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Transmissibility of epidemic diseases caused by delay with local proportional fractional derivative

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Abstract

Epidemiological models have been playing a vital role in different areas of biological sciences for the analysis of various contagious diseases. Transmissibility of virulent diseases is being portrayed in the literature through different compartments such as susceptible, infected, recovered (SIR), susceptible, infected, recovered, susceptible (SIRS) or susceptible, exposed, infected, recovered (SEIR), etc. The novelty in this endeavor is the addition of compartments of latency and treatment with vaccination, so the system is designated as susceptible, vaccinated, exposed, latent, infected, treatment, and recovered (SVELITR). The contact of a susceptible individual to an infective individual firstly makes the individual exposed, latent, and then completely infection carrier. Innovatively, the assumption that exposed, latent, and infected individuals enter the treatment compartment at different rates after a time lag is also deliberated through the existence of time delay. The rate of change and constant solutions of each compartment are studied with incorporation of a special case of proportional fractional derivative (PFD). In addition, existence and uniqueness of the system are also comprehensively elaborated. Moreover, novel dynamic assessment of the system is carried out in context with the fractional order index. Succinctly, the manuscript accomplishes cyclic epidemiological behavior of the infectious disease due to the delay in treatment of the infected individuals.

Keywords: Proportional fractional derivative; Hopf bifurcation; Stability; Limit cycles

1 Introduction

Mathematical modelling is a significant tool in epidemiology that helps healthcare researchers and policymakers to make public health and socioeconomic decisions. These models are designed by using data collected from clinicians and health workers to make prophecies about a disease's development. A tremendous number of models have been formulated, analyzed, and applied, which improved our understanding and predictive ability about a variety of infectious diseases [1–5]. Many of these models also consider time delay in the process of transmissibility of any infection to further study the effect of delays on the spread out of diseases. For instance, in [6] authors discussed the traveling wave solutions of age-structured susceptible, exposed, infective, recovered, susceptible with time

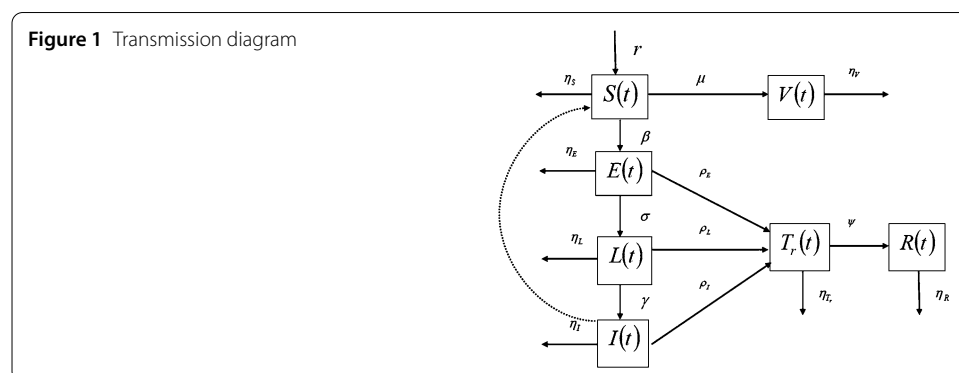
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delay. Zhang et al. [7] generated significant results elaborating the influence of time delay on the stability and measures of synthetic drugs transmission. Liu et al. [8] incorporated incubation time delay in the susceptible, infected, recovered vector, bone epidemic model and discussed its stability with the logistic population growth. Rifhat et al. [9] designed a stochastic SIRV model and explained the effects of environmental noises on the disease transmission. Ameen et al. [10] accumulated an SIRV epidemiological model with fractional optimal control problem, taking proportions of vaccination and treatment as suggested controlling parameters. Thus, literature comprehends many such contributions in the area of epidemiological studies [11–15].

Calculus of noninteger orders has a long and exciting history beginning with communication by two scientists Leibniz and de L'Hopital. Equations with fractional order derivatives are a powerful tool for unfolding processes with nonlocality and memory. Some new definitions of fractional derivatives have a conventional view of nonstandard things that contains violations of the Leibniz rule for the derivative of a product function and the chain rule that describe the composition of a function. Some of these new operators and results are just formal noninteger calculus of the known conventional theories, often without any justification and motivation. There are different types of noninteger derivatives, which describe the processes of nonlocality and memory, that were suggested by Liouville, Caputo, and Reisz [16]. Recently, fractional calculus has been widely applied in many fields such as engineering and sciences [17–20].

Treating an infection sometimes undergoes a delay. Bearing this in mind, in this endeavor a compartment for treated individuals with time delay is integrated in the model. A pictorial representation of the transmission process of the disease among compartments of the proposed model is described in Fig. 1. Accordingly, constructive novel features of the proposed model can be outlined as follows:

- Growth of susceptible individuals is defined with a threshold through the perception of logistic growth.
- A group of vaccinated individuals is also defined in the system.
- Elaboration of the rate of exposed, latent, and infected individuals entering in the treated compartment after a time lag.
- The dynamic system of the seven equations is scrutinized in the context with a special case of proportional fractional derivative [19, 21].
- Memory effect on the basic reproduction number and the stability of equilibrium points.



- Investigation of Hopf bifurcation critical point [11] for the existence of cycles in phase diagrams of the system.

Sequentially, the whole paper is described in seven sections: Sect. 2 comprises details of modelling the aforementioned assumptions with essential theorems. Epidemiological properties, such as basic reproduction number, dynamics of the proposed system, and Hopf bifurcation, are comprehended in Sects. 3, 4, and 5, respectively. Moreover, the analytical discussions are numerically validated through tables and graphs in Sect. 6, which are later concluded in Sect. 7.

2 Modelling

The SIR and SEIR models have undergone different modifications, according to variations of contemporary diseases and developed theories of mathematical modelling. Since these systems deal with the human population, the basic biological assumptions are retained in such models. Here, we adapted a model that consists of seven compartments, where individuals are categorized on the basis of their compartment to ailment. The constructed system is based on the following assumptions:

- All individuals, except vaccinated, have equal likelihood to catch infection when they are exposed to the infection.
- The vaccinated individuals do not get infected even if they are exposed to the disease.
- Population growth of the susceptible is defined with a maximum sustainability to survive in the available resources in an environment.
- The susceptible population undergoes two phases of disease diagnosis, i.e., exposed and latency, before becoming completely infective.
- Different rates of treatment for exposed, latent, and infected individuals are considered due to a different degree of ailment.
- Individuals go for treatment after the contact with infection, therefore the treating compartment will show a delay in dynamics.
- After being treated, recovered individuals do not participate in transmitting the disease.
- Dynamic changes are evaluated, in particular, with respect to change in the rate of fractional index in time for the considered region.

Articulating these assumptions, the following system of seven nonlinear fractional order differential equations with delay is attained, i.e., susceptible, vaccinated, exposed, latent, infected, treatment, and recovered (SVELITR):

$$\begin{aligned}
 {}^{PF}D_t^\alpha S(t) &= rS(t)\left(1 - \frac{S(t)}{k}\right) - \mu S(t) - \beta S(t)I(t) - \eta_S S(t), \\
 {}^{PF}D_t^\alpha V(t) &= \mu S(t) - \eta_V V(t), \\
 {}^{PF}D_t^\alpha E(t) &= \beta S(t)I(t) - \sigma E(t) - \rho_E E(t) - \eta_E E(t), \\
 {}^{PF}D_t^\alpha L(t) &= \sigma E(t - \tau) - \gamma L(t) - \rho_L L(t) - \eta_L L(t), \\
 {}^{PF}D_t^\alpha I(t) &= \gamma L(t - \tau) - \rho_I I(t) - \eta_I I(t), \\
 {}^{PF}D_t^\alpha T_r(t) &= \rho_E E(t - \tau) + \rho_L L(t - \tau) + \rho_I I(t - \tau) - \psi T_r(t) - \eta_L T_r(t), \\
 {}^{PF}D_t^\alpha R(t) &= \psi T_r(t - \tau) - \eta_R R(t).
 \end{aligned}
 \tag{1}$$

Table 1 Details of symbols of SVELITR model (6)

Symbol	Description	Units (population = '000s & time = days)
$\aleph(t)$	Total population	Population/day
$S(t)$	No. of susceptible individuals at any time t	Population/day
$V(t)$	No. of vaccinated individuals at any time t	Population/day
$E(t)$	No. of individuals exposed to the infection at any time t	Population/day
$L(t)$	No. of individuals latent to the infection at any time t	Population/day
$I(t)$	No. of infected individuals at any time t	Population/day
$T_r(t)$	No. of treated individuals at any time t	Population/day
$R(t)$	No. of recovered individuals at any time t	Population/year
τ	Time delay	days
α	Order of fractional derivative	dimensionless
r	Rate intrinsic growth of susceptible individuals	Individuals/(individuals \times year)
k	Surviving capacity of susceptible individuals	Individuals/(area \times year)
μ	Rate of vaccinated susceptible	Individuals/(individuals \times year)
β	Contact rate of susceptible with infected	Individuals/(individuals \times year)
σ	Rate of susceptible exposed to infection	Individuals/(individuals \times year)
γ	Rate of exposed latent to infection	Individuals/(individuals \times year)
ψ	Recovery rate of the treated individuals	Individuals/(individuals \times year)
$\eta_{(\bullet)}$	Natural death rate of individuals in a compartment	Individuals/(individuals \times year)
$\rho_{(\bullet)}$	Rate of treated exposed, latent, or infected individuals	Individuals/(individuals \times year)

Initial conditions and history functions are assumed to be as follows:

$$\begin{aligned}
 S(\vartheta) &= \xi_1(\vartheta), & V(\vartheta) &= \xi_2(\vartheta), & E(\vartheta) &= \xi_3(\vartheta), & L(\vartheta) &= \xi_4(\vartheta), \\
 I(\vartheta) &= \xi_5(\vartheta), & T_r(\vartheta) &= \xi_6(\vartheta), & R(\vartheta) &= \xi_7(\vartheta),
 \end{aligned}
 \tag{2}$$

where $\xi_i(\vartheta) \geq 0$ and $\xi_i(0) > 0$ for $i = 1, 2, \dots, 7$ such that $(\xi_1(\vartheta), \xi_2(\vartheta), \dots, \xi_7(\vartheta)) \in C([-\tau, 0], \mathfrak{R}_+^7)$. All the variables and parameters in system (1) are elaborated in Table 1. Moreover, let $\aleph(t)$ be the population of the considered region expressed as

$$\aleph(t) = S(t) + V(t) + E(t) + L(t) + I(t) + T_r(t) + R(t).
 \tag{3}$$

After the exploration of conformable derivative through the concept of proportional controller by Anderson et al. [21], proportional fractional derivative has been undergone with different simplifying properties [19]. The special case of this derivative considered in [19] has great potential to convert the fractional order derivative into an integer order one, which can be scrutinized without any ambiguity [22–24]. According to this development, the fractional operator of any continuous function $y(t)$ expands to

$${}^{PF}D_t^\alpha y(t) = \kappa_0(\alpha, t)\dot{y}(t) + \kappa_1(\alpha, t)y(t), \quad 0 < \alpha \leq 1,
 \tag{4}$$

where $\kappa_0(\alpha, t) \neq 0$ for $\alpha \in (0, 1]$, with $\lim_{\alpha \rightarrow 0^+} \kappa_0(\alpha, t) = 0$ and $\lim_{\alpha \rightarrow 1^-} \kappa_0(\alpha, t) = 1$. Additionally, $\kappa_1(\alpha, t) \neq 0$ for $\alpha \in [0, 1)$, with $\lim_{\alpha \rightarrow 0^+} \kappa_1(\alpha, t) = 1$ and $\lim_{\alpha \rightarrow 1^-} \kappa_1(\alpha, t) = 0$. Taking the same case as defined, let $\kappa_0(\alpha, t) = \alpha$ and $\kappa_1(\alpha, t) = 1 - \alpha$, so Eq. (4) becomes

$${}^{PF}D_t^\alpha y(t) = \alpha\dot{y}(t) + (1 - \alpha)y(t).
 \tag{5}$$

Applying the above expansion on system (1), we get the following expression of the governing model:

$$\begin{aligned}
 \dot{S}(t) &= \frac{1}{\alpha} \left(rS(t) \left(1 - \frac{S(t)}{k} \right) - \mu S(t) - \beta S(t)I(t) - \eta_S S(t) - (1 - \alpha)S(t) \right), \\
 \dot{V}(t) &= \frac{1}{\alpha} (\mu S(t) - \eta_V V(t) - (1 - \alpha)V(t)), \\
 \dot{E}(t) &= \frac{1}{\alpha} (\beta S(t)I(t) - \sigma E(t) - \rho_E E(t) - \eta_E E(t) - (1 - \alpha)E(t)), \\
 \dot{L}(t) &= \frac{1}{\alpha} (\sigma E(t - \tau) - \gamma L(t) - \rho_L L(t) - \eta_L L(t) - (1 - \alpha)L(t)), \\
 \dot{I}(t) &= \frac{1}{\alpha} (\gamma L(t - \tau) - \rho_I I(t) - \eta_I I(t) - (1 - \alpha)I(t)), \\
 \dot{T}_r(t) &= \frac{1}{\alpha} (\rho_E E(t - \tau) + \rho_L L(t - \tau) + \rho_I I(t - \tau) - \psi T_r(t) - \eta_L T_r(t) - (1 - \alpha)T_r(t)), \\
 \dot{R}(t) &= \frac{1}{\alpha} (\psi T_r(t - \tau) - \eta_R R(t) - (1 - \alpha)R(t)),
 \end{aligned} \tag{6}$$

with initial conditions being the same as defined in Eq. (2). Since model (6) is equivalent to Eq. (1) and is computationally easy to discuss, further dynamic analysis is carried out using model (6) to attain effective solutions. Subsequently, the following theorems elucidate the existence and uniqueness of the solutions for system (6), which expands the validity of the governing model mathematically.

Theorem 2.1 *Let $\Lambda \in \mathfrak{R}_+^7$ be the set of all possible solutions of system (6), then it is a uniformly bounded subset of \mathfrak{R}_+^7 such that*

$$\Lambda = \left\{ (S, V, E, L, I, T_r, R) \in \mathfrak{R}_+^7; \aleph \leq \frac{r}{\eta^* k} \right\}. \tag{7}$$

Proof Utilizing expansion (5) on Eq. (3), we get the expression

$$\dot{\aleph}(t) = \frac{1}{\alpha} (\dot{S}(t) + \dot{V}(t) + \dot{E}(t) + \dot{L}(t) + \dot{I}(t) + \dot{T}_r(t) + \dot{R}(t) - (1 - \alpha)\aleph(t)). \tag{8}$$

On simplifying by substituting values from system (6) and taking η^* as a total proportion of natural deaths defined as

$$\eta^* \aleph(t) = \eta_S S(t) + \eta_V V(t) + \eta_E E(t) + \eta_L L(t) + \eta_I I(t) + \eta_{T_r} T_r(t - \tau) + \eta_R R(t), \tag{9}$$

since $0 < \alpha \leq 1$, therefore Eq. (8) can be converted into

$$\dot{\aleph}(t) \leq rS(t) \left(1 - \frac{S(t)}{k} \right) - \eta^* \aleph(t). \tag{10}$$

As $0 < \frac{S(t)}{k} \leq 1$, so the above inequality reduces to

$$\dot{\aleph}(t) \leq \frac{r}{k} - \eta^* \aleph(t). \tag{11}$$

On integrating, we get

$$\aleph(t) \leq e^{-t\eta^*} \aleph(\vartheta) + \frac{r}{\eta^*k}. \tag{12}$$

Therefore, as $t \rightarrow \infty$, we obtain the final statement of boundedness as follows:

$$\aleph(t) \leq \frac{r}{\eta^*k}. \tag{13}$$

□

Theorem 2.2 Assume $\Omega(M(t)) : \aleph_+^7 \rightarrow \aleph_+^7$ to be the matrix of the right-hand side of system (6) such that $\Omega(M(t))$ and $\frac{\partial \Omega(M(t))}{\partial M(t)}$ are continuous and

$$\begin{aligned} \|\Omega(M(t))\| &\leq \left(\frac{\delta}{|\alpha|} - \tilde{\alpha} \right) \|M(t)\| + \|A_4\| \|M(t - \tau)\|, \\ \forall M(t) \in \aleph_+^7 \text{ and } 0 < \alpha \leq 1. \end{aligned} \tag{14}$$

Then, for given initial conditions (2), there exists a unique, nonnegative, and bounded solution of system (6) for all $t \in [-\tau, \infty)$.

Proof Boundedness of system (6) can be followed from Theorem 2.1. Here, the remaining elements of the above statement are proved. Consider

$$\dot{M}(t) = \Omega(M(t), M(t - \tau)),$$

where

$$M(t) = \begin{bmatrix} S(t) & V(t) & E(t) & L(t) & I(t) & T_r(t) & R(t) \end{bmatrix}^T, \tag{15}$$

$$\begin{aligned} M(t - \tau) &= \begin{bmatrix} S(t - \tau) & V(t - \tau) & E(t - \tau) & L(t - \tau) & I(t - \tau) & T_r(t - \tau) & R(t - \tau) \end{bmatrix}^T \tag{16} \end{aligned}$$

and

$$\begin{aligned} &\Omega(M(t), M(t - \tau)) \\ &= \frac{1}{\alpha} \begin{bmatrix} rS(t)(1 - \frac{S(t)}{k}) - \mu S(t) - \beta S(t)I(t) - \eta_S S(t) - (1 - \alpha)S(t) \\ \mu S(t) - \eta_V V(t) - (1 - \alpha)V(t) \\ \beta S(t)I(t) - \sigma E(t) - \rho_E E(t) - \eta_E E(t) - (1 - \alpha)E(t) \\ \sigma E(t - \tau) - \gamma L(t) - \rho_L L(t) - \eta_L L(t) - (1 - \alpha)L(t) \\ \gamma L(t - \tau) - \rho_I I(t) - \eta_I I(t) - (1 - \alpha)I(t) \\ \rho_E E(t - \tau) + \rho_L L(t - \tau) + \rho_I I(t - \tau) - \psi T_r - \eta_L T_r(t) - (1 - \alpha)T_r(t) \\ \psi T_r(t - \tau) - \eta_R R(t) - (1 - \alpha)R(t) \end{bmatrix}. \end{aligned} \tag{17}$$

Equation (17) can be further expanded into

$$\begin{aligned} &\Omega(M(t), M(t - \tau)) \\ &= \frac{1}{\alpha} (A_1 M(t) + S(t)A_2 M(t) + I(t)A_3 M(t) + A_4 M(t - \tau) - (1 - \alpha)M(t)) \end{aligned} \tag{18}$$

such that

$$A_1 = \begin{bmatrix} r - \mu - \eta_S & 0 & 0 & \dots & \dots & \dots & 0 \\ \mu & -\eta_V & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\sigma - \eta_E - \rho_E & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -\gamma - \eta_L - \rho_L & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & -\eta_I - \rho_I & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & -\eta_{T_r} - \psi & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & -\eta_R \end{bmatrix}_{7 \times 7},$$

$$A_2 = \begin{bmatrix} r/k & \mathbf{0} \end{bmatrix}_{1 \times 7}, \quad A_3 = \begin{bmatrix} -\beta & 0 & \beta & \mathbf{0} \end{bmatrix}_{1 \times 7},$$

and

$$A_4 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A'_4 \end{bmatrix}_{7 \times 7},$$

where

$$A'_4 = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 \\ \rho_E & \rho_L & \rho_I & 0 & 0 \\ 0 & 0 & 0 & \psi & 0 \end{bmatrix}.$$

Equation (18) can be manipulated as follows:

$$\begin{aligned} & \|\Omega(M(t), M(t - \tau))\| \\ &= \left\| \frac{1}{\alpha} (A_1 M(t) + S(t)A_2 M(t) + I(t)A_3 M(t) + A_4 M(t - \tau) + (\alpha - 1)M(t)) \right\| \\ &\leq \left| \frac{1}{\alpha} \right| \left((\|A_1\| + \|A_2\| + \|A_3\| + |\alpha - 1|) \|M(t)\| + \|A_4\| \|M(t - \tau)\| \right). \end{aligned}$$

Let $\delta = \|A_1\| + \|A_2\| + \|A_3\|$, so the required statement for uniqueness is achieved as

$$\|\Omega(M(t), M(t - \tau))\| \leq \left(\frac{\delta}{|\alpha|} - \tilde{\alpha} \right) \|M(t)\| + \|A_4\| \|M(t - \tau)\|,$$

where $\tilde{\alpha} = |(\frac{1}{\alpha} - 1)|$ for $0 < \alpha \leq 1$. Next, to prove the nonnegativity of the solutions, we use the positivity of initial conditions (2), i.e., $\xi_i(\vartheta) \geq 0$ and $\xi_i(0) > 0$ for $i = 1, 2, \dots, 7$. Consider the first equation of system (6), it can be deduced to

$$\begin{aligned} \dot{S}(t) &= \frac{1}{\alpha} \left(rS(t) \left(1 - \frac{S(t)}{k} \right) - \mu S(t) - \beta S(t)I(t) - \eta_S S(t) - (1 - \alpha)S(t) \right) \\ &\geq -\frac{1}{\alpha} (\mu + \eta_S + (1 - \alpha))S(t). \end{aligned}$$

On manipulating, we get

$$S(t) \geq \xi_1(\vartheta)e^{-((\mu+\eta_S+(1-\alpha))/\alpha)t}. \tag{19}$$

Since $0 \leq e^{-((\mu+\eta_S+(1-\alpha))/\alpha)t} \leq 1$ for $t > 0$, therefore, Eq. (19) reduces to $S(t) \geq 0$.

Thus, the nonnegativity of $S(t)$ is proved. Analogously, all the remaining equations of system (6) can be ascertained to have nonnegative solutions with the assumption of positive initial conditions. \square

3 Basic reproduction number

The basic reproduction number, mostly denoted by R_0 , has a significant importance in epidemiological studies as it measures the expected number of secondary cases produced by the contact of a single infected individual in a completely susceptible population. If $R_0 < 1$, then the disease dies out after a time period, whereas if $R_0 > 1$, then the disease spreads out and becomes endemic. It is dimensionless and can be calculated either directly by the product of transmissibility, average rate of contact, and duration of infectiousness or by using the next generation method [25, 26].

Here, using the next generation method, a submodel of SVELITR model (6) is considered to only contain the infectious compartments. Therefore, for the case of model (6), it includes the exposed, latent, and infected individuals, i.e., the equations of $E, L,$ and I . Let

$$\frac{d\vec{X}}{dt} = F(\vec{X}) - V(\vec{X}), \tag{20}$$

where $\vec{X} = [E \ L \ I]^t$ and $F(\vec{X}) = [\beta SI/\alpha \ 0 \ 0]^t$ define new infections entering each compartment. Since no new infected individual enters L and I compartments directly, rather they transit from the E compartment into L and then I compartments, therefore, the second and third element of $F(\vec{X})$ are zero. On the other hand, $V(\vec{X})$ can be further fragmented down as

$$V(\vec{X}) = V^-(\vec{X}) - V^+(\vec{X}),$$

where $V^-(\vec{X})$ and $V^+(\vec{X})$ contain all other outputs and inputs of each $E, L,$ and I compartments, respectively. Hence $V(\vec{X})$ can be outlined as follows:

$$V(\vec{X}) = \begin{bmatrix} (\sigma + \rho_E + \eta_E + (1 - \alpha))E(t)/\alpha \\ (\gamma + \rho_L + \eta_L + (1 - \alpha))L(t)/\alpha \\ (\rho_I + \eta_I + (1 - \alpha))I(t)/\alpha \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma E(t)/\alpha \\ \gamma L(t)/\alpha \end{bmatrix}.$$

Taking Jacobian matrix of Eq. (20) at a disease-free equilibrium point

$$\Delta_1(k(r - \eta_S - \mu + \alpha - 1)/r, k(1 - r + \eta_S + \mu - \alpha)\mu/r(\alpha - \eta_V - 1), 0, 0, 0, 0, 0),$$

we get

$$\begin{aligned} J \left[\frac{d\vec{X}}{dt} \right] \\ = \mathbf{F} - \mathbf{V} \end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & \beta k(r - \eta_S - \mu + \alpha - 1)/\alpha r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} (\sigma + \rho_E + \eta_E + (1 - \alpha))/\alpha & 0 & 0 \\ -\sigma/\alpha & (\gamma + \rho_L + \eta_L + (1 - \alpha))/\alpha & 0 \\ 0 & -\gamma/\alpha & (\rho_I + \eta_I + (1 - \alpha))/\alpha \end{bmatrix}. \tag{21}$$

From Eq. (21), we can extract the following next generation matrix:

$$\mathbf{K} = \mathbf{FV}^{-1}. \tag{22}$$

On calculating the eigenvalues of Eq. (22), the eigenvalue with the largest magnitude is the required value of the basic reproduction number. So, after some simplification, we get

$$R_0 = \frac{k\beta\gamma\sigma(r + \alpha - \eta_S - \mu - 1)}{r(1 - \alpha + \eta_L + \gamma + \rho_L)(1 - \alpha + \eta_E + \sigma + \rho_E)(1 - \alpha + \eta_I + \rho_I)}. \tag{23}$$

4 System dynamics

In this section, the equilibria of system (6) along with its dynamic properties are discussed in particular. Let the Jacobian matrix at an equilibrium point $(\hat{S}, \hat{V}, \hat{E}, \hat{L}, \hat{I}, \hat{T}_r, \hat{R})$ be outlined as $\mathbf{J}_{(\hat{S}, \hat{V}, \hat{E}, \hat{L}, \hat{I}, \hat{T}_r, \hat{R})} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$, where

$$M_1 = \begin{bmatrix} r(1 - \frac{2\hat{S}}{k}) - \beta\hat{I} - (\mu + \eta_S + (1 - \alpha)) & 0 & 0 & 0 \\ \mu & -(\eta_V + (1 - \alpha)) & 0 & 0 \\ \beta\hat{I} & 0 & -(\sigma + \rho_E + \eta_E + (1 - \alpha)) & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -\beta\hat{S} & 0 & 0 \\ 0 & 0 & 0 \\ \beta\hat{S} & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & \sigma e^{-\lambda\tau} & -(\gamma + \rho_L + \eta_L + (1 - \alpha)) \\ 0 & 0 & 0 & \gamma e^{-\lambda\tau} \\ 0 & 0 & \rho_E e^{-\lambda\tau} & \rho_L e^{-\lambda\tau} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$M_4 = \begin{bmatrix} 0 & 0 & 0 \\ -(\rho_I + \eta_I + (1 - \alpha)) & 0 & 0 \\ \rho_I e^{-\lambda\tau} & -(\psi + \eta_{T_r} + (1 - \alpha)) & 0 \\ 0 & \psi e^{-\lambda\tau} & -(\eta_R + (1 - \alpha)) \end{bmatrix}.$$

So, the corresponding characteristic equation for the eigenvalues λ can be articulated as

$$Q(\lambda) = \det(\lambda\mathbf{I} - \mathbf{J}) = 0. \tag{24}$$

Theorem 4.1 *The trivial equilibrium solution, $\Lambda_0(0, 0, 0, 0, 0, 0, 0) \in \mathfrak{R}_+^7$, of system (6) is asymptotically unstable for all $0 \leq \tau < \infty$ and $0 < \alpha \leq 1$.*

Proof It can be easily proved by eigenvalues of J , i.e., for all $0 \leq \tau < \infty$,

$$\begin{aligned} \lambda_1 &= \frac{-1 - \eta_V + \alpha}{\alpha}, & \lambda_2 &= \frac{-1 - \eta_R + \alpha}{\alpha}, & \lambda_3 &= \frac{-1 - \eta_S - \mu + \alpha + r}{\alpha}, \\ \lambda_4 &= \frac{-1 - \eta_L - \gamma - \rho_L + \alpha}{\alpha}, \\ \lambda_5 &= \frac{-1 - \eta_I - \rho_I + \alpha}{\alpha}, & \lambda_6 &= \frac{-1 - \eta_E - \sigma - \rho_E + \alpha}{\alpha}, & \lambda_7 &= \frac{-1 - \eta_{T_r} - \psi + \alpha}{\alpha}. \end{aligned}$$

Since $r > 1 + \eta_S + \mu - \alpha$, it is clear that $\lambda_7 > 0$ for $0 < \alpha \leq 1$. Thus, $\Lambda_0 \in \mathfrak{R}_+^7$ is unstable for all $0 \leq \tau < \infty$. □

Theorem 4.2 *The disease-free equilibrium solution*

$$\Lambda_1(k(r - \eta_S - \mu + \alpha - 1)/r, k(1 - r + \eta_S + \mu - \alpha)\mu/r(\alpha - \eta_V - 1), 0, 0, 0, 0, 0) \in \mathfrak{R}_+^7$$

of system (6), for $r > 1 + \eta_S + \mu - \alpha$, is locally asymptotically stable if $R_0 < 1$ and unstable when $R_0 > 1$ for all $0 \leq \tau < \infty$.

Proof Solving J for the equilibrium point

$$\Lambda_1(k(r - \eta_S - \mu + \alpha - 1)/r, k(r - \eta_S - \mu + \alpha - 1)\mu/r(1 + \eta_V - \alpha), 0, 0, 0, 0, 0) \in \mathfrak{R}_+^7,$$

we get the negative real eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{-1 - \eta_V + \alpha}{\alpha}, & \lambda_2 &= \frac{-1 - \eta_R + \alpha}{\alpha}, \\ \lambda_3 &= \frac{-r + \eta_S + \mu - \alpha + 1}{\alpha}, & \lambda_4 &= \frac{-1 - \eta_{T_r} - \psi + \alpha}{\alpha}, \end{aligned}$$

with the transcendental equation

$$Q(\lambda) = \lambda^3 + C_2\lambda^2 + C_1\lambda + C_0(1 - e^{-2\lambda\tau}R_0) = 0, \tag{25}$$

where

$$\begin{aligned} C_2 &= \frac{1}{\alpha}(3 - 3\alpha + \gamma + \sigma + \eta_E + \eta_L + \eta_I + \rho_E + \rho_L + \rho_I), \\ C_1 &= \frac{1}{\alpha^2} \left(\begin{aligned} &3 + 3\alpha^2 + 2(\eta_E + \eta_L + \eta_I + \rho_E + \rho_L + \rho_I) + \eta_E\eta_L + \eta_E\eta_I + \eta_I\eta_L \\ &+ \rho_E(\eta_L + \eta_I) + \rho_L(\eta_E + \eta_I + \rho_E) + \rho_I(\eta_E + \eta_L + \rho_E + \rho_L) \\ &+ (2 + \eta_I + \eta_L + \rho_L + \rho_I)\sigma + (2 + \eta_E + \eta_I + \rho_E + \rho_I + \sigma)\gamma \\ &- 2\alpha(3 + \gamma + \sigma + \eta_E + \eta_L + \eta_I + \rho_E + \rho_L + \rho_I) \end{aligned} \right), \\ C_0 &= \frac{1}{\alpha^3}(1 - \alpha + \eta_L + \gamma + \rho_L)(1 - \alpha + \eta_E + \sigma + \rho_E)(1 - \alpha + \eta_I + \rho_I). \end{aligned}$$

From the Routh–Hurwitz condition [27] for cubic polynomial, if $C_2 > 0$, $C_0(1 - e^{-2\lambda\tau}R_0) > 0$ and $C_1C_2 > C_0(1 - e^{-2\lambda\tau}R_0)$, then $Q(\lambda) > 0$ for all nonnegative real eigenvalues, and thus all the real parts of the eigenvalues must be negative. Therefore, it can be clearly seen that $C_1 > 0$, $C_2 > 0$, and $C_0 > 0$, so the only thing left to be proved is that $(1 - e^{-2\lambda\tau}R_0) > 0$.

Accordingly, when $\tau = 0$, the only possibility for $(1 - e^{-2\lambda\tau} R_0) > 0$ is that $R_0 < 1$. Hence, $\Lambda_1 \in \mathfrak{N}_+^7$ is asymptotically stable if $R_0 < 1$ for $\tau = 0$. If $R_0 > 1$ and $(1 - e^{-2\lambda\tau} R_0) < 0$ implies $Q(0) < 0$, then Eq. (25) must have a nonnegative real part, thus it becomes unstable.

Now, $\forall \tau \in \mathfrak{N}_+$, we prove that if $\lambda = a \pm ib$ is the complex solution of $Q(\lambda)$, then $\Lambda_1 \in \mathfrak{N}_+^7$ is stable if $a \leq 0$. On substituting the complex value of λ , for $a \geq 0$ and $b^2 > 0$, into Eq. (21) and separating the real and imaginary parts, we get the following polynomial:

$$\begin{aligned}
 & b^6 + b^4(3a^2 - 2C_1 + 2aC_2 + C_2^2) \\
 & + b^2(3a^4 + C_1^2 + 4a^3C_2 - 2C_0C_2 + 2a^2C_2^2 + a(-6C_0 + 2C_1C_2)) \\
 & + (C_0 + a(C_1 + a(a + C_2)))^2 - C_0^2e^{-4a\tau}R_0^2 = 0.
 \end{aligned} \tag{26}$$

$(C_0 + a(C_1 + a(a + C_2)))^2 > C_0^2e^{-4a\tau}R_0^2$, whenever $R_0 < 1, \forall \tau \in \mathfrak{N}_+$ since there is no solution for $a \geq 0$ and $b^2 > 0$, thus there is no complex number with a nonnegative real part. Hence, $\Lambda_1 \in \mathfrak{N}_+^7$ is asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1, \forall \tau \in \mathfrak{N}_+$. \square

Theorem 4.3 *If $R_0 > 1$, then the endemic equilibrium $\Lambda_2(\check{S}, \check{V}, \check{E}, \check{L}, \check{I}, \check{T}_r, \check{R}) \in \mathfrak{N}_+^7$ is locally asymptotically stable and unstable if and only if either it has a zero solution or has a complex conjugate pair of purely imaginary solutions when $R_0 > 1$ for some $\tau > \tau_0$.*

Proof Solving J for the equilibrium point $\Lambda_2(\check{S}, \check{V}, \check{E}, \check{L}, \check{I}, \check{T}_r, \check{R}) \in \mathfrak{N}_+^7$, we get the negative real eigenvalues

$$\lambda_1 = \frac{-1 - \eta_V + \alpha}{\alpha}, \quad \lambda_2 = \frac{-1 - \eta_R + \alpha}{\alpha},$$

with the transcendental equation

$$Q(\lambda) = \lambda^4 + U_3\lambda^3 + U_2\lambda^2 + U_1\lambda + U_0 + (-C_0\lambda + U_a)e^{-2\lambda\tau} = 0, \tag{27}$$

where

$$\begin{aligned}
 U_0 &= -\frac{C_0U_b}{\alpha}, \\
 U_1 &= C_0 - \frac{C_1U_b}{\alpha}, \\
 U_2 &= C_1 - \frac{C_2U_b}{\alpha}, \\
 U_3 &= C_2 - \frac{U_b}{\alpha}, \\
 U_a &= \frac{C_0U_b}{\alpha^3}(2 - R_0)
 \end{aligned}$$

such that

$$U_b = -\frac{(r + \alpha - \eta_S - \mu - 1)}{R_0}.$$

For $\tau = 0$, $Q(\lambda)$ reduces into

$$Q(\lambda)|_{\tau=0} = \lambda^4 + U_3\lambda^3 + U_2\lambda^2 + (U_1 - C_0)\lambda + U_0 + U_a = 0 \tag{28}$$

such that $U_3 > 0, U_2 > 0, (U_1 - C_0) = -\frac{C_1 U_b}{\alpha} > 0$, and $(U_0 + U_a) = -\frac{C_0 U_b}{\alpha^2}(\alpha^2 + R_0 - 2) > 0$ as $R_0 > 1$ and $U_b < 0$. It can be easily manipulated and investigated that Eq. (28) satisfies the Routh–Hurwitz condition of fourth order polynomial, i.e., in addition to the above inequalities, $U_3 U_2 > (U_1 - C_0)$ and $U_1 U_2 U_3 > (U_0 + U_a) U_3^2 + (U_1 - C_0)^2$. Thus $Q(\lambda) > 0$, therefore when $R_0 > 1$, all the real parts of the eigenvalues are negative for $\tau = 0$. Next, it is also clear from Eq. (27) that $Q(0) = U_0 + U_a \neq 0$, so the only possibility of $\Lambda_2 \in \mathfrak{N}_+^7$ to be unstable is that a complex conjugate pair of purely imaginary solution exists for some critical value $\tau_0 > 0$. Now consider $\lambda = \pm ib$, and on substituting in Eq. (27) and separating real and imaginary parts, we get

$$b^4 + U_0 - b^2 U_2 + U_a \cos(2b\tau) - bC_0 \sin(2b\tau) \tag{29}$$

and

$$bU_1 - b^3 U_3 - U_a \sin(2b\tau) - bC_0 \cos(2b\tau). \tag{30}$$

On simplifying and assuming $\varpi = b^2$, we get

$$Y(\varpi) = \varpi^4 + v_3 \varpi^3 + v_2 \varpi^2 + v_1 \varpi + v_0 = 0, \tag{31}$$

where $v_3 = -2U_2 + U_3^2, v_2 = 2U_0 + U_2^2 - 2U_1 U_3, v_1 = -C_0^2 + U_1^2 - 2U_0 U_2, v_0 = U_0^2 - U_a^2$. The existence of at least one positive real root will lead Eq. (31) to a complex conjugate pair of purely imaginary solutions and also a Hopf bifurcation might exist. Since it is straightforward from the constant of Eq. (31) that $v_0 < 0$, when $R_0 > 1$ and $\lim_{\varpi \rightarrow \infty} Y(\varpi) = +\infty$, therefore $Y(\varpi)$ must have at least one positive real root. Hence, Eq. (27) produces a complex conjugate pair of purely imaginary solutions for some $\tau_0 > 0$ and $\Lambda_2 \in \mathfrak{N}_+^7$ becomes unstable when $R_0 > 1$. □

5 Existence of Hopf bifurcation

From Theorem 4.3, the endemic equilibrium point is unstable if and only if $Q(\lambda) = 0$ of Eq. (27) produces a complex conjugate pair of purely imaginary solutions for some $\tau_0 > 0$, which might lead to the existence of the Hopf bifurcation [11]. As it is possible that Eq. (31) can have four positive real roots, i.e., ϖ_p for $p = 1, 2, 3, 4$, thus there are four purely imaginary pairs of eigenvalues $\lambda_p = ib_p = \pm i\sqrt{\varpi_p}$ for $p = 1, 2, 3, 4$. Now, we may find values of the time delay τ_p corresponding to the value of b_p . By substituting b_p into Eqs. (29)–(30) and solving for τ_p , we get

$$\begin{aligned} \cos(2b_p \tau) &= -\frac{b_p^4 C_0 U_3 - b_p^2 C_0 U_1 + b_p^4 U_a + U_0 U_a - b_p^2 U_2 U_a}{b_p^2 C_0^2 + U_a^2} \quad \text{and} \\ \sin(2b_p \tau) &= -\frac{b_p^3 C_0 U_2 - b_p^5 C_0 - b_p C_0 U_0 - b_p U_1 U_a + b_p^3 U_3 U_a}{b_p^2 C_0^2 + U_a^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \tau_p &= \frac{1}{2b_p} \arctan\left(\frac{b_p^3 C_0 U_2 - b_p^5 C_0 - b_p C_0 U_0 - b_p U_1 U_a + b_p^3 U_3 U_a}{b_p^4 C_0 U_3 - b_p^2 C_0 U_1 + b_p^4 U_a + U_0 U_a - b_p^2 U_2 U_a}\right) \\ &\quad + \frac{j\pi}{b_p} \quad \text{for } p = 1, 2, 3, 4. \end{aligned} \tag{32}$$

The minimum time delay at which purely imaginary eigenvalues $\lambda_0 = ib_0$ occur is

$$\tau_0 = \min_{1 \leq p \leq 4, j \geq 0} \tau_p^j, \quad \tau_p^j > 0.$$

Now, for the transversality condition for the existence of Hopf bifurcation, we have to prove $\frac{d}{d\tau} \operatorname{Re}(\lambda)|_{\tau=\tau_0} \neq 0$. So, taking the derivative of Eq. (27) and simplifying yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{U_1 + 2U_2\lambda + 3U_3\lambda^2 + 4\lambda^3}{2\lambda e^{-2\tau\lambda}(U_a - C_0\lambda)} - \frac{\tau}{\lambda} - \frac{C_0}{2\lambda(U_a - C_0\lambda)},$$

with $\lambda_0 = ib_0$, we get

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_0} = \frac{1}{2(b_0^2 C_0^2 + U_a^2)} \left(-C_0^2 - \frac{-C_0 U_1 + 3b_0^3 C_0 U_3 + 4b_0^2 U_a - 2U_2 U_a}{U_d} \right) - \frac{U_c(4b_0^4 C_0 + U_1 U_a - b_0^2(2C_0 U_2 + 3U_3 U_a))}{2(b_0^2 C_0^2 + U_a^2) U_d},$$

where

$$\begin{aligned} U_c &= (U_0 U_a + b_0^4(C_0 U_3 + U_a) - b_0^2(C_0 U_1 + U_2 U_a)), \\ U_f &= (b_0^4 C_0 + C_0 U_0 + U_1 U_a - b_0^2(C_0 U_2 + U_3 U_a)), \\ U_d &= \sqrt{1 + \frac{b_0^2 U_f^2}{U_c^2}}. \end{aligned}$$

Since $\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\big|_{\tau=\tau_0} \neq 0$, thus $\frac{d}{d\tau} \operatorname{Re}(\lambda)|_{\tau=\tau_0} \neq 0$, with the critical point τ_0 as a Hopf bifurcation point.

6 Numerical findings

In this section, numerical investigations of system (6) are carried out by considering some numerical values of the parameters, as shown in the form of three sets in Table 2. Taking the initial conditions as

$$\begin{aligned} S(\vartheta) &= 100, & V(\vartheta) &= 0, & E(\vartheta) &= 0, & L(\vartheta) &= 0, \\ I(\vartheta) &= 1, & T_r(\vartheta) &= 0, & R(\vartheta) &= 0, \end{aligned}$$

for $\vartheta \in [-\tau, 0)$, all the manipulation and plotting of the dynamics are undergone by using *Mathematica 10.0*. Stability of the equilibrium states and phase portraits is discussed in detail through graphs and tables.

On using set 1 of Table 2, the basic reproduction numbers R_0 and constant solutions of $S(t)$, $V(t)$, $E(t)$, $L(t)$, $I(t)$, $T_r(t)$, and $R(t)$ for different values of α are calculated and represented in Table 3. Since at each value of α , $R_0 < 1$, therefore the plots obtained for some values of $\alpha = 0.8, 0.95$ and 1 , $\tau = 1$ and $t \in [0, 200]$ in Fig. 2, verify the stability of disease-free equilibrium states for $R_0 = 0.37668, 0.545976, 0.62441$, respectively.

Subsequently, for the values of parameters in set 2 of Table 2, the basic reproduction number along with the constant solutions of $S(t)$, $V(t)$, $E(t)$, $L(t)$, $I(t)$, $T_r(t)$, and $R(t)$ for

Table 2 Values of the parameters for SVELITR model (6)

Parameters	Set	
	1	2
r	19	19
k	8.5	8.5
μ	0.3	0.043
β	0.35	0.95
σ	0.6	0.6
γ	0.5	0.96
ψ	0.58	0.005
η_S	0.1	0.01
η_V	0.1	0.1
η_E	0.035	0.015
η_L	0.1	0.016
η_I	0.2	0.5
η_{T_r}	0.42	0.42
η_R	0.035	0.01
ρ_E	0.6	0.041
ρ_L	0.5	0.044
ρ_I	0.83	1

Table 3 Basic reproduction number R_0 and disease-free equilibrium point Λ_1 for set 1 and $\tau = 1$ at different values of α

α	R_0	$S(t)$	$V(t)$	$E(t)$	$L(t)$	$I(t)$	$T_r(t)$	$R(t)$
0.4	0.166286	8.05263	3.45113	0	0	0	0	0
0.5	0.200181	8.09737	4.04868	0	0	0	0	0
0.6	0.24377	8.14211	4.88526	0	0	0	0	0
0.7	0.300758	8.18684	6.14013	0	0	0	0	0
0.8	0.37668	8.23158	8.23158	0	0	0	0	0
0.9	0.480049	8.27632	12.4145	0	0	0	0	0
1.	0.624411	8.32105	24.9632	0	0	0	0	0

different values of α are obtained and represented in Table 4. As at these values $R_0 > 1$ for all $0 < \alpha \leq 1$, then, according to Theorems 4.2 and 4.3, the disease-free equilibrium state becomes unstable. On the other hand, endemic equilibrium will become stable for some $\tau < \tau_0$. Therefore, calculating the critical Hopf bifurcation point τ_0 by using Eq. (32), we have $\tau_0 = 0.380397$ for the values of parameters in set 2 of Table 2. Consequently, the plots of system (6) in Fig. 3 at different values of $\alpha = 0.8, 0.95, 1$ and $t \in [0, 200]$, the stability of endemic equilibrium state is evident due the reason that $\tau = 0.035 < \tau_0$ with $R_0 = 2.58500, 3.95079, 4.62122$; whereas Fig. 4 shows unstability of the endemic equilibrium states for the same values of all parameters and R_0 due to the reason that $\tau = 0.9 > \tau_0$. In addition, the phase diagrams are also portrayed in Figs. 5 and 6, at $\tau = 1.5$ and $\tau = 3$, by using values of set 2 in Table 2, for $\alpha = 0.8, 0.95, 1$, and $t \in [0, 5000]$. These figures further illustrate that as the values of τ and α are increased system (6) loses stability and generates the limit cyclic patterns of the infectious compartments, i.e., $E(t)$, $L(t)$, and $I(t)$.

The abovementioned graphical and tabulated results are successfully attained at different values of fractional order α , which greatly enables us to illustrate the equilibrium states and the basic reproduction number R_0 of system (6) with memory effect. Tables 3 and 4 significantly elaborate the effect of α on R_0 , through which it can be clearly seen in detail that within the time bar a fractional variation in the system changes the value of the basic reproductive number.

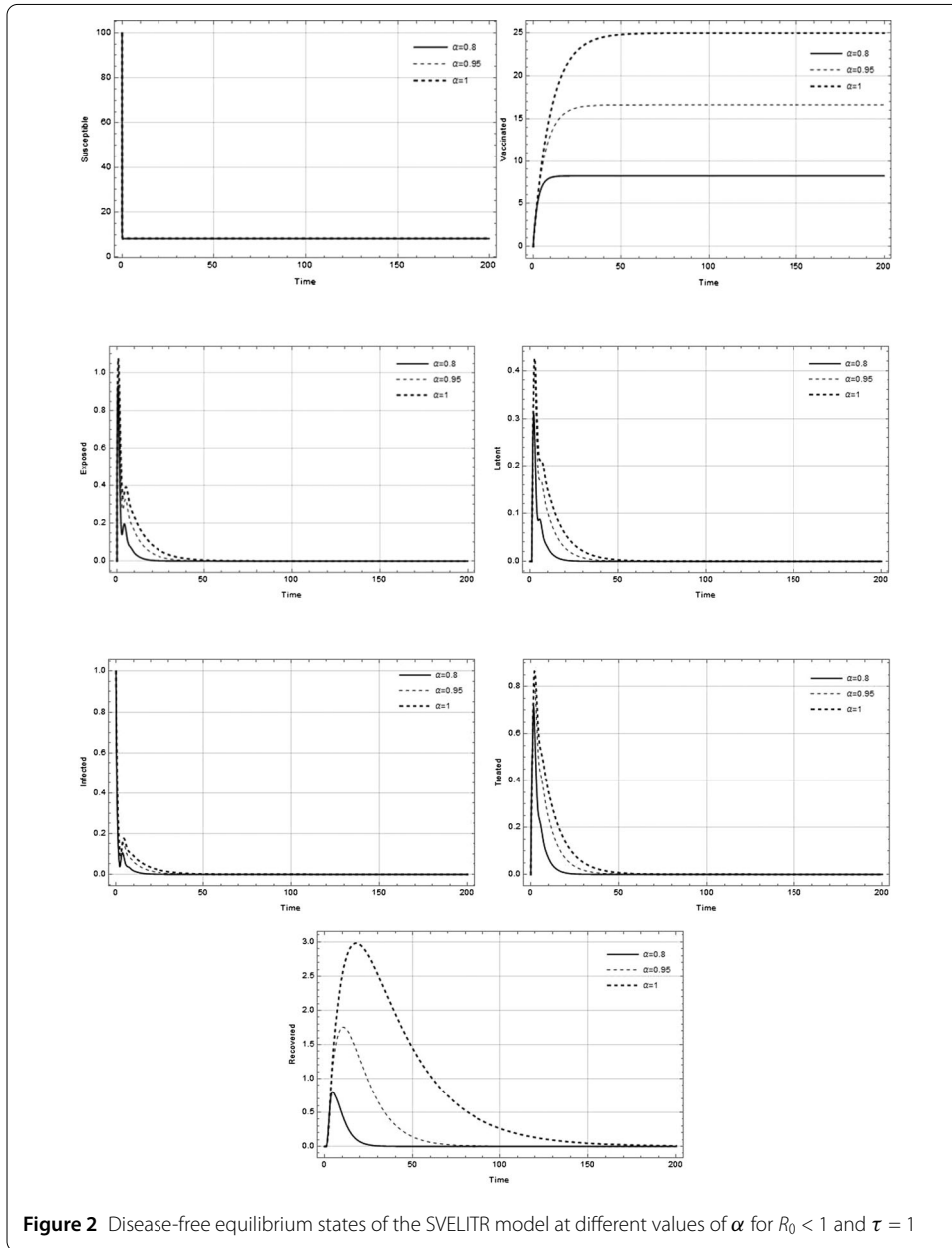
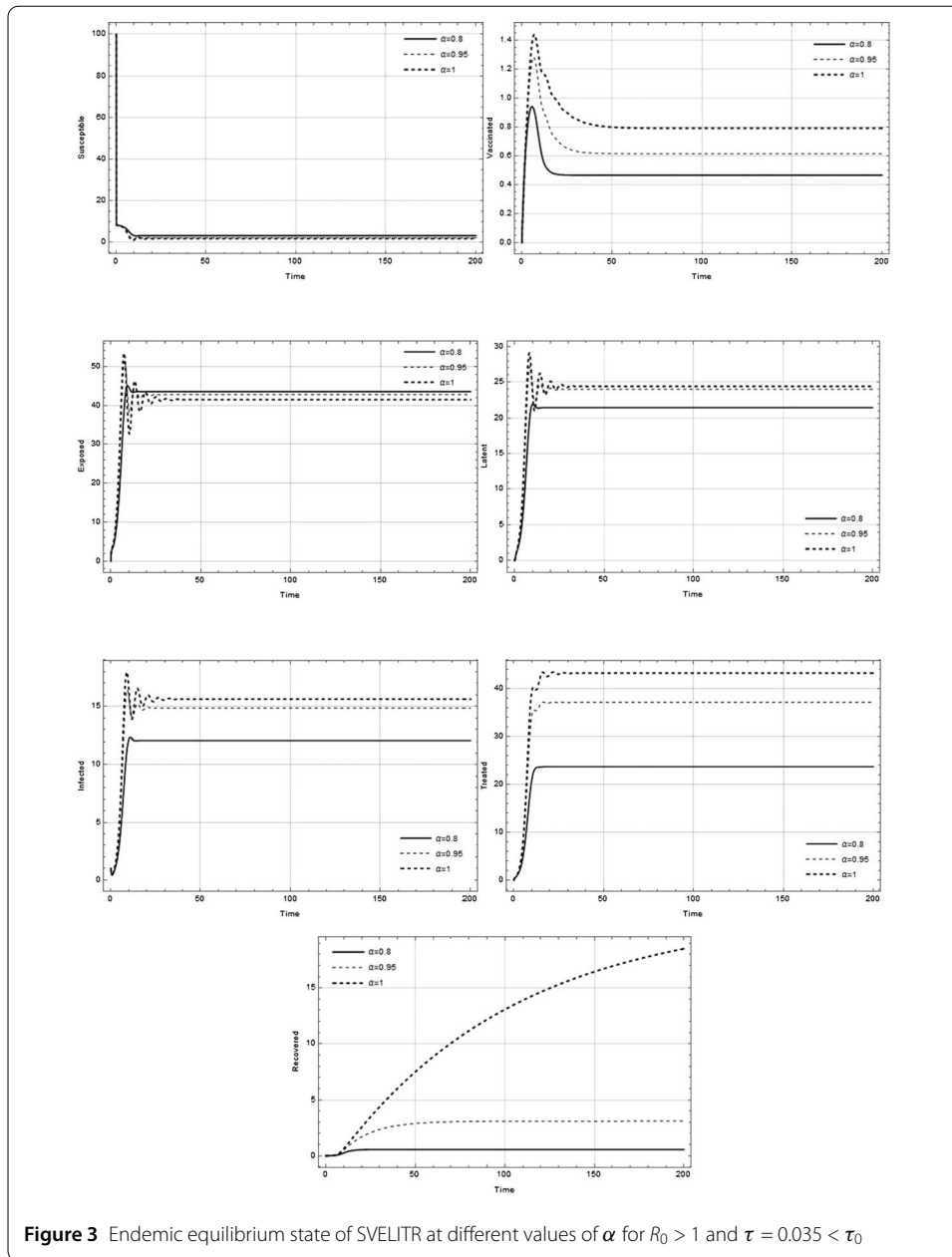


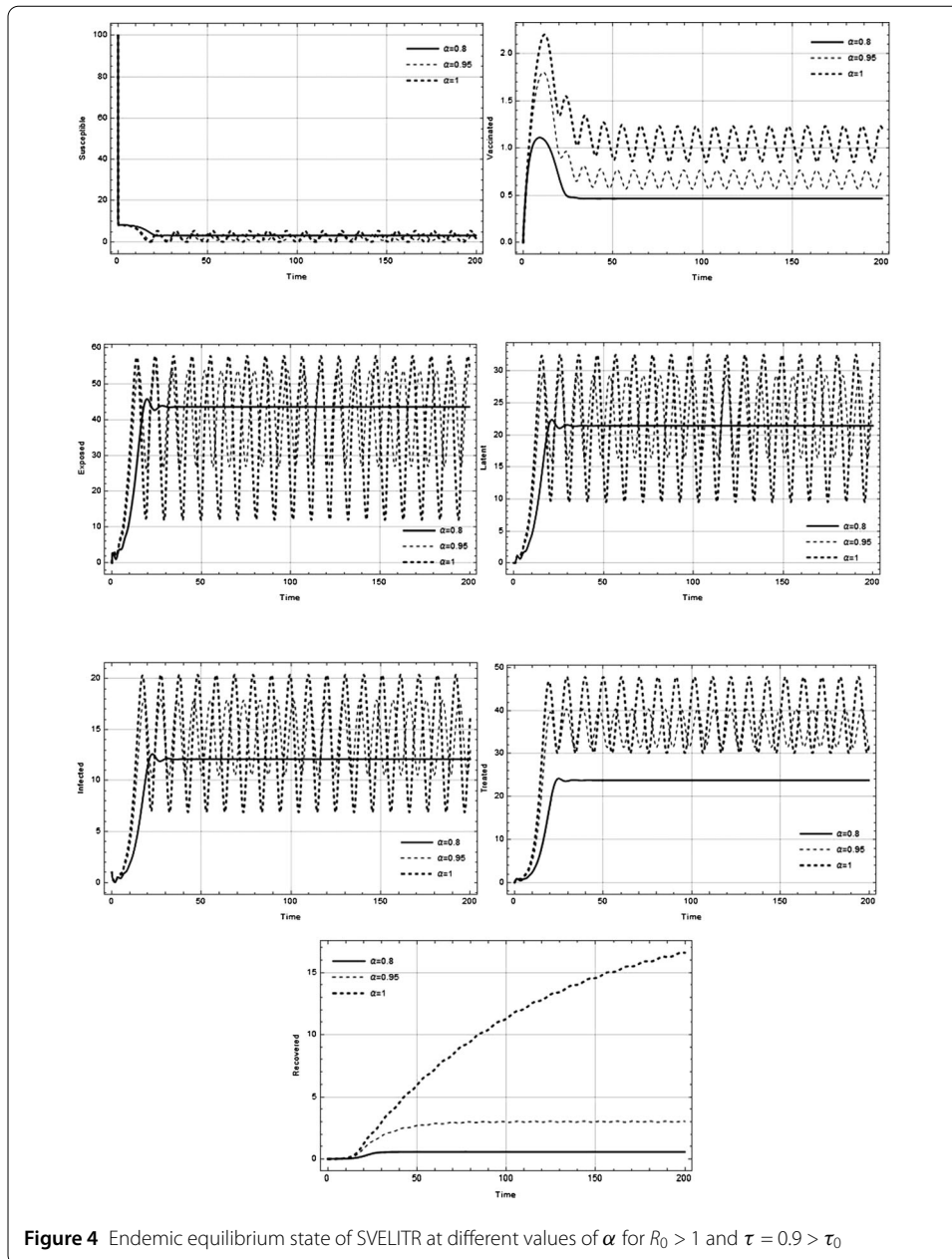
Table 4 Basic reproduction number R_0 and endemic equilibrium point Λ_2 for set 2 and $\tau = 0.035$ at different values of α

α	R_0	$S(t)$	$V(t)$	$E(t)$	$L(t)$	$I(t)$	$T_r(t)$	$R(t)$
0.4	1.05112	7.80868	0.479676	5.54749	2.05462	0.939257	1.22645	0.0100528
0.5	1.28501	6.42222	0.460259	22.7302	8.97247	4.30678	6.09028	0.0597087
0.6	1.5936	5.20667	0.447773	34.0631	14.3929	7.27218	11.2752	0.137503
0.7	2.00963	4.15105	0.446238	40.6776	18.4898	9.86124	17.0242	0.274584
0.8	2.585	3.24442	0.465033	43.5676	21.4267	12.0998	23.7261	0.564907
0.9	3.4056	2.47579	0.532295	43.5977	23.3559	14.0136	32.0547	1.45703
1.	4.62122	1.83421	0.788711	41.513	24.4194	15.6284	43.3057	21.6528



7 Conclusions

A system of seven fractional order differential equations was designed with time delay to illustrate the seven epidemiological compartments. These compartments elaborated each stage of any individual from being susceptible and exposed to infection till its treatment and recovery or being vaccinated and safe. In addition, time delay was also added in the treatment compartment to show the dynamics of the individuals of exposed, latent, or infected stages going through the treatment with different rates and being recovered later. Moreover, we applied a special case of the proportional fractional derivative on the system of equations. This simplified form made it computationally easy to discuss the variations and equilibrium states of the functions by converting the system into an integer order without losing the originality of the fractional index. This definition also enabled to study



the significant impact of the fractional index change on the basic reproduction number and the stability of the equilibrium states. Furthermore, the point-to-point outcomes of the study can be summarized as follows:

- The basic reproduction number R_0 does not remain constant throughout the dynamic process of an epidemic in a particular time interval, but a fractional index variation in the system will also cause change in its value within the time interval.
- For any delay in time taken by an individual for the treatment and recovery, if the basic reproduction number of the infection for each fractional index step is less than 1, then a disease-free state can be achieved.
- If $R_0 > 1$ for each fractional index value and time delay by an individual is less than the Hopf bifurcation point, then the infection becomes endemic with a constant rate.

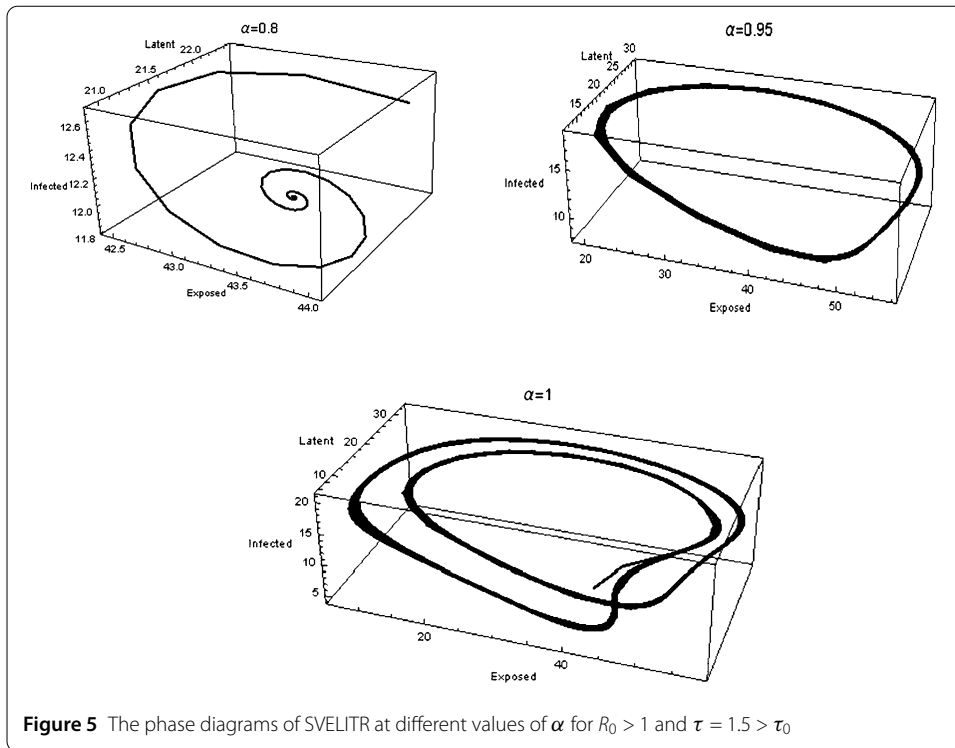


Figure 5 The phase diagrams of SVELITR at different values of α for $R_0 > 1$ and $\tau = 1.5 > \tau_0$

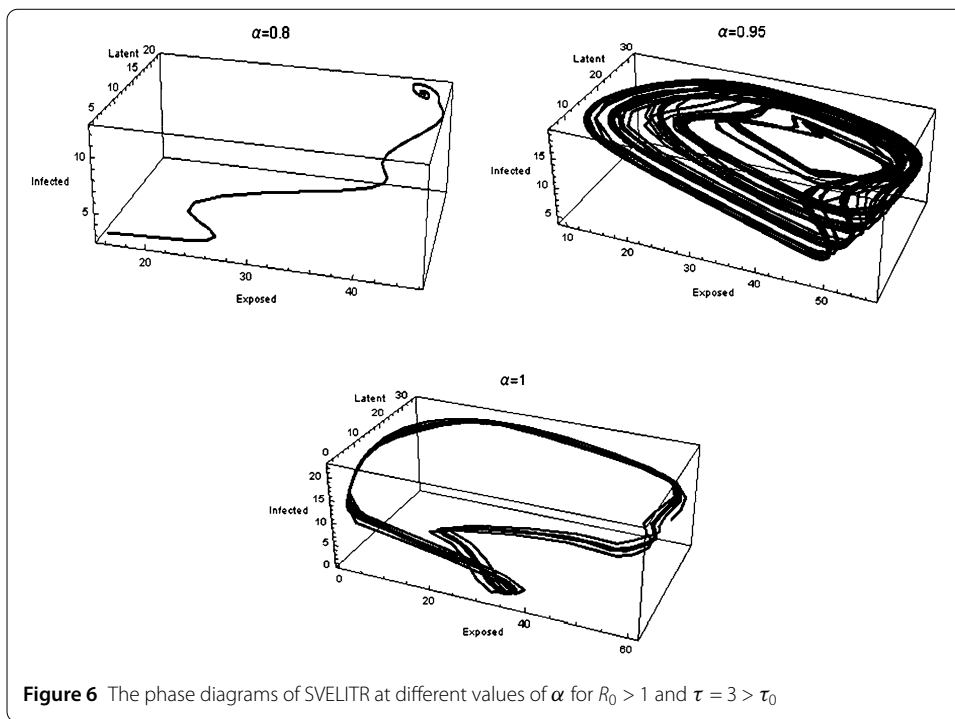


Figure 6 The phase diagrams of SVELITR at different values of α for $R_0 > 1$ and $\tau = 3 > \tau_0$

- If $R_0 > 1$ for each fractional value and time delay by an individual is greater than the Hopf bifurcation point, then the system moves towards an instability state bit by bit, and so the infection outbreaks.

As the proposed epidemiological model for the transmission of an infection that shows the impact of fractional operator on the basic reproduction number of any infection has not been found in the literature yet, it will be greatly advantageous for the epidemiologist to cope with the infection at each fractional index value before the occurrence of a complete change. In future, we will design a model that will demonstrate the age classification and impact of fractional index variation of an infection on different age groups.

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Not applicable.

Ethics approval and consent to participate

This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors carried out the proofs and conceived of the study. All authors read and approved the final manuscript.

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