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Some properties of degenerate complete and partial Bell polynomials

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Abstract
In this paper, we study degenerate complete and partial Bell polynomials and establish some new identities for those polynomials. In addition, we investigate the connections between modified degenerate complete and partial Bell polynomials, which are closely related to the degenerate complete and partial Bell polynomials, and the joint distribution of weighted sums of independent degenerate Poisson random variables.

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1 Introduction

The recent research on degenerate versions of some special numbers and polynomials have led us to introduce the fascinating degenerate gamma functions (see [18]), and λ-umbral calculus which is about the study of λ-Sheffer sequences (see [14]). Thus we may say that studying degenerate versions of many special polynomials and numbers is by now well justified.

The complete Bell polynomials and the partial Bell polynomials are, respectively, multivariate versions for Bell polynomials and Stirling numbers of the second kind. They have applications in such diverse areas as combinatorics, probability, algebra and analysis. For example, higher-order derivatives of composite functions can be expressed in terms of the partial Bell polynomials, which is known as the Faà di Bruno formula and the nth moment of a random variable is the nth complete Bell polynomial in the first n cumulants. The number of monomials appearing in the partial Bell polynomial $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ (see (6), (7)) is the number of partitionings of a set with n elements into k blocks and the coefficient of each monomial is the number of partitioning a set with n elements as the corresponding k blocks.

The aim of this paper is to further study the recently introduced degenerate complete and partial Bell polynomials which are degenerate versions of the complete and partial Bell polynomials (see (12), (13)). In more detail, we derive several identities connected with such Bell polynomials whose arguments are given by the sum of two 'variable-vectors' (see Theorems 1–4). Further, we obtain a recurrence relation for the degenerate partial Bell polynomials in Theorem 5. Also, we mention three results for the degenerate partial Bell polynomials which can be derived by the same method as for the partial Bell polynomials (see [9]). Then, as applications to probability theory, we show the connections between the modified degenerate complete and partial Bell polynomials, which are slightly different from the degenerate complete and partial Bell polynomials (see (27), (29)) and the joint distributions of weighted sums of independent degenerate Poisson random variables (see Theorems 6 and 7).

Even though there are a vast number of papers on Bell polynomials in the literature, degenerate versions of complete and partial Bell polynomials are first introduced in [17] and [19]. The contribution of the present paper is twofold. The first one is the derivation of further results on degenerate complete and incomplete Bell polynomials. The second one is the applications to probability theory which shows certain connections between the modified degenerate complete and partial Bell polynomials and the joint distributions of weighted sums of independent degenerate Poisson random variables. Some of the recent work on Bell polynomials can be found in [1, 3, 4, 6, 7, 9, 10, 12, 25].

For the rest of this section, we recall the necessary facts that are needed throughout this paper. For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_\lambda^x(t) = \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}, \tag{1}$$

where

$$\begin{aligned} &(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\bigl(x-(n-1)\lambda\bigr) \quad (n \geq 1), \\ &e_\lambda(t) = e_\lambda^1(t) = \sum_{l=0}^{\infty} (1)_{l,\lambda} \frac{t^l}{l!} \quad (\text{see [13, 15–17, 19–24, 26]}). \end{aligned} \tag{2}$$

Recently, Kim–Kim introduced the degenerate Stirling numbers of the second kind given by

$$\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \text{ (see [13])}. \tag{3}$$

Note that $(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, (n \geq 0)$, and $\lim_{\lambda\to 0} S_{2,\lambda}(n,l) = S_2(n,l)$, where $S_2(n,l)$ are the Stirling numbers of the second kind.

In [19], the degenerate Bell polynomials are defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!} \quad (\text{see [2, 5, 8, 9, 11, 13, 15–17, 19–24]}). \tag{4}$$

Thus, by (3) and (4), we get

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{l=0}^{n} S_{2,\lambda}(n,l)x^l \quad (\text{see [19]}). \tag{5}$$

For any integers with $n \geq k \geq 0$, the partial Bell polynomials are given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!} \quad \text{(see [8])}. \tag{6}$$

Thus, we note that

$$\begin{aligned} &B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \\ &\quad = \sum_{\substack{l_1+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}} \frac{n!}{l_1!l_2!\cdots l_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{l_1}\left(\frac{x_2}{2!}\right)^{l_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{l_{n-k+1}}. \end{aligned} \tag{7}$$

In [9], it was found that

$$\begin{aligned} &B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \\ &\quad = \frac{1}{x_1}\frac{1}{n-k}\sum_{\alpha=1}^{n-k}\binom{n}{\alpha}\left[(k+1) - \frac{n+1}{\alpha+1}\right]x_{\alpha+1}B_{n-\alpha,k}(x_1, x_2, \dots, x_{n-\alpha-k+1}), \end{aligned} \tag{8}$$

$$\begin{aligned} &B_{n,k_1+k_2}(x_1, x_2, \dots, x_{n-k_1-k_2+1}) \\ &\quad = \frac{k_1!k_2!}{(k_1+k_2)!}\sum_{\alpha=0}^{n}\binom{n}{\alpha}B_{\alpha,k_1}(x_1, \dots, x_{\alpha-k_1+1})B_{n-\alpha,k_2}(x_1, x_2, \dots, x_{n-\alpha-k_2+1}) \end{aligned} \tag{9}$$

and

$$\begin{aligned} &B_{n,k+1}(x_1, x_2, \dots, x_{n-k}) \\ &\quad = \frac{1}{(k+1)!}\sum_{\alpha_1=k}^{n-1}\sum_{\alpha_2=k-1}^{\alpha_1-1}\cdots\sum_{\alpha_k=1}^{\alpha_{k-1}-1}\binom{n}{\alpha_1}\binom{\alpha_1}{\alpha_2}\cdots\binom{\alpha_{k-1}}{\alpha_k}x_{n-\alpha_1}x_{\alpha_1-\alpha_2}\cdots x_{\alpha_{k-1}-\alpha_k}x_{\alpha_k} \end{aligned} \tag{10}$$

$(n \geq k+1, k = 1, 2, \dots)$.

From (6), we note that $B_{n,k}(1, 1, \dots, 1) = S_2(n,k)$, $(n, k \geq 0)$.

Let X be the Poisson random variable with parameter $\alpha > 0$. Then the probability mass function of X is given by

$$p(i) = P\{X = i\} = \frac{\alpha^i}{i!}e^{-\alpha} (i = 0, 1, 2, \dots) \quad \text{(see [26])}. \tag{11}$$

Note that the nth moment of X is given by

$$\begin{aligned} E\left[X^n\right] &= \sum_{k=0}^{\infty} k^n p(k) = e^{-\alpha}\sum_{k=0}^{\infty}\frac{k^n}{k!}\alpha^k \quad \text{(see [26])} \\ &= \mathrm{Bel}_n(\alpha) \quad (n \geq 0), \end{aligned}$$

where $\mathrm{Bel}_n(\alpha)$ are the ordinary Bell polynomials defined by

$$e^{\alpha(e^t-1)} = \sum_{n=0}^{\infty}\mathrm{Bel}_n(\alpha)\frac{t^n}{n!} \quad \text{(see [26])}.$$

Let g be a real valued function. Then the function of $E[g(X)]$ is defined as

$$E\big[g(X)\big]=\sum_{k=0}^{\infty}g(k)p(k)\quad(\text{see [26]}),$$

where $p(k)$ is the probability mass function of the discrete random variable X.

For $\lambda\in(0,1)$, X is the degenerate Poisson random variable with parameter $\alpha(>0)$ if the probability mass function of X is given by

$$p_\lambda(i)=P\{X=i\}=e_\lambda^{-1}(\alpha)(1)_{i,\lambda}\frac{\alpha^i}{i!}\quad(\text{see [13, 15]}).$$

Note that $\lim_{\lambda\to 0}P_\lambda(i)=e^{-\alpha}\frac{\alpha^i}{i!}$ is the probability mass function of the Poisson random variable with parameter $\alpha>0$.

Recently, the degenerate partial Bell polynomials are defined by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)^k=\sum_{n=k}^{\infty}B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!}\quad(\text{see [17, 22–24]}),\tag{12}$$

where k is a nonnegative integer.

By (3), we get

$$B_{n,k}^{(\lambda)}(1,1,\ldots,1)=S_{2,\lambda}(n,k)\quad(n\geq k\geq 0).$$

In [17, 19, 24], the degenerate complete Bell polynomials are introduced by

$$\exp\left(\sum_{i=1}^{\infty}x_i(1)_{i,\lambda}\frac{t^i}{i!}\right)=\sum_{n=0}^{\infty}B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}.\tag{13}$$

From (4) and (13), we note that

$$B_n^{(\lambda)}(x,x,\ldots,x)=\mathrm{Bel}_{n,\lambda}(x)\quad(n\geq 0).\tag{14}$$

In particular, by (12) and (13), we get

$$B_n^{(\lambda)}(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{n}B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1}).\tag{15}$$

2 Degenerate complete and degenerate partial Bell polynomials

In this section, we will derive several properties of the degenerate complete and partial Bell polynomials. From (13), we note that

$$\begin{aligned}&\sum_{n=0}^{\infty}B_n^{(\lambda)}(x_1+y_1,x_2+y_2,\ldots,x_n+y_n)\frac{t^n}{n!}\\&\quad=\exp\left(\sum_{i=1}^{\infty}(x_i+y_i)(1)_{i,\lambda}\frac{t^i}{i!}\right)\end{aligned}\tag{16}$$

$$= \exp\left(\sum_{i=1}^{\infty} x_i (1)_{i,\lambda} \frac{t^i}{i!}\right) \exp\left(\sum_{i=1}^{\infty} y_i (1)_{i,\lambda} \frac{t^i}{i!}\right)$$
$$= \sum_{j=0}^{\infty} B_j^{(\lambda)}(x_1, x_2, \ldots, x_j) \frac{t^j}{j!} \sum_{m=0}^{\infty} B_m^{(\lambda)}(y_1, y_2, \ldots, y_m) \frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} B_j^{(\lambda)}(x_1, x_2, \ldots, x_j) B_{n-j}^{(\lambda)}(y_1, y_2, \ldots, y_{n-j}) \right) \frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (16), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$B_n^{(\lambda)}(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(\lambda)}(x_1, x_2, \ldots, x_j) B_{n-j}^{(\lambda)}(y_1, y_2, \ldots, y_{n-j}).$$

Thus, by Theorem 1 and (15), we get

$$\sum_{n=k}^{\infty} B_{n,k}^{(\lambda)}(x_1 + y_1, x_2 + y_2, \ldots, x_{n-k+1} + y_{n-k+1}) \frac{t^n}{n!} \tag{17}$$
$$= \frac{1}{k!} \left(\sum_{m=1}^{\infty} (1)_{m,\lambda} x_j \frac{t^m}{m!} + \sum_{m=1}^{\infty} (1)_{m,\lambda} y_m \frac{t^m}{m!} \right)^k$$
$$= \sum_{i=0}^{k} \frac{1}{i!} \left(\sum_{m=1}^{\infty} (1)_{m,\lambda} y_m \frac{t^m}{m!} \right)^i \frac{1}{(k-i)!} \left(\sum_{m=1}^{\infty} (1)_{m,\lambda} x_m \frac{t^m}{m!} \right)^{k-i}$$
$$= \sum_{i=0}^{k} \sum_{j=i}^{\infty} B_{j,i}^{(\lambda)}(y_1, y_2, \ldots, y_{j-i+1}) \frac{t^j}{j!} \sum_{l=k-i}^{\infty} B_{l,k-i}^{(\lambda)}(x_1, x_2, \ldots, x_{l-k+i+1}) \frac{t^l}{l!}$$
$$= \sum_{i=0}^{k} \sum_{n=k}^{\infty} \sum_{j=i}^{n-k+i} \binom{n}{j} B_{j,i}^{(\lambda)}(y_1, y_2, \ldots, y_{j-i+1}) B_{n-j,k-i}^{(\lambda)}(x_1, x_2, \ldots, x_{n-j-k+i+1}) \frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty} \left(\sum_{i=0}^{k} \sum_{j=i}^{n-k+i} \binom{n}{j} B_{j,i}^{(\lambda)}(y_1, \ldots, y_{j-i+1}) B_{n-j,k-i}^{(\lambda)}(x_1, x_2, \ldots, x_{n-j-k+i+1}) \right) \frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (17), we get the following theorem.

Theorem 2 *For any integers with $n \geq k \geq 0$, we have*

$$B_{n,k}^{(\lambda)}(x_1 + y_1, x_2 + y_2, \ldots, x_{n-k+1} + y_{n-k+1})$$
$$= \sum_{i=0}^{k} \sum_{j=i}^{n-k+i} \binom{n}{j} B_{j,i}^{(\lambda)}(y_1, y_2, \ldots, y_{j-i+1}) B_{n-j,k-i}^{(\lambda)}(x_1, x_2, \ldots, x_{n-j-k+i+1}).$$

From (12) with $k = 0$, we have

$$B_{n,0}^{(\lambda)}(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{18}$$

From Theorem 2 and (18), we note that

$$\begin{aligned}
&B_{n,k}^{(\lambda)}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1}) \\
&= \sum_{i=0}^{k}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,\dots,y_{j-i+1})B_{n-j,k-i}(x_1,x_2,\dots,x_{n-j-k+i+1}) \\
&= \sum_{j=0}^{n-k}\binom{n}{j}B_{j,0}^{(\lambda)}(y_1,y_2,\dots,y_{j+1})B_{n-j,k}^{(\lambda)}(x_1,x_2,\dots,x_{n-j-k+1}) \\
&\quad + \sum_{i=1}^{k}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,y_2,\dots,y_{j-i+1})B_{n-j,k-i}^{(\lambda)}(x_1,x_2,\dots,x_{n-k-j+i+1}) \\
&= B_{n,k}^{(\lambda)}(x_1,x_2,\dots,x_{n-k+1}) \\
&\quad + \sum_{i=1}^{k}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,\dots,y_{j-i+1})B_{n-j,k-i}^{(\lambda)}(x_1,x_2,\dots,x_{n-j-k+i+1}) \\
&= B_{n,k}^{(\lambda)}(x_1,x_2,\dots,x_{n-k+1}) + \sum_{j=k}^{n}\binom{n}{j}B_{j,k}^{(\lambda)}(y_1,y_2,\dots,y_{j-k+1})B_{n-j,0}^{(\lambda)}(x_1,x_2,\dots,x_{n-j+1}) \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,y_2,\dots,y_{j-i+1})B_{n-j,k-i}^{(\lambda)}(x_1,x_2,\dots,x_{n-j-k+i+1}) \\
&= B_{n,k}^{(\lambda)}(x_1,x_2,\dots,x_{n-k+1}) + B_{n,k}^{(\lambda)}(y_1,y_2,\dots,y_{n-k+1}) \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,y_2,\dots,y_{j-i+1})B_{n-j,k-i}^{(\lambda)}(x_1,x_2,\dots,x_{n-j-k+i+1}).
\end{aligned} \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 3 *For $n,k\in\mathbb{Z}$ with $n\geq k$ and $k\geq 2$, we have*

$$\begin{aligned}
&\sum_{i=1}^{k-1}\sum_{j=i}^{n-k+i}\binom{n}{j}B_{j,i}^{(\lambda)}(y_1,y_2,\dots,y_{j-i+1})B_{n-j,k-i}^{(\lambda)}(x_1,x_2,\dots,x_{n-j-k+i+1}) \\
&= B_{n,k}^{(\lambda)}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1}) - B_{n,k}^{(\lambda)}(x_1,x_2,\dots,x_{n-k+1}) \\
&\quad - B_{n,k}^{(\lambda)}(y_1,y_2,\dots,y_{n-k+1}).
\end{aligned}$$

From Theorem 1, we have

$$\begin{aligned}
B_n^{(\lambda)}(x_1+y_1,x_2+y_2,\dots,x_n+y_n) &= \sum_{j=0}^{n}\binom{n}{j}B_{n-j}^{(\lambda)}(x_1,x_2,\dots,x_{n-j})B_j^{(\lambda)}(y_1,y_2,\dots,y_j) \\
&= B_n^{(\lambda)}(x_1,\dots,x_n) + B_n^{(\lambda)}(y_1,y_2,\dots,y_n) + \sum_{j=1}^{n-1}\binom{n}{j}B_{n-j}^{(\lambda)}(x_1,\dots,x_{n-j})B_j^{(\lambda)}(y_1,y_2,\dots,y_j).
\end{aligned} \tag{20}$$

Therefore, by (20), we obtain the following theorem.

Theorem 4 *For $n \geq 2$, we have*

$$\sum_{j=1}^{n-1} \binom{n}{j} B_{n-j}^{(\lambda)}(x_1, \ldots, x_{n-j}) B_j^{(\lambda)}(y_1, y_2, \ldots, y_j)$$
$$= B_n^{(\lambda)}(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) - B_n^{(\lambda)}(x_1, \ldots, x_n) - B_n^{(\lambda)}(y_1, y_2, \ldots, y_n).$$

From (12), we have

$$\begin{aligned} &\sum_{n=k}^{\infty} k B_{n,k}^{(\lambda)}(x_1, \ldots, x_{n-k+1}) \frac{t^n}{n!} \\ &\quad = \frac{1}{(k-1)!} \left(\sum_{j=1}^{\infty} (1)_{j,\lambda} x_j \frac{t^j}{j!} \right)^{k-1} \sum_{j=1}^{\infty} (1)_{j,\lambda} x_j \frac{t^j}{j!} \\ &\quad = \sum_{j=k-1}^{\infty} B_{j,k-1}^{(\lambda)}(x_1, x_2, \ldots, x_{j-k+2}) \frac{t^j}{j!} \sum_{l=1}^{\infty} (1)_{l,\lambda} x_l \frac{t^l}{l!} \\ &\quad = \sum_{n=k}^{\infty} \left(\sum_{j=k-1}^{n-1} \binom{n}{j} B_{j,k-1}^{(\lambda)}(x_1, x_2, \ldots, x_{j-k+2}) (1)_{n-j,\lambda} x_{n-j} \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

Theorem 5 *For $n, k \geq 1$ we have*

$$k B_{n,k}^{(\lambda)}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{j=k-1}^{n-1} \binom{n}{j} B_{j,k-1}^{(\lambda)}(x_1, x_2, \ldots, x_{j-k+2}) (1)_{n-j,\lambda} x_{n-j}.$$

From (12), we can derive the following equation:

$$\begin{aligned} &B_{n,k}^{(\lambda)}(x_1, x_2, \ldots, x_{n-k+1}) \\ &\quad = \sum \frac{n!}{i_1! i_2! \cdots i_{n-k+1}!} \left(\frac{(1)_{1,\lambda} x_1}{1!} \right)^{i_1} \left(\frac{(1)_{2,\lambda} x_2}{2!} \right)^{i_2} \cdots \left(\frac{(1)_{n-k+1,\lambda} x_{n-k+1}}{(n-k+1)!} \right)^{i_{n-k+1}}, \end{aligned} \tag{22}$$

where the summation is over all integers $i_1, i_2, \ldots, i_{n-k+1} \geq 0$ such that $i_1 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$.

Thus, by using (22) and proceeding with a similar argument to the ones used in deriving (8), (9) and (10) (see [9]), we get the following identities:

$$\begin{aligned} &B_{n,k}^{(\lambda)}(x_1, x_2, \ldots, x_{n-k+1}) \\ &\quad = \frac{1}{x_1} \frac{1}{n-k} \sum_{j=1}^{n-k} \binom{n}{j} \left[(k+1) - \frac{n+1}{j+1} \right] (1)_{j+1,\lambda} x_{j+1} B_{n-j,k}^{(\lambda)}(x_1, x_2, \ldots, x_{n-j-k+1}), \end{aligned} \tag{23}$$

$$\begin{aligned} &B_{n,k_1+k_2}^{(\lambda)}(x_1, x_2, \ldots, x_{n-k_1-k_2+1}) \\ &\quad = \frac{k_1! k_2!}{(k_1+k_2)!} \sum_{j=0}^{n} \binom{n}{j} B_{j,k_1}^{(\lambda)}(x_1, x_2, \ldots, x_{j-k_1+1}) B_{n-j,k_2}^{(\lambda)}(x_1, x_2, \ldots, x_{n-j-k_2+1}), \end{aligned} \tag{24}$$

and

$$\begin{aligned} &B_{n,k+1}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k}) \\ &\quad = \frac{1}{(k+1)!}\sum_{j_1=k}^{n-1}\sum_{j_2=k=1}^{j_1-1}\cdots\sum_{j_k=1}^{j_{k-1}-1}\binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{k-1}}{j_k} \\ &\qquad \times (1)_{n-j_1,\lambda}x_{n-j_1}(1)_{j_1-j_2,\lambda}x_{j_1-j_2}\cdots(1)_{j_{k-1}-j_k,\lambda}x_{j_{k-1}-j_k}(1)_{j_k,\lambda}x_{j_k}, \end{aligned} \tag{25}$$

where $n \geq k+1$, $k = 1,2,\ldots$.

From (13), we note that

$$\begin{aligned} &B_n^{(\lambda)}(x_1,x_2,\ldots,x_n) \\ &\quad = \sum_{l_1+2l_2+\cdots+nl_n=n}\frac{n!}{l_1!l_2!\cdots l_n!}\left(\frac{x_1(1)_{1,\lambda}}{1!}\right)^{l_1}\left(\frac{x_2(1)_{2,\lambda}}{2!}\right)^{l_2}\cdots\left(\frac{x_n(1)_{n,\lambda}}{n!}\right)^{l_n}, \end{aligned} \tag{26}$$

where n is a nonnegative integer.

3 Further remarks

For any integers n,k with $n \geq k$, we define the modified degenerate partial Bell polynomials as

$$\begin{aligned} &B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}|\lambda) \\ &\quad = \sum_{\substack{l_1+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}}\frac{n!}{l_1!l_2!\cdots l_{n-k+1}!}\left(\prod_{i=1}^{n-k+1}\frac{x_i}{i!}\right)^{l_i}\left(\prod_{i=1}^{n-k+1}(1)_{l_i,\lambda}\right). \end{aligned} \tag{27}$$

Here one should observe the difference between the modified degenerate partial Bell polynomials and the degenerate partial Bell polynomials which are given by

$$\begin{aligned} &B_{n,k}^{(\lambda)}(x_1,x_2,\ldots,x_{n-k+1}) \\ &\quad = \sum_{\substack{l_1+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}}\frac{n!}{l_1!l_2!\cdots l_{n-k+1}!}\left(\prod_{i=1}^{n-k+1}\frac{x_i}{i!}\right)^{l_i}\left(\prod_{i=1}^{n-k+1}(1)_{i,\lambda}\right)^{l_i}. \end{aligned}$$

Note that $\lim_{\lambda\to 0}B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}|\lambda) = B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$.

Assume that X_i $(i = 1,2,\ldots,n)$ are identically independent degenerate Poisson random variables with parameter $\alpha_i(>0)$ $(i = 1,2,\ldots,n)$, and let n,k be integers with $n \geq k \geq 2$. Then we have

$$\begin{aligned} &P\{X_1+X_2+\cdots+X_n = k, X_1+2X_1+\cdots+nX_n = n\} \\ &\quad = \sum_{\substack{k_1+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} P\{X_1 = k_1, X_2 = k_2, \ldots, X_n = k_n\} \\ &\quad = \sum_{\substack{k_1+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} P\{X_1 = k_1\}\cdot P\{X_2 = k_2\}\cdots P\{X_n = k_n\} \end{aligned} \tag{28}$$

$$
\begin{aligned}
&= e_{\lambda}^{-1}(\alpha_1)e_{\lambda}^{-1}(\alpha_2)\cdots e_{\lambda}^{-1}(\alpha_n) \\
&\quad\times \sum_{\substack{k_1+\cdots+k_{n-k+1}=k \\ k_1+2k_2+\cdots+(n-k+1)k_{n-k+1}=n}} \frac{(1)_{k_1,\lambda}(1)_{k_2,\lambda}\cdots(1)_{k_{n-k+1},\lambda}}{k_1!k_2!\cdots k_{n-k+1}!}\alpha_1^{k_1}\cdots\alpha_{n-k+1}^{k_{n-k+1}} \\
&= \frac{P\{X_1+X_2+\cdots+X_n=0\}}{n!}B_{n,k}\bigl(1!\alpha_1,2!\alpha_2,\ldots,(n-k+1)!\alpha_{n-k+1}|\lambda\bigr).
\end{aligned}
$$

Therefore, by (28), we obtain the following theorem.

Theorem 6 *Let $X_1, X_2, \ldots, X_n$ be identically independent degenerate Poisson random variables with parameters $\alpha_1(>0), \alpha_2(>0), \ldots, \alpha_n(>0)$. For any integers n, k with $n \geq k \geq 2$, we have*

$$
\begin{aligned}
&B_{n,k}\bigl(1!\alpha_1,2!\alpha_2,\ldots,(n-k+1)!\alpha_{n-k+1}|\lambda\bigr) \\
&= \frac{n!}{P\{X_1+X_2+\cdots+X_n=0\}}P\{X_1+X_2+\cdots+X_n=k, X_1+2X_1+\cdots+nX_n=n\}.
\end{aligned}
$$

For any positive integer n, we define the modified degenerate complete Bell polynomials by

$$
\begin{aligned}
&B_n(x_1,x_2,\ldots,x_n|\lambda) \\
&= \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\prod_{i=1}^{n}\frac{x_i}{i!}\right)^{k_i}\left(\prod_{i=1}^{n}(1)_{k_i,\lambda}\right).
\end{aligned} \tag{29}
$$

Again, one should observe the difference between the modified degenerate complete Bell polynomials and the degenerate complete Bell polynomials which are given by

$$
\begin{aligned}
&B_n^{(\lambda)}(x_1,x_2,\ldots,x_n) \\
&= \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\prod_{i=1}^{n}\frac{x_i}{i!}\right)^{k_i}\left(\prod_{i=1}^{n}(1)_{i,\lambda}\right)^{k_i}.
\end{aligned}
$$

Suppose that X_i $(i = 1, 2, \ldots, n)$ are identically independent degenerate Poisson random variables with parameters $\alpha_i(>0)$ $(i = 1, 2, \ldots, n)$. We have

$$
\begin{aligned}
&P\{X_1+2X_2+3X_3+\cdots+nX_n=n\} \\
&= \sum_{k_1+2k_2+\cdots+nk_n} P\{X_1=k_1, X_2=k_2, X_3=k_3, \ldots, X_n=k_n\} \\
&= e_{\lambda}^{-1}(\alpha_1)e_{\lambda}^{-1}(\alpha_2)\cdots e_{\lambda}^{-1}(\alpha_n)\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(1)_{k_1,\lambda}(1)_{k_2,\lambda}\cdots(1)_{k_n,\lambda}}{k_1!k_2!\cdots k_n!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n} \\
&= \frac{P\{X_1+X_2+\cdots+X_n=0\}}{n!}B_n(1!\alpha_1,2!\alpha_2,\ldots,n!\alpha_n|\lambda) \quad (n \geq 0).
\end{aligned}
$$

Therefore, we obtain the following theorem.

Theorem 7 *Let X_i $(i = 1, 2, \dots, n)$ be identically independents degenerate Poisson random variables with parameters $\alpha_i > 0$ $(i = 1, 2, \dots, n)$. Then we have*

$$\begin{aligned} &B_n(1!\alpha_1, 2!\alpha_2, \dots, n!\alpha_n|\lambda) \\ &\quad = \frac{n!}{P\{X_1 + X_2 + \cdots + X_n = 0\}} P\{X_1 + 2X_2 + 3X_3 + \cdots + nX_n = n\}. \end{aligned}$$

Now, we consider X_i $(i = 1, 2, \dots, n)$ to be identically independent Poisson random variables with parameters

$$\frac{\alpha_i}{i!}(1)_{i,\lambda}(> 0) \quad (i = 1, 2, \dots, n). \tag{30}$$

Then we have

$$\begin{aligned} &P\{X_1 + 2X_2 + \cdots + nX_n = n\} \\ &\quad = \sum_{k_1+2k_2+\cdots+nk_n=n} P\{X_1 = k_1, X_2 = k_2, \dots, X_n = k_n\} \\ &\quad = e^{-(\frac{\alpha_1}{1!}(1)_{1,\lambda} + \frac{\alpha_2}{2!}(1)_{2,\lambda} + \cdots + \frac{\alpha_n}{n!}(1)_{n,\lambda})} \\ &\qquad \times \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{1}{k_1!k_2!\cdots k_n!} \left(\frac{\alpha_1(1)_{1,\lambda}}{1!}\right)^{k_1} \cdots \left(\frac{\alpha_n(1)_{n,\lambda}}{n!}\right)^{k_n} \\ &\quad = \frac{P\{X_1 + X_2 + \cdots + X_n = 0\}}{n!} B_n^{(\lambda)}(x_1, x_2, \dots, x_n). \end{aligned}$$

By (30), we get

$$B_n^{(\lambda)}(x_1, x_2, \dots, x_n) = \frac{n!}{P\{X_1 + X_2 + \cdots + X_n = 0\}} P\{X_1 + 2X_2 + \cdots + nX_n = n\}.$$

Also, we have

$$\begin{aligned} &P\{X_1 + X_2 + \cdots + X_n = k, X_1 + 2X_2 + \cdots + nX_n = n\} \\ &\quad = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} P\{X_1 = k_1, X_2 = k_2, \dots, X_n = k_n\} \\ &\quad = e^{-\sum_{j=1}^n \frac{\alpha_j}{j!}(1)_{j,\lambda}} \sum_{\substack{k_1+k_2+\cdots+k_{n-k+1}=k \\ k_1+2k_2+\cdots+(n-k+1)k_{n-k+1}=n}} \frac{1}{k_1!k_2!\cdots k_{n-k+1}!} \left(\prod_{j=1}^{n-k+1} \frac{(1)_{j,\lambda}}{j!} x_j\right)^{l_j} \\ &\quad = \frac{P\{X_1 + \cdots + X_n = 0\}}{n!} B_{n,k}^{(\lambda)}(x_1, x_2, \dots, x_{n-k+1}). \end{aligned}$$

Thus, we have

$$\begin{aligned} &B_{n,k}^{(\lambda)}(x_1, x_2, \dots, x_{n-k+1}) \\ &\quad = \frac{n!}{P\{X_1 + X_2 + \cdots + X_n = 0\}} P\{X_1 + X_2 + \cdots + X_n = k, X_1 + 2X_2 + \cdots + nX_n = n\}. \end{aligned}$$

4 Conclusion

The complete Bell polynomials and the partial Bell polynomials are, respectively, multivariate versions for Bell polynomials and Stirling numbers of the second kind. They have applications in such diverse areas as combinatorics, probability, algebra and analysis.

In this paper, we studied the recently introduced degenerate complete and partial Bell polynomials which are degenerate versions of the complete and partial Bell polynomials. In more detail, we derived several identities connected with such Bell polynomials whose arguments are given by the sum of two 'variable-vectors.' Further, we obtained a recurrence relation for the degenerate partial Bell polynomials. Also, we mentioned three results for the degenerate partial Bell polynomials which can be derived by the same method as for the partial Bell polynomials. Then, as applications to probability theory, we showed the connections between the modified degenerate complete and partial Bell polynomials, which are slightly different from the degenerate complete and partial Bell polynomials, and the joint distributions of weighted sums of independent degenerate Poisson random variables.

It is one of our future projects to continue to explore applications to probability theory of some special numbers and polynomials.

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Authors' contributions
DSK and TK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK, HL and SHP checked the results of the paper and typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

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