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A certain q -Ruscheweyh type derivative operator and its applications involving multivalent functions

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Abstract

In the present paper, by using the concept of convolution and q -calculus, we define a certain q -derivative (or q -difference) operator for analytic and multivalent (or p -valent) functions. This presumably new q -derivative operator is an extension of the known q -analogue of the Ruscheweyh derivative operator. We also give some interesting applications of this q -derivative operator for multivalent functions by using the method of differential subordination. Relevant connections with a number of earlier works on this subject are also pointed out.

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1 Introduction, definitions, and motivation

Let $\mathcal{A}(p)$ denote the class of multivalent (or p -valent) functions of the form:

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in the open unit disk \mathbb{U} given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We note that

$$\mathcal{A}(1) =: \mathcal{A}.$$

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Definition 1 The Hadamard product or convolution of the following two functions $f_j(z) \in \mathcal{A}(p)$ ($j = 1, 2$)

$$f_j(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1,j} z^{n+p-1} \quad (p \in \mathbb{N})$$

is given by

$$(f_1 * f_2)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1,1} a_{n+p-1,2} z^{n+p-1}.$$

Definition 2 For two analytic functions f_j ($j = 1, 2$) in \mathbb{U} , the function f_1 is said to be subordinate to the function f_2 , which is written as follows:

$$f_1 \prec f_2 \quad \text{or} \quad f_1(z) \prec f_2(z) \quad (z \in \mathbb{U}),$$

if we can find a Schwartz function w , analytic in \mathbb{U} , with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f_1(z) = f_2(w(z)).$$

Further, if the function f_2 is univalent in \mathbb{U} , then the following equivalence relation holds true:

$$f_1(z) \prec f_2(z) \quad (z \in \mathbb{U}) \iff f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

We denote by $\mathcal{P}(\beta)$ the class of functions of the form

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \dots, \tag{1.1}$$

which are analytic in \mathbb{U} and satisfy the following inequalities:

$$\Re(\varphi(z)) > \beta \quad (0 \leq \beta < 1).$$

It can be seen that

$$\mathcal{P}(0) = \mathcal{P},$$

where \mathcal{P} is the well-known class of Carathéodory functions (see, for details, [6] and [49]).

We next define the function $h(A, B; z)$ given by (see [3])

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}; -1 \leq B \leq A \leq 1).$$

This function $h(A, B; z)$ is known to be the conformal map of the unit disk \mathbb{U} to a circle which is symmetrical with respect to the real axis having the center at $\frac{1-AB}{1-B^2}$ ($B \neq \pm 1$) and the radius equal to $\frac{A-B}{1-B^2}$ ($B \neq \pm 1$).

In order to present some of the noteworthy and useful details of the definitions and principles of q -difference calculus, we assume throughout this article that

$$0 < q < 1 \quad \text{and} \quad p \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

Definition 3 For $f \in \mathcal{A}$, the q -difference (or the q -derivative) operator \mathcal{D}_q in a given subset of the set \mathbb{C} of complex numbers is defined by (see [9] and [10])

$$(\mathcal{D}_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & (z \neq 0), \\ f'(0) & (z = 0), \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists.

It is readily observed from equation (1.2) that

$$\lim_{q \rightarrow 1^-} (\mathcal{D}_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z)-f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function f in a given subset of the complex space \mathbb{C} (see also [37], [39] and [40] for some recent applications of the q -difference operators in the theory of q -series and q -polynomials).

Definition 4 Let $q \in (0, 1)$ and define the q -number $[\tau]_q$ as follows:

$$[\tau]_q = \begin{cases} \frac{1-q^\tau}{1-q} & (\tau \in \mathbb{C}), \\ \sum_{k=0}^{\tau-1} q^k & (\tau = n \in \mathbb{N}), \\ 0 & (\tau = 0). \end{cases}$$

It follows from Definition 4 with $\tau = n$ ($n \in \mathbb{N}$) that

$$[n]_{q!} = \begin{cases} 1 & (n = 0), \\ \prod_{k=1}^n [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 5 The generalized q -Pochhammer symbol given by

$$[\tau]_{n,q} \quad (\tau \in \mathbb{C}; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; |q| < 1)$$

is defined as follows:

$$[\tau]_{n,q} = \frac{(q^\tau; q)_n}{(1-q)^n} = \begin{cases} 1 & (n = 0) \\ [\tau]_q [\tau + 1]_q [\tau + 2]_q \cdots [\tau + n - 1]_q & (n \in \mathbb{N}). \end{cases}$$

Moreover, the q -gamma function $\Gamma_q(z)$ satisfies the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z) \quad \text{and} \quad \Gamma_q(1) = 1,$$

where

$$\Gamma_q(z) = (1-q)^{1-z} \prod_{n=0}^{\infty} \left(\frac{1-q^{n+1}}{1-q^{n+z}} \right).$$

The intensive applications of the q -calculus in exploring new directions in various diverse areas of mathematics and physics have fascinated a number of researchers to work in several distinctive areas of the mathematical and physical sciences. The q -derivative (\mathfrak{D}_q) operator's versatile applications makes it significantly more important. Initially, in the year 1990, Ismail *et al.* [8] presented the idea of a q -extension of the class \mathcal{S}^* of starlike functions. However, historically speaking, in the article [32] published in 1989, Srivastava gave a firm footing of the usages of the q -calculus and the basic (or q -) hypergeometric functions:

$${}_v\Phi_s(v, s \in \mathbb{N}_0 = \{0, 1, 2, \dots\})$$

in the study of geometric function theory (GFT) (see, for details, [32]). More recently, the state-of-the-art survey and applications of the operators of the q -calculus and the fractional q -calculus, such as the q -derivative operator and the fractional q -derivative operators in geometric function theory of complex analysis, were investigated in a survey-cum-expository review article by Srivastava [33]. In this same survey-cum-expository review article, Srivastava [33] revealed and exposed the triviality of the so-called (p, q) -calculus associated with an obviously redundant and inconsequential additional parameter p (see, for details, [33, p. 340]).

The aforementioned works [8] and [33] have inspired a number of researchers to contribute significantly in geometric function theory of complex analysis. Several convolution and fractional q -operators, which have been already defined, were surveyed in the above-cited work [33]. For example, Kanas and Răducanu [11] introduced the q -analogue of Ruscheweyh's derivative operator and, by using the concept of the Hadamard product (or convolution), Srivastava *et al.* [45] introduced a q -extension of Noor's integral operator and studied some of its applications. Aldweby [2] and Sokól [21] studied some classes of analytic functions defined by means of the q -analogue of Ruscheweyh's derivative operator. Many q -derivative and q -integral operators can be written in terms of the Hadamard product (or convolution). For details, we refer the reader to the earlier works [1, 7, 18, 21, 23]. Moreover, several authors (see, for example, [13, 19, 42, 47, 48]) have concentrated upon the classes of q -starlike functions related with the Janowski and other functions from several different viewpoints and the references cited therein. For some more recent investigations involving q -calculus, we may refer the interested reader to [4, 5, 12, 14–17, 20, 22, 26–28, 31, 34–36, 38, 43, 46].

In this paper, we first define an extended q -analogue of Ruscheweyh's derivative operator for multivalent (or p -valent) functions. In order to define this extended q -analogue of Ruscheweyh's derivative operator, we use the concepts of the Hadamard product (or

convolution). We then give some interesting applications of this operator for multivalent functions by making use of the method of differential subordination. The extended q -analogue of Ruscheweyh's derivative operator is defined below.

Definition 6 For $f \in \mathcal{A}(p)$, the extended q -derivative operator $\mathcal{R}_q^{\lambda+p-1} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ for multivalent functions is defined as follows:

$$\begin{aligned} \mathcal{R}_q^{\lambda+p-1}f(z) &= Q_{\lambda,q}^p(z) * f(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{[\lambda + p]_{n-p,q}}{[n-p]_q!} a_{n+p-1} z^{n+p-1} \quad (\lambda > -1), \end{aligned} \tag{1.3}$$

where

$$Q_{\lambda,q}^p(z) = z^{p-1} \left(z + \sum_{n=2}^{\infty} \frac{[\lambda + p]_{n-p,q}}{[n-p]_q!} z^n \right).$$

The following identity can easily be verified by using (1.3):

$$q^\lambda z \mathcal{D}_q \mathcal{R}_q^{\lambda+p-1}f(z) = [\lambda + p]_q \mathcal{R}_q^{\lambda+p}f(z) - [\lambda]_q \mathcal{R}_q^{\lambda+p-1}f(z). \tag{1.4}$$

It can also be seen that, by putting $p = 1$ in (1.4), we have

$$q^\lambda z \mathcal{D}_q \mathcal{R}_q^\lambda f(z) = [\lambda + 1]_q \mathcal{R}_q^{\lambda+1}f(z) - [\lambda]_q \mathcal{R}_q^\lambda f(z),$$

which is the well-known relation studied by Kanas [11].

Remark 1 It is easily seen that, upon setting $p = 1$, the extended q -analogue of Ruscheweyh's derivative operator $\mathcal{R}_q^{\lambda+p-1}f(z)$ reduces to the q -Ruscheweyh derivative operator which was studied by Kanas [11]. For $p = 1$ and $q \rightarrow 1-$, the extended q -analogue of Ruscheweyh's derivative operator $\mathcal{R}_q^{\lambda+p-1}f(z)$ reduces to the familiar derivative operator introduced by Ruscheweyh [30]. Moreover, if we put $p = 1$ and $\lambda = 0$, we have

$$\mathcal{R}_q^{0+1-1}f(z) = \mathcal{R}_q^0f(z) = f(z).$$

2 A set of lemmas

To prove our main results, we need the following lemmas.

Lemma 1 (see [25]) *Let $\varphi_j \in \mathcal{P}(\beta_j)$ be given by (1.1) for $(0 \leq \beta_j < 1; j = 1, 2)$. Then*

$$\varphi_1 * \varphi_2 \in \mathcal{P}(\beta_3),$$

where

$$\beta_3 = 1 - (1 - \beta_1)(1 - \beta_2).$$

Lemma 2 (see [24]) *Let the function φ , given by (1.4), be in the class $\mathcal{P}(\beta)$. Then*

$$\Re(\varphi(z)) > 2\beta - 1 + \frac{2(1 - \beta)}{1 + |z|} \quad (0 \leq \beta < 1).$$

Lemma 3 (see [29]) *The function given by*

$$(1 - z)^\gamma = e^{\gamma \log(1-z)} \quad (\gamma \neq 0)$$

is univalent in \mathbb{U} if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.

Lemma 4 (see [3]) *Let the function $h(z)$ be analytic and convex univalent in \mathbb{U} with $h(0) = 1$. Also, let the function $g(z)$ given by*

$$g(z) = 1 + b_1z + b_2z^2 + \dots$$

be analytic in \mathbb{U} . If

$$g(z) + \frac{z \mathcal{D}_q g(z)}{c} \prec h(z) \quad (z \in \mathbb{U}; c \neq 0),$$

then, for $\Re(c) \geq 0$, the following subordination relation holds true:

$$g(z) \prec \frac{c}{z^c} \int_0^z t^{c-1} h(t) dt.$$

Lemma 5 (see [3]) *Let the function $u(z)$ be univalent in \mathbb{U} , and let the functions $\theta(w)$ and $\varphi(w)$ be analytic in the domain \mathbb{D} containing $u(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in u(\mathbb{U})$. Set*

$$Q(z) = z \mathcal{D}_q(u(z))\varphi(u(z)) \quad \text{and} \quad h(z) = \theta(u(z) + Q(z))$$

and suppose that

- (i) $Q(z)$ is starlike univalent in \mathbb{U} .
- (ii) $\Re\left(\frac{z \mathcal{D}_q h(z)}{Q(z)}\right) = \Re\left(\frac{z \mathcal{D}_q \theta(u(z))}{\varphi(u(z))}\right) + \left(\frac{z \mathcal{D}_q Q(z)}{Q(z)}\right) > 0$.

If $m(z)$ is analytic in \mathbb{U} ,

$$m(z) = u(0) \quad \text{and} \quad m(\mathbb{U}) \subset \mathbb{D}$$

and

$$\theta(m(z)) + z \mathcal{D}_q(m(z))\varphi(m(z)) \prec \theta(u(z)) + z \mathcal{D}_q(u(z))\varphi(u(z)) = h(z),$$

then

$$m(z) \prec u(z) \quad (z \in \mathbb{U})$$

and $u(z)$ is the best dominant.

3 Main results

Our first main result in this section is stated as Theorem 1.

Theorem 1 *Let $\lambda > 0, \alpha > 0$, and $-1 \leq B \leq A < 1$. If $f \in \mathcal{A}(p)$ satisfies the following subordination relation:*

$$(1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} f(z)}{z^p} \prec h(A, B, z),$$

then

$$\Re \left(\left\{ \frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} \right\}^{\frac{1}{n}} \right) \geq \left(\frac{[\lambda + p]_q}{\alpha [p]_q q^\lambda} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 - Au}{1 - Bu} \right) du \right)^{\frac{1}{n}} \quad (n \geq 1). \tag{3.1}$$

The result is sharp.

Proof Let

$$g(z) = \frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p}. \tag{3.2}$$

Then, for $f \in \mathcal{A}(p)$, the function g given by

$$g(z) = 1 + b_1 z + b_2 z^2 + \dots \quad (z \in \mathbb{U})$$

is analytic in \mathbb{U} . By using the logarithmic q -differentiation on both sides of (3.2) and multiplying the resulting equation by z , we have

$$\frac{z \mathcal{D}_q(g(z))}{g(z)} = \frac{z \mathcal{R}_q^{\lambda+p-1} f(z)}{\mathcal{R}_q^{\lambda+p-1} f(z)} - [p]_q.$$

By making use of (1.4), we obtain

$$\frac{z \mathcal{D}_q(g(z))}{g(z)} = \frac{[\lambda + p]_q}{q^\lambda} \frac{\mathcal{R}_q^{\lambda+p} f(z)}{\mathcal{R}_q^{\lambda+p-1} f(z)} - \frac{[\lambda]_q}{q^\lambda} - [p]_q.$$

Taking into account that

$$[\lambda + p]_q = [\lambda]_q + [p]_q q^\lambda,$$

we get

$$\frac{q^\lambda}{[\lambda + p]_q} z \mathcal{D}_q(g(z)) + g(z) = \frac{\mathcal{R}_q^{\lambda+p} f(z)}{z^p}. \tag{3.3}$$

Also, from (1.4), (3.2), and (3.3), we have

$$g(z) + \frac{\alpha q^\lambda [p]_q}{[\lambda + p]_q} z \mathcal{D}_q g(z) \prec h(A, B, z). \tag{3.4}$$

Now, by applying Lemma 4, we find that

$$g(z) < \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} z^{-\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 + At}{1 + Bt}\right) dt. \tag{3.5}$$

Also, by making use of the concept of subordination on (3.5), we have

$$\left(\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p}\right) = \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 + Auw(z)}{1 + Buw(z)}\right) du. \tag{3.6}$$

It follows from (3.6), together with $-1 \leq B < A \leq 1$ and $\lambda > 0$, that

$$\Re\left(\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p}\right) > \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 - Au}{1 - Bu}\right) du. \tag{3.7}$$

Since

$$\Re(w^{\frac{1}{n}}) \geq \{\Re(w)\}^{1/n} \quad (\Re(w) > 0; n \geq 1), \tag{3.8}$$

by making use of concept (3.8), the inequality in (3.1) follows directly from (3.7).

In order to show the sharpness of (3.1), we define the function $f \in \mathcal{A}(p)$ by

$$\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} = \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 + Auz}{1 + Buz}\right) du. \tag{3.9}$$

For this function f involved in (3.9), we find that

$$(1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} f(z)}{z^p} = \frac{1 + Az}{1 + Bz}$$

and

$$\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} \rightarrow \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha [p]_q q^\lambda}\right)-1} \left(\frac{1 - Au}{1 - Bu}\right) du \quad (z \rightarrow -1).$$

This completes the proof of Theorem 1. □

Remark 2 If we set $p = 1$ in Theorem 1, we are led to the results similar to those given by Aldweby and Darus [3].

We next state and prove Theorem 2.

Theorem 2 Let $A = 1 - 2\alpha, B = -1, \alpha, \lambda > 1, n \geq 1$, and $0 \leq \beta < 1$. If the function $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$(1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} f(z)}{z^p} < h(1 - 2\alpha, -1, z),$$

then

$$\Re\left(\left\{\frac{\mathcal{R}_q^{\lambda+p-1}f(z)}{z^p}\right\}^{\frac{1}{n}}\right) > \left((2\beta - 1)u^{\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}} + \frac{2(1 - \beta)[\lambda + p]_q}{\alpha[p]_q q^\lambda} \int_0^1 \frac{u^{\left(\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}\right)-1}}{1 + u} du\right)^{\frac{1}{n}}.$$

Proof Following the same steps as those in the proof of Theorem 1 and considering the function g given by

$$g(z) = \frac{\mathcal{R}_q^{\lambda+p-1}f(z)}{z^p},$$

the differential subordination in (3.4) becomes

$$g(z) + \frac{\alpha q^\lambda [p]_q}{[\lambda + p]_q} z \mathcal{D}_q g(z) < \frac{1 + (2\beta - 1)z}{1 + z}.$$

Therefore, we have

$$\begin{aligned} &\Re\left(\left\{\frac{\mathcal{R}_q^{\lambda+p-1}f(z)}{z^p}\right\}^{\frac{1}{n}}\right) \\ &> \left(\frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}\right)-1} \left(\frac{1 + (2\beta - 1)u}{1 + u}\right) du\right)^{\frac{1}{n}} \\ &= \left(\frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}\right)-1} \left((2\beta - 1) + \frac{2(1 - \beta)u}{1 + u}\right) du\right)^{\frac{1}{n}} \\ &= \left((2\beta - 1)u^{\left(\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}\right)-1} + \frac{2(1 - \beta)[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 \frac{u^{\left(\frac{[\lambda+p]_q}{\alpha[p]_q q^\lambda}\right)-1}}{1 + u} du\right)^{\frac{1}{n}}. \end{aligned}$$

Our demonstration of Theorem 2 is now completed. □

Theorem 3 Let $\lambda > 0$ and $0 \leq \rho < 1$. Also, let the parameter $\gamma \in \mathbb{C} \setminus \{0\}$ satisfy either

$$\left|2\gamma(1 - \rho)\left(\frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} - 1\right)\right| \leq 1$$

or

$$\left|2\gamma(1 - \rho)\left(\frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} + 1\right)\right| \leq 1.$$

If $f \in \mathcal{A}(p)$ satisfies the following inequality:

$$\Re\left(\frac{\mathcal{R}_q^{\lambda+p}f(z)}{\mathcal{R}_q^{\lambda+p-1}f(z)}\right) > \rho, \tag{3.10}$$

then

$$\Re \left(\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} \right)^\gamma < \frac{1}{(1-z)^{2\gamma(1-\rho) \frac{[\lambda+p]_q}{\alpha q^\lambda [p]_q}}}, \tag{3.11}$$

where $u(z)$ is the best dominant.

Proof Let

$$m(z) = \left(\frac{\mathcal{R}_q^{\lambda+p-1} f(z)}{z^p} \right)^\gamma.$$

Then, by making use of (1.4), (3.10), and (3.11), we obtain

$$1 + \frac{q^\lambda [p]_q}{\gamma [\lambda + p]_q} \frac{z \mathcal{D}_q(g(z))}{g(z)} < \frac{1 + (1 - 2\rho)z}{1 - z}. \tag{3.12}$$

We now assume that

$$u(z) = \frac{1}{(1-z)^{2\gamma(1-\rho) \frac{[\lambda+p]_q}{q^\lambda [p]_q}}}, \quad \theta(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{q^\lambda [p]_q}{\gamma [\lambda + p]_q w}.$$

Then, by Lemma 3, $u(z)$ is univalent in \mathbb{U} . Further, it is easy to show that the functions $u(z)$, $\theta(w)$, and $\varphi(w)$ satisfy the condition of Lemma 5. We note also that the functions given by

$$Q(z) = z \mathcal{D}_q(u(z)) \varphi(u(z)) = \frac{2(1-\rho)z}{1-z}$$

and

$$h(z) = \theta(u(z) + Q(z)) = \frac{1 + (1 - 2\rho)z}{1 - z}$$

are univalent starlike in \mathbb{U} . Hence, by finally combining (3.12) and Lemma 5, we get the assertion of Theorem 3. \square

Remark 3 If we set $p = 1$ in Theorem 3, we are led to the results similar to those given by Aldweby and Darus [3].

Theorem 4 Let $\lambda > 0, \alpha < 1$, and $-1 \leq B_j \leq A_j < 1$ ($j = 1, 2$). If each of the functions $f_j \in \mathcal{A}(p)$ ($j = 1, 2$) satisfies the following subordination condition:

$$(1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} f_j(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} f_j(z)}{z^p} < h(A_j, B_j, z) \quad (j = 1, 2),$$

then

$$(1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} \Theta(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} \Theta(z)}{z^p} < h(1 - 2\gamma, -1, z),$$

where

$$\Theta(z) = \mathcal{R}_q^{\lambda+p-1}(f_1 * f_2)(z) \tag{3.13}$$

and

$$\gamma = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \cdot \left(1 - \frac{[\lambda + p]_q}{\alpha [p]_q q^\lambda} \int_0^1 \frac{u^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1}}{1 + u} du \right).$$

Proof We define the functions h_j ($j = 1, 2$) by

$$h_j = (1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} f_j(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} f_j(z)}{z^p}, \tag{3.14}$$

where $f_j \in \mathcal{A}(p)$ ($j = 1, 2$). We have $h_j \in \mathcal{P}(\beta_j)$ ($j = 1, 2$), where

$$\beta_j = \frac{1 - A_j}{1 - B_j} \quad (j = 1, 2).$$

By making use of (1.4) and (3.14), we obtain

$$\mathcal{R}_q^{\lambda+p-1} f_j(z) = \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} z^{1 - \left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)} \int_0^1 t^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} h_j(t) dt \quad (j = 1, 2),$$

which, in light of (3.13), shows that

$$\mathcal{R}_q^{\lambda+p-1} \Theta(z) = \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} z^{1 - \left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)} \int_0^1 t^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} h_0(t) dt,$$

where, for convenience,

$$\begin{aligned} h_0 &= (1 - \alpha) \frac{\mathcal{R}_q^{\lambda+p-1} \Theta(z)}{z^p} + \alpha \frac{\mathcal{R}_q^{\lambda+p} \Theta(z)}{z^p} \\ &= \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} z^{1 - \left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)} \int_0^1 t^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} (h_1 * h_2)(t) dt. \end{aligned}$$

If we apply Lemma 1, we get $(h_1 * h_2) \in \mathcal{P}(\beta_3)$, where

$$\beta_3 = 1 - 2(1 - \beta_1)(1 - \beta_2).$$

Now, with an application of Lemma 2, we have

$$\begin{aligned} \Re(h_0(z)) &= \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} \Re(h_1 * h_2)(uz) du \\ &\geq \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} \left((2\beta_3 - 1) + \frac{2(1 - \beta_3)}{1 + u|z|} \right) du \end{aligned}$$

$$\begin{aligned}
 &> \frac{[\lambda + p]_q}{\alpha q^\lambda [p]_q} \int_0^1 u^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1} \left((2\beta_3 - 1) + \frac{2(1 - \beta_3)}{1 + u} \right) du \\
 &= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{[\lambda + p]_q}{\alpha [p]_q q^\lambda} \int_0^1 \frac{u^{\left(\frac{[\lambda+p]_q}{q^\lambda [p]_q}\right)-1}}{1 + u} du \right) \\
 &= \gamma,
 \end{aligned}$$

which leads us to the desired assertion of Theorem 4. □

Remark 4 If we set $p = 1$ in Theorem 4, we are led to the results similar to those given by Aldweby and Darus [3].

4 Conclusion

In our present work, we are motivated by the well-established usage of the basic (or q -) calculus and fractional basic (or q -) calculus in geometric function theory of complex analysis as described by Srivastava’s survey-cum-expository review article [33]. The extended q -version of the q -Ruscheweyh type derivative operator for p -valent functions in \mathbb{U} has been introduced here. We have also derived several interesting results for this newly defined q -operator. The importance of the results demonstrated in this paper lies in the demonstrated fact that these results would generalize and extend various previously known results derived in many earlier works.

Basic (or q -) polynomials and basic (or q -) series, especially the basic (or q -) hypergeometric functions and basic (or q -) hypergeometric polynomials, are relevant specially in many areas (see, for example, [41, pp. 350–351]; see also [28, 44], and [16]). Moreover, as we remarked above and in the introductory Sect. 1, based upon the recently-published survey-cum-expository review article by Srivastava [33], the so-called (p, q) -calculus is a relatively insignificant and inconsequential translation of the traditional q -calculus, the extra parameter p being redundant or superfluous (see, for details, [33, p. 340]). This observation by Srivastava [33] will indeed apply also to any attempt to produce the rather straightforward (p, q) -variations of the results which we have presented in this paper.

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