp) \in M \times M$;
(d_3) If $(\hbar,\wp) \in M \times M, (u_i)_{i=1}^N \subset M$ such that $(u_1,u_N) = (\hbar,\wp), N \in \mathrm{N}, N \geq 2$ we have

$$d(\hbar,\wp) > 0 \quad \text{implies} \quad \psi\big(d(\hbar,\wp)\big) \leq \psi\left(\sum_{i=1}^{N-1} d(u_i,u_{i+1})\right) + a.$$

Then d is called an $\mathfrak{F}$-metric on M, and the pair (M,d) is said to be a $\mathfrak{F}$-metric space.

If we have the following condition for a sequence $\{\hbar_n\}$

$$\lim_{n \to \infty} d(\hbar_n, \hbar) = 0,$$

then we say $\{\hbar_n\}$ is convergent to $\hbar$ with respect to $\mathfrak{F}$-metric d. Similar to the common definitions $\{\hbar_n\}$ is $\mathfrak{F}$-Cauchy in (M, d) if

$$\lim_{n,m \to +\infty} d(\hbar_n, \hbar_m) = 0.$$

Similarly the $\mathfrak{F}$-completeness of (M, d) can be defined.

Essential are the next definition and also the next fixed point theorem: let Γ be the set of $\ell : [0, \infty) \to [0, \infty)$ such that

(ℓ_1) ℓ is nondecreasing;
(ℓ_2) $\sum_{n=1}^{\infty} \ell^n(p) < \infty$ for $p \in R^+$, where ℓ^n is the nth iterate of ℓ.

Definition 2.6 ([31]) Suppose that $\psi : M \to M$ and $\alpha : M \times M \to [0, \infty)$, if for $p \in M$

$$\alpha(p, \psi p) \geq 1 \quad \Rightarrow \quad \alpha\big(\psi p, \psi^2 p\big) \geq 1. \tag{8}$$

Then ψ is said to be an α-orbital admissible.

Theorem 2.7 ([13]) *Suppose (M, d) be a $\mathfrak{F}$-complete metric space and $\psi : M \to M$ such that*

$$\alpha(\hbar, \wp) d(\psi \hbar, \psi \wp) \leq \ell\big(d(\hbar, \wp)\big),$$

for $\hbar, \wp \in M$, where $\ell \in \Gamma$. Also assume

($n1$) ψ *is α-orbital admissible;*
($n2$) *there exists $\hbar_0 \in M$ with $\alpha(\hbar_0, \psi \hbar_0) \geq 1$;*
($n3$) $\psi \in \mathfrak{F}$ *verifying (d_3) is continuous and for a continuous function ℓ we have $\psi(u) > \psi(\ell(u)) + a$, $0 < u < \infty$, such that a is the same appeared in (d_3).*
Then there exists a fixed point for f.

Now let us define the $\mathfrak{F}$-metric $d : M \times M \to [0, \infty)$ given by

$$d(\hbar, \wp) = \begin{cases} e^{|\hbar - \wp|} & \text{if } \hbar \neq \wp, \\ 0 & \text{if } \hbar = \wp, \end{cases}$$

where $M = \mathcal{C}(\mathfrak{J}, \mathrm{N})$ and continues function ψ on $(0, \infty)$ by $\psi(p) = -\frac{1}{p}$ for $p > 0$. Since $-\frac{1}{u} > \frac{1}{\ell(u)} > 1$, it is obvious that $\psi(u) > \psi(\ell(u)) + a$, $u > 0$, in the sense that ℓ has the following conditions:

$$\ell(u) < \frac{u}{u+1},$$

$$e^{\ell(p)} \leq \ell\big(e^p\big), \quad p \in \{0, 1, 2, 3, \ldots\}.$$

Now we are ready to present the following theorem.

Theorem 2.8 *Assume*

(*n*1) *there exists* $\jmath : R^2 \to R$ *with*

$$\left| f\big(p, \hbar(p)\big) - f\big(p, \wp(p)\big) \right| \le \frac{(2-\varsigma)M(\kappa)}{2} \ell\big(\big|\hbar(p) - \wp(p)\big|\big),$$

where $s \in \mathfrak{J}$ *and* $\hbar, \wp \in R$ *such that* $\jmath(\hbar, \wp) \ge 0$;

(*n*2) *there exists* $\hbar_1 \in \mathcal{C}(\mathfrak{J})$ *with* $\jmath(\hbar_1(p), \top\hbar_1(p)) \ge 0$ *where* $p \in \mathfrak{J}$ *and* $\top : \mathcal{C}(\mathfrak{J}) \to \mathcal{C}(\mathfrak{J})$ *is defined by*

$$(\top\hbar)(p) = \hbar_0 + {}_0^{CF}I^\varsigma f\big(p, \hbar(p)\big); \tag{9}$$

(*n*3) *for* $p \in \mathfrak{J}$ *and* $\hbar \in \mathcal{C}(\mathfrak{J})$, $\jmath(\hbar(p), \top\hbar(p)) \ge 0$ *implies* $\jmath(\top\hbar(p), \top^2\hbar(p)) \ge 0$.

Then there exists at least one fixed point for $\top$ *which is the solution of* (6).

Proof We have

$$
\begin{aligned}
\left| \top\hbar(p) - \top\wp(p) \right| = \left| {}^{CF}_0 I\big[f\big(p, \hbar(p)\big) - f\big(p, \wp(p)\big)\big] \right| \\
\le \left| \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} \ell \big|\hbar(p) - \wp(p)\big| \right. \\
\left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} \int_0^p \ell\big|\hbar(p) - x(p)\big| \, dp \right| \\
\le (1 - \varsigma + \varsigma)(\sup_{p \in \mathfrak{J}} \big|\hbar(p) - \wp(p)\big| \\
\le \ell\big(\big|\hbar(p) - \wp(p)\big|\big).
\end{aligned}
$$

Hence for $\hbar, \wp \in \mathcal{C}(\mathfrak{J}), p \in \mathfrak{J}$ with $\jmath(\hbar(p), \wp(p)) \ge 0$, we have

$$d(\top\hbar, \top\wp) = e^{|\top\hbar(p) - \top\wp(p)|} \le e^{\ell(|\hbar(p) - \wp(p)|)} \le \ell\big(e^{|\hbar(p) - \wp(p)|}\big) = \ell\big(d(\hbar, \wp)\big).$$

Put $\alpha : \mathcal{C}(\mathfrak{J}) \times \mathcal{C}(\mathfrak{J}) \to [0, \infty)$ by

$$
\alpha(\hbar, \wp) = \begin{cases} 1 & \jmath(\hbar(p), \wp(p)) \ge 0, p \in \mathfrak{J}, \\ 0 & \text{else.} \end{cases}
$$

Therefore $\alpha(\hbar, \wp)d(\top\hbar, \top\wp)) \le d(\top\hbar, \top\wp)) \le \ell(d(\hbar, \wp)), \hbar, \wp \in M$ with $d(\top\hbar, \top\wp) > 0$. From (n3),

$$
\begin{aligned}
\alpha(\hbar, \top\hbar) \ge 1 &\implies \jmath\big(\hbar(p), \top\hbar(p)\big) \ge 0 \implies \jmath\big(\top(\hbar), \top^2(\hbar)\big) \ge 0 \\
&\implies \alpha\big(\top(\hbar), \top^2(\hbar)\big) \ge 1,
\end{aligned}
$$

for $\hbar \in \mathcal{C}(\mathfrak{J})$. Therefore, we conclude that $\top$ is orbital α-admissible. From (n2), we can choose $\hbar_1 \in \mathcal{C}(\mathfrak{J})$ such that $\alpha(\hbar_1, \top\hbar_1) \ge 1$. Also from (n3) and Theorem 2.7, we get $\wp^* \in \mathcal{C}(\mathfrak{J})$ with $\hbar^* = \top\hbar^*$. Hence, we obtain $\hbar^*$ as a solution of the problem and this completes the proof. □

Now, for a positive integer n we denote by h^n the nth iterate of h, so that $y = h^0 y$ and $h^{n+1} y = h(h^n y)$ for $y \in X$ and $n \in N$. The triplet (X, d, h) represent a metric space (X, d) with a self-mapping h on it. We shall use $(X*, d, h)$ to indicate the corresponding metric space is complete. Also on (X, d, h) an orbit of $y_0 \in X$ is the set

$$O(y_0) = \{h^n y_0 : n = 0, 1, 2, \dots\},$$

and $\rho(y_0)$ denote to the diameter of the set $O(y_0)$. Note that for any subset B of X, $\rho(B) = \sup\{d(u, y) : u, y \in B\}$ is the diameter of B. We shall use the triplet (X_{0*}, d, h) if for some $y \in X$, every Cauchy sequence from $O(y)$ converges in X. In this case, the corresponding space is called orbitally complete.

Corollary 2.9 *For $(X, d*, h)$ with $p : X \to N$, we suppose there exists $\tau > 0$ such that for $v, w \in X$*

$$d\big(h^{p(y)} y, h^{p(y)} w\big) \le e^{-\tau} d(y, w). \tag{10}$$

Assume there exists $y_0 \in X$ such that $0 < \rho < \infty$. Moreover, (X, d) is h-orbitally complete. Then h has a unique fixed point.

Theorem 2.10 *On account of* (6), *we assume that*

$$\big|h\big(p, \hbar(p)\big) - h\big(p, \wp(p)\big)\big| \le \frac{(2 - \varsigma)M(\varsigma)}{2} e^{-y} \left| \sqrt{\big|\hbar(p)\big|} - \sqrt{\big|\wp(p)\big|} \right|$$

and

$$\big|h\big(p, \hbar(p)\big)\big| + \big|h\big(p, \wp(p)\big)\big| \le \frac{(2 - \varsigma)M(\varsigma)}{2} e^{-y} \left| \sqrt{\big|\hbar(p)\big|} + \sqrt{\big|\wp(p)\big|} \right|,$$

for $p \in \mathfrak{J}$, then the problem (6) *possesses a unique solution.*

Proof We consider $d : M \times M \to [0, \infty)$ given by $d(\hbar, \wp) = \sup_{p \in \mathfrak{J}} |\hbar(p) - \wp(p)|$ and, applying the Caputo–Fabrizio integral to both sides of (6), we get

$$(\top \hbar)(p) = \hbar_0 + {}_0^{CF} I^\varsigma h\big(p, \hbar(p)\big).$$

We demonstrate that (6) has a unique solution. We get

$$\begin{aligned}
\big|\top \hbar(p) - \top \wp(p)\big| &= \big|{}^{CF} I\big[h\big(p, \hbar(p)\big) - h\big(p, \wp(p)\big)\big]\big| \\
&\le \frac{2(1 - \varsigma)}{(2 - \varsigma)M(\varsigma)} \big|h\big(p, \hbar(p)\big) - h\big(p, \wp(p)\big)\big| \\
&\quad + \frac{2\varsigma}{(2 - \varsigma)M(\varsigma)} \int_0^p \big|h\big(p, \hbar(p)\big) - h\big(p, \wp(p)\big)\big|\, dp \\
&\le \frac{2(1 - \varsigma)}{(2 - \varsigma)M(\varsigma)} \cdot \frac{(2 - \varsigma)M(\varsigma)}{2} e^{-y} \left| \sqrt{\big|\hbar(p)\big|} - \sqrt{\big|\wp(p)\big|} \right| \\
&\quad + \frac{2\varsigma}{(2 - \varsigma)M(\varsigma)} \cdot \frac{(2 - \varsigma)M(\varsigma)}{2} e^{-y} \int_0^p \left| \sqrt{\big|\hbar(p)\big|} - \sqrt{\big|\wp(p)\big|} \right| dp
\end{aligned}$$

$$\le e^{-y} \sup \left| \sqrt{\left|\hbar(p)\right|} - \sqrt{\left|\wp(p)\right|} \right|,$$

also

$$\left|\top\hbar(p)\right| + \left|\top\wp(p)\right| = \left|{}^{CF}I\left[h\left(p, \hbar(p)\right)\right]\right| + \left|{}^{CF}I\left[h\left(p, \wp(p)\right)\right]\right|$$

$$\le {}^{CF}I\left[\left|h\left(p, \hbar(p)\right)\right| + \left|h\left(p, \wp(p)\right)\right|\right]$$

$$\le \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} e^{-y} \left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right|$$

$$+ \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \cdot \frac{(2-\varsigma)M(\varsigma)}{2} e^{-y} \int_0^p \left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right| dp$$

$$\le e^{-y} \sup \left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right| \le \sup \left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right|.$$

On the other hand

$$\sup\left(\left|\top\hbar(p)\right| + \left|\top\wp(p)\right|\right) \le \sup\left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right|,$$

also we have

$$d\left(\top^2\hbar, \top^2\wp\right) = \sup\left(\left|\top^2\hbar(p) - \top^2\wp(p)\right|\right)$$

$$= \sup\left(\left|\top\hbar(p) - \top\wp(p)\right|\right) \times \sup\left(\left|\top\hbar(p) + \top\wp(p)\right|\right)$$

$$\le \sup\left(\left|\top\hbar(p) - \top\wp(p)\right|\right) \times \sup\left(\left|\top\hbar(p)\right| + \left|\top\wp(p)\right|\right)$$

$$\le e^{-y} \sup \left|\sqrt{\left|\hbar(p)\right|} - \sqrt{\left|\wp(p)\right|}\right| \times \sup \left|\sqrt{\left|\hbar(p)\right|} + \sqrt{\left|\wp(p)\right|}\right|$$

$$= e^{-y} \sup \left|\left|\hbar(p)\right| - \left|\wp(p)\right|\right|$$

$$\le e^{-y} \sup \left|\hbar(p) - \wp(p)\right|$$

$$= e^{-y} d(\hbar, \wp).$$

Hence condition (10) holds with $p : X \to N$ such that $p(\hbar) = 2$, $\hbar \in X$. Accordingly all axioms of Corollary 2.9 are verified and consequently $\top$ possesses a unique fixed point. So (6) possesses a unique solution. □

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