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Existence and optimal controls for nonlocal fractional evolution equations of order (1,2) in Banach spaces

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Abstract

In this paper, we mainly investigate the existence, continuous dependence, and the optimal control for nonlocal fractional differential evolution equations of order (1,2) in Banach spaces. We define a competent definition of a mild solution. On this basis, we verify the well-posedness of the mild solution. Meanwhile, with a construction of Lagrange problem, we elaborate the existence of optimal pairs of the fractional evolution systems. The main tools are the fractional calculus, cosine family, multivalued analysis, measure of noncompactness method, and fixed point theorem. Finally, an example is propounded to illustrate the validity of our main results.

MSC: 26A33; 34K37

Keywords: Fractional evolution equations; Mild solution; Existence; Continuous dependence; Optimal control

1 Introduction

Fractional differential equations are a branch of concentration in the field of applied mathematics. In last decades, fractional calculus has been more and more popular and has a significant development in the theory and applications of ordinary differential equations (ODEs), partial differential equations (PDEs), evolution equations and inclusions, owing to its memory character. It is a generalization of classical calculus and can characterize many phenomena in various branches of science and engineering that the latter cannot depict; see the monographs of Podlubny [1], Kilbas et al. [2], Zhou [3, 4] and papers [5–13] and the references therein.

The Cauchy problem for fractional differential and integro-differential equations of order $\alpha \in (1,2)$ has been paid more and more attention in recent years. By using the concept of fractional resolvent family Kexue Li et al. [14] established two fractional evolution problems in the Riemann–Liouville sense. Shu and Wang [15] derived the existence and uniqueness of mild solutions for a class of nonlocal fractional differential equations via the constructed concept of sectorial operator. Yaning Li [16] dealt with the regularity of mild solutions for fractional abstract Cauchy problem of order $\alpha \in (1,2)$ based on some properties of solution operators and analytic solution operators. Moreover, using eigenfunction expansions, Kian and Yamamoto [17] showed the well-posedness of solutions for semilinear fractional wave equations. Based on some properties of an introduced operator, Li et al. [18] explored the existence, uniqueness, and regularity of the mild solution of fractional abstract Cauchy problems of order $\alpha \in (1,2)$.

The control systems have occupied a significant place and played an important role in manufacturing, vehicles, computers, and regulated environment. Among the classical control issues, the controllability, exact controllability, approximate controllability, and optimal control are an essential characteristic to a control framework, namely adjustment of unsteady frameworks by input control. With the application of fractional calculus to multifarious fields of science and engineering, the optimal control problem has been reducing into a fractional-order one, especially for fractional differential evolution systems. The authors of [19] considered the optimality and relaxation of multiple control problems for nonlinear fractional differential equations with nonlocal control conditions in Banach spaces. The authors of [20] proved a Noether-like theorem in the more general context of the fractional optimal control by means of a new Noether theorem, the Lagrange multiplier method, and the fractional Euler–Lagrange equations. In cylindrical coordinates, the fractional optimal control (FOC) of a distributed system was investigated in [21]. By means of singular version Gronwall inequality and Leray–Schauder fixed point theorem, Wang and Zhou [22] presented the existence and uniqueness of α-mild solutions and the optimal control of a class of fractional evolution equations. Wang et al. [23] also investigated fractional evolution systems with finite-time delay and obtained the well-posedness of mild solutions by means of a new Gronwall inequality and derived the optimal control through introducing the fractional Lagrange problem. Looking at the existing literature, the fractional order among the most ones belongs to $(0,1)$. Based on this, some researchers work on the fractional evolution systems of order $(1,2)$. They contributed their works to extend the order by $(1,2)$. For instance, by using the Sadovskii fixed point theorem and vector-valued operator theory, the paper [24] established sufficient conditions for controllability of fractional differential systems of order $\alpha \in (1, 2]$ with nonlocal conditions in infinite-dimensional Banach spaces. Via fractional resolvent operator family and approximating minimizing sequences of suitable functions twice, the authors of [25] proposed the existence and optimal control of a class of Sobolev-type time fractional differential equations in the Caputo and Riemann–Liouville sense, respectively. The authors of [26] derived some results about the mild solutions and optimal controls for a class of Sobolev-type fractional stochastic evolution equations of order [1,2] via some compactness results of the corresponding fractional operators. Yan and Jia [27] studied optimal control for fractional stochastic functional differential equations of order (1, 2) in a Hilbert space via the fixed point theorem, approximation technique, and properties of the solution operator.

Motivated by the above discussion, this paper is devoted to analyzing the existence and optimal controls for the following nonlocal fractional evolution system:

$$\begin{cases} {}_0^cD_t^{\gamma} Ex(t) = Gx(t) + J(t,x(t)) + K(t)u(t), & t \in \Omega := [0,S], \\ x(0) = x_0 + g(x), \qquad x'(0) = x_1, \end{cases} \tag{1.1}$$

where ${}_0^cD_t^{\gamma}$ denotes the fractional derivative in the Caputo sense of order $\gamma \in (1,2)$, $GE^{-1} : D(GE^{-1}) \to X$ is the infinitesimal generator of continuous cosine family $\{C(t)\}_{t\geq 0}$

on a separable reflexive Banach space X, the state $x(\cdot)$ takes values in X as well as $x_0, x_1 \in X$, the control function $u(\cdot)$ is given in another separable reflexive Banach space U of admissible control functions, the nonlinear function $J : \Omega \times X^\alpha \to X$, where $X^\alpha = D(G_b^\alpha E^{-1})$ is a Banach space with $\|x\|_\alpha = \|G_b^\alpha x\|$, the fractional power operator G_b^α has a dense domain $D(G_b^\alpha)$ [28], K is a linear operator and maps U into X, $g : C(\Omega; X) \to X$, $G : D(G) \subset X \to X$ is a closed linear operator with, which is the infinitesimal generator of strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ in X, and $E : D(E) \subset D(G) \to X$ is a bijective linear operator such that $E^{-1} : X \to D(E) \subset X$ is compact. From the above hypotheses we obtain that E^{-1} is bounded, closed, and injective, which yields that E is also closed and that the linear operator GE^{-1} is bounded.

The remainder of our paper is arranged as follows. Section 2 collects some notations and useful concepts for fractional calculus and cosine family. In Sect. 3, based on changed hypothesis, we present the mild solution for system (1.1), which is correlated to probability density function and cosine families. Unlike the operators in some previously mentioned papers, we propose a linear bounded operator. In Sect. 4, we elaborate the well-posedness of the system. Furthermore, in Sect. 4, we prove that the system is mildly solvable and a mild solution is unique and continuously depends of the solution. In Sect. 5, we derive the FOC for the Lagrange problem. Finally, in Sect. 6, an illustrative example proves the validity of our results.

Notations: Let X and U be Banach spaces equipped with norms $\|\cdot\|$ and $\|\cdot\|_U$, respectively. $\mathfrak{L}(X,U)$ denotes the space of all bounded linear operators from X to U with norm $\|\cdot\|_{\mathfrak{L}(X,U)}$. In particular, when $X = U$, we have $\mathfrak{L}(X,U) = \mathfrak{L}(X)$ and $\|\cdot\|_{\mathfrak{L}(X,U)} = \|\cdot\|_{\mathfrak{L}(X)}$. We denote by $\mathfrak{C}_\alpha$ the Banach space $C(\Omega, X^\alpha)$ endowed with the sup-norm $\|x\|_\Omega^\alpha = \sup_{t\in\Omega} \|x(t)\|_\alpha$. Additionally, the resolvent set of G is defined by $\rho(G)$, and the resolvent of G by $R(\mu, G) = (\mu I - GE^{-1})^{-1} \in \mathfrak{L}(X)$.

2 Preliminaries

In this section, we first introduce some definitions and results of fractional calculus.

Definition 2.1 ([2]) The right-hand side fractional integral of order $q \in \mathbb{R}_+$ with the lower limit zero for a function $g \in C[a,b]$ is given by

$$ {}_0D_t^{-q} g(t) = (k_q * g)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} g(\xi)\, \mathrm{d}\xi, $$

where $Re(q) > 0$, and Γ is the classical gamma function, and, as usual, the symbol $*$ denotes convolution, and

$$ k_q(t) = \begin{cases} \frac{t^{q-1}}{\Gamma(q)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases} $$

Moreover, $\lim_{q\to 0} k_q(t) = \delta(t)$ with the delta Dirac function δ, and the Dirac measure is concentrated at the origin.

Definition 2.2 ([2]) The Caputo fractional derivative of order $q \in \mathbb{R}_+$ of a function $g(t) \in C^n[a,b]$ is represented by

$$ {}_0^cD_t^q g(t) = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^t (t-\xi)^{n-q-1} g^{(n)}(\xi)\,\mathrm{d}\xi & \text{if } q \notin \mathbb{N}, \\ g^{(n)}(t) & \text{if } q = n \in \mathbb{N}, \end{cases} $$

where $g^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(t)$, $Re(q) \geq 0$, $n = [Re(q)] + 1$, and $\mathbb{N} = \{0, 1, \ldots\}$.

Lemma 2.3 ([29]) *Suppose that* $x \in \mathfrak{C}_\alpha$ *satisfies the inequality*:

$$ \|x(t)\|_\alpha \leq c + d \int_0^t (t-\theta)^{z-1} \|x(\theta)\|_\alpha \, d\theta, \quad t \in \Omega, $$

with constants $c, d \geq 0$, $z > 0$. *Then there exists a constant* $M > 0$, *independent of* c, *such that*

$$ \|x(t)\|_\alpha \leq M(c) \quad \textit{for all } t \in \Omega. $$

Definition 2.4 ([30]) A function $f \in L^p(\Omega, U)$ with $1 \leq p < +\infty$ is said to be globally continuous if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$ \left(\int \left| f(t+z) - f(t) \right|^p dt \right)^{\frac{1}{p}} < \varepsilon \quad \text{for all } z : |z| < \delta(\varepsilon), $$

where the region of integration is the whole space and $f(t) = 0$ for $t \notin \Omega$.

Let us review the following definition and some properties of a cosine family. For more detail, we refer to [31–33].

Definition 2.5 ([31]) A one-parameter family $\{C(t)\}_{t \in \mathbb{R}}$ of bounded linear operators mapping the Banach space U into itself is called a strongly continuous cosine family if

(i) $C(0) = I$;
(ii) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$;
(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in U$.

With the cosine family $\{C(t)\}_{t \in \mathbb{R}}$, the strongly continuous sine family $\{S(t)\}_{t \in \mathbb{R}}$ is defined by

$$ S(t)x = \int_0^t C(s)x \, ds, \quad x \in U, t \in \mathbb{R}. $$

3 Existence and uniqueness

For simplicity, throughout this paper, we set $z = \gamma/2$ with $\gamma \in (1, 2)$ and suppose that the linear operators $\{C(t)\}_{t \geq 0}$ are uniformly bounded, that is, there exists a constant $M \geq 1$ such that $\|C(t)\|_{\mathfrak{L}(U)} \leq M$ for $t \geq 0$. Before proving the existence and uniqueness of system (1.1), we pose the following hypotheses:

(H1) $J : \Omega \times X^\alpha \to X$ satisfies
  (i) for all $x \in X^\alpha$, $t \to J(t, x(t))$ is measurable;

(ii) for all $x_1, x_2 \in X^\alpha$ satisfying $\|x_1\|_\alpha, \|x_2\|_\alpha \le \sigma$, there exists a constant $L_J(\sigma) > 0$ such that

$$\left\|J(t,x_1) - J(t,x_2)\right\| \le L_J(\sigma)\|x_1 - x_2\|_\alpha, \quad \forall t \in \Omega.$$

In particular,

$$\left\|J(t,x)\right\| \le C_J\left(1 + \|x\|_\alpha\right), \quad \forall x \in X^\alpha, t \in \Omega,$$

where C_J is a positive constant.

(H2) $g : C(\Omega, X) \to X$, and for all $x_1, x_2 \in C(\Omega, X)$ satisfying $\|x_1\|, \|x_2\| \le \sigma$, there exists a constant $L_g(\sigma) > 0$ such that

$$\left\|g(x_1) - g(x_2)\right\| \le L_g(\sigma)\|x_1 - x_2\|.$$

In particular,

$$\left\|g(x)\right\| \le C_{g1}\|x\| + C_{g2}, \quad \forall x \in C(\Omega, X),$$

where C_{g1} and C_{g2} are positive constants.

Suppose that U is another separable reflexive Banach space from which the controls u take the values. We denote the class of nonempty closed convex subsets of U by $W_f(U)$. The multifunction $\omega : \Omega \to W_f(U)$ is measurable, and $\omega(\cdot) \subset F$ where F is a bounded set of U, and the admissible control set $U_{ad} = S_\omega^p = \{u \in L^p(F) | u(t) \in \omega(t) \text{ a.e.}\}$, $1 < p < \infty$.

(H3) The operator $K \in L_\infty(\Omega, \mathfrak{L}(U, X^\alpha))$, and $\|K\|_\infty$ stands for the norm of operator K on the Banach space $L_\infty(\Omega, \mathfrak{L}(U, X^\alpha))$.

Obviously, $Ku \in L^p(\Omega, X^\alpha)$ for all $u \in U_{ad}$.

By the method of [2] we can find that equation (1.1) has the representation

$$\begin{aligned} Ex(t) &= E\left(x_0 + g(x)\right) + Ex_1 t \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}\left[Gx(\theta) + J\left(s, x(s)\right) + K(s)u(s)\right] ds, \quad t \in \Omega, \end{aligned} \tag{3.1}$$

provided that the right-hand side of the above equation holds.

We will use the probability density function $\vartheta_z(\theta)$ defined on $(0, \infty)$ as

$$\begin{aligned} \vartheta_z(\theta) &= \frac{1}{z\theta^{(1+1/z)}} \varpi_z\left(\theta^{-1/z}\right) \ge 0, \quad z \in (0,1), \\ \varpi_z(\theta) &= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} (\theta)^{-zn-1} \frac{\Gamma(nz+1)}{n!} \sin(n\pi z). \end{aligned} \tag{3.2}$$

Lemma 3.1 ([34]) *If formula (3.1) holds, then for $t \in \Omega$ and $z = \gamma/2$,*

$$\begin{aligned} x(t) &= \mathcal{S}_E(t)E\left(x_0 + g(x)\right) + \chi_E(t)Ex_1 + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\left(s, x(s)\right) ds \\ &\quad + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u(s)\, ds, \end{aligned} \tag{3.3}$$

where

$$\mathcal{S}_E(t) = \int_0^\infty E^{-1}\vartheta_z(s)\mathcal{C}(t^z s)\,ds, \qquad \chi_E(t) = \int_0^t \mathcal{S}_E(\xi)\,d\xi,$$
$$\mathcal{P}_E(t) = z\int_0^\infty E^{-1}s\vartheta_z(s)\mathcal{S}(t^z s)\,ds.$$

Proof For $\mu > 0$, applying the Laplace transform to Eq. (3.1), we get

$$El(\mu) = \frac{1}{\mu}E\big(x_0 + g(x)\big) + \frac{1}{\mu^2}Ex_1 + \frac{1}{\mu^\gamma}Gl(\mu) + \frac{1}{\mu^\gamma}m(\mu) + \frac{1}{\mu^\gamma}n(\mu), \tag{3.4}$$

where

$$l(\mu) = \int_0^\infty e^{-\mu s}x(s)\,ds, \qquad m(\mu) = \int_0^\infty e^{-\mu s}J\big(s, x(s)\big)\,ds,$$
$$n(\mu) = \int_0^\infty e^{-\mu s}K(s)u(s)\,ds,$$

which implies

$$E\big(\mu^\gamma I - GE^{-1}\big)l(\mu) = \mu^{\gamma-1}E\big(x_0 + g(x)\big) + \mu^{\gamma-2}Ex_1 + m(\mu) + n(\mu). \tag{3.5}$$

Therefore, by the relationship between the resolvent and cosine function, that is, for $Re(\mu) > 0$,

$$\mu R\big(\mu^2; G\big)x = \int_0^\infty e^{-\mu s}\mathcal{C}(s)x\,ds, \qquad R\big(\mu^2; A\big)x = \int_0^\infty e^{-\mu s}\mathcal{S}(s)x\,ds \quad \text{for } x \in U,$$

we first have

$$\begin{aligned} El(\mu) &= \mu^{\gamma-1}\big(\mu^\gamma I - GE^{-1}\big)^{-1}E\big(x_0 + g(x)\big) + \mu^{\gamma-2}\big(\mu^\gamma I - GE^{-1}\big)^{-1}Ex_1 \\ &\quad + \big(\mu^\gamma I - GE^{-1}\big)^{-1}m(\mu) + \big(\mu^\gamma I - GE^{-1}\big)^{-1}n(\mu) \\ &= \mu^{\frac{\gamma}{2}-1}\int_0^\infty E^{-1}e^{-\mu^{\frac{\gamma}{2}}s}\mathcal{C}(s)E\big(x_0 + g(x)\big)\,ds + \mu^{-1}\mu^{\frac{\gamma}{2}-1}\int_0^\infty E^{-1}e^{-\mu^{\frac{\gamma}{2}}s}\mathcal{C}(s)Ex_1\,ds \\ &\quad + \int_0^\infty E^{-1}e^{-\mu^{\frac{\gamma}{2}}s}\mathcal{S}(s)m(\mu)\,ds + \int_0^\infty E^{-1}e^{-\mu^{\frac{\gamma}{2}}s}\mathcal{S}(s)n(\mu)\,ds. \end{aligned} \tag{3.6}$$

With $z = \gamma/2 \in (1/2, 1)$, we have

$$\begin{aligned} l(\mu) &= \mu^{z-1}\int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{C}(s)E\big(x_0 + g(x)\big)\,ds + \mu^{-1}\mu^{z-1}\int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{C}(s)x_1\,ds \\ &\quad + \int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{S}(s)m(\mu)\,ds + \int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{S}(s)n(\mu)\,ds. \end{aligned} \tag{3.7}$$

Considering the Laplace transform for the one-sided probability density function Eq. (3.2),

$$\int_0^\infty e^{-\mu s}\varpi_z(s)\,ds = e^{-\mu^z}, \quad z \in (0, 1), \tag{3.8}$$

and combining Eqs. (3.7) and (3.8), we obtain

$$\begin{aligned}
&\mu^{z-1}\int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{C}(s)E\big(x_0+g(x)\big)\,ds \\
&= \int_0^\infty \mu^{z-1}E^{-1}e^{-(\mu\eta)^z}\mathcal{C}\big(\eta^z\big)z\eta^{z-1}E\big(x_0+g(x)\big)\,d\eta \\
&= \int_0^\infty z(\mu\eta)^{z-1}E^{-1}e^{-(\mu\eta)^z}\mathcal{C}\big(\eta^z\big)E\big(x_0+g(x)\big)\,d\eta \\
&= \int_0^\infty \frac{-1}{\mu}\frac{d}{d\eta}\big(e^{-(\mu\eta)^z}\big)E^{-1}\mathcal{C}\big(\eta^z\big)E\big(x_0+g(x)\big)\,d\eta \\
&= \int_0^\infty\int_0^\infty \frac{-1}{\mu}\frac{d}{d\eta}\big(e^{-\mu\eta\xi}\varpi_z(\xi)\big)E^{-1}\mathcal{C}\big(\eta^z\big)E\big(x_0+g(x)\big)\,d\xi\,d\eta \\
&= \int_0^\infty e^{-\mu\eta}\int_0^\infty E^{-1}\varpi_z(\xi)\mathcal{C}\bigg(\frac{\eta^z}{\xi^z}\bigg)E\big(x_0+g(x)\big)\,d\xi\,d\eta \qquad (3.9)\\
&= \int_0^\infty e^{-\mu\eta}\int_0^\infty E^{-1}\frac{1}{z\xi^{(1+1/z)}}\varpi_z\big(\xi^{-1/z}\big)\mathcal{C}\big(\eta^z\xi\big)E\big(x_0+g(x)\big)\,d\xi\,d\eta \\
&= \int_0^\infty e^{-\mu\eta}\int_0^\infty E^{-1}\vartheta_z(\xi)\mathcal{C}\big(\eta^z\xi\big)E\big(x_0+g(x)\big)\,d\xi\,d\eta \\
&= \int_0^\infty e^{-\mu\eta}\big[\mathcal{S}_E(\eta)E\big(x_0+g(x)\big)\big]\,d\eta \\
&= \mathbb{L}\big[\mathcal{S}_E(\eta)E\big(x_0+g(x)\big)\big](\mu),
\end{aligned}$$

where $\mathbb{L}$ denotes the Laplace transform. Additionally, denoting $\mathcal{L}[g_1(t)](\mu)=\mu^{-1}$, we have

$$\begin{aligned}
\mu^{-1}\mu^{z-1}\int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{C}(s)Ex_1\,ds &= \mathbb{L}\big[g_1(x)\big](\mu)\cdot\mathbb{L}\big[\mathcal{S}_E(t)Ex_1\big](\mu) \qquad (3.10)\\
&= \mathbb{L}\big[(g_1*\mathcal{S}_E)(t)Ex_1\big](\mu).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^\infty E^{-1}e^{-\mu^z s}\mathcal{S}(s)m(\mu)\,ds \\
&= \int_0^\infty e^{-\mu\eta}\bigg[z\int_0^\eta\int_0^\infty E^{-1}\varpi_z(\xi)\mathcal{S}\bigg(\frac{(\eta-s)^z}{\xi^z}\bigg)J\big(s,x(s)\big)\frac{(\eta-s)^{z-1}}{\xi^z}\,d\xi\,ds\bigg]\,d\eta \\
&= \mathbb{L}\bigg[z\int_0^\eta(\eta-s)^{z-1}\int_0^\infty E^{-1}\varpi_z(\xi)\mathcal{S}\bigg(\frac{(\eta-s)^z}{\xi^z}\bigg)J\big(s,x(s)\big)\frac{1}{\xi^z}\,d\xi\,ds\bigg](\mu) \qquad (3.11)\\
&= \mathbb{L}\bigg[\int_0^\eta(\eta-s)^{z-1}\mathcal{P}_E(\eta-s)J\big(s,x(s)\big)\,ds\bigg](\mu),
\end{aligned}$$

and

$$\int_0^\infty E^{-1}e^{-\mu^z y}\mathcal{S}(y)n(\mu)\,dy = \mathcal{L}\bigg[\int_0^t(t-y)^{z-1}\mathcal{P}_E(t-y)K(y)\nu(y)\,dy\bigg](\mu). \qquad (3.12)$$

Combining (3.9), (3.10), (3.11), and (3.12), we complete the proof. □

Definition 3.2 ([22]) For any $u \in L^p(\Omega, U)$ $(1 \le p \le \infty)$, system (1.1) is said to be mildly solvable with respect to (w.r.t.) u on $[0, S]$ if there exists $x \in C(\Omega, X^\alpha)$ such that for $t \in \Omega$,

$$\begin{aligned} x(t) = \mathcal{S}_E(t)E\bigl(x_0 + g(x)\bigr) + \chi_E(t)Ex_1 + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\bigl(s, x(s)\bigr)\,ds \\ + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u(s)\,ds. \end{aligned} \tag{3.13}$$

Lemma 3.3 ([22]) *The operators $\mathcal{S}_E(\cdot)$, $\chi_E(\cdot)$, and $\mathcal{P}_E(\cdot)$ have the following properties*:

(i) *For any fixed $t \ge 0$, the operators $\mathcal{S}_E(t)$, $\chi_E(t)$ and $\mathcal{P}_E(t)$ are linear and bounded, that is, for any $x \in X$,*

$$\begin{aligned} &\bigl\|\mathcal{S}_E(t)x\bigr\| \le M\bigl\|E^{-1}\bigr\|\|x\|, \qquad \bigl\|\chi_E(t)x\bigr\| \le Mt\bigl\|E^{-1}\bigr\|\|x\|, \\ &\bigl\|\mathcal{P}_E(t)x\bigr\| \le \frac{M\|E^{-1}\|}{\Gamma(2z)}t^z\|x\|; \end{aligned}$$

(ii) *The operators $\{\mathcal{S}_E(t)\}_{t\ge0}$, $\{\chi_E(t)\}_{t\ge0}$, and $\{\mathcal{P}_E(t)\}_{t\ge0}$ are strongly continuous*;

(iii) *For every $t > 0$, the operators $\mathcal{S}_E(t)$, $\chi_E(t)$, and $\mathcal{P}_E(t)$ are compact.*

Proof (i) For any fixed $t \ge 0$, the operators $\mathcal{S}_E(t)$, $\chi_E(t)$, and $\mathcal{P}_E(t)$ are also linear because E^{-1} and $C(t)$ are linear. By [34], for any fixed $t \ge 0$ and any $x \in X$,

$$\bigl\|\mathcal{S}_E(t)x\bigr\| \le \int_0^\infty \vartheta_z(\xi)\bigl\|E^{-1}\bigr\|\bigl\|\mathcal{C}\bigl(t^z\xi\bigr)x\bigr\|\,d\xi \le M\bigl\|E^{-1}\bigr\|\|x\|, \tag{3.14}$$

which yields

$$\bigl\|\chi_E(t)x\bigr\| \le Mt\bigl\|E^{-1}\bigr\|\|w\| \tag{3.15}$$

and

$$\begin{aligned} \bigl\|\mathcal{P}_E(t)x\bigr\| &\le \int_0^\infty z\xi\vartheta_z(\xi)\bigl\|E^{-1}\bigr\|\bigl\|\mathcal{S}\bigl(t^z\xi\bigr)w\bigr\|\,d\xi \le \int_0^\infty z\xi\vartheta_z(\xi)\int_0^{t^z\xi}\bigl\|\mathcal{C}(\eta)x\bigr\|\,d\eta\,d\xi \\ &\le Mz\bigl\|E^{-1}\bigr\|\|x\|t^z\int_0^\infty \xi^2\vartheta_z(\xi)\,d\xi = \frac{M\|E^{-1}\|}{\Gamma(2z)}\|x\|t^z. \end{aligned} \tag{3.16}$$

(ii) From Definition 2.5 (iii) we know that $\{C(t)\}_{t\in\mathbb{R}}$ is strongly continuous for every $x \in U$, that is, for any $\varepsilon > 0$ and $t_1, t_2 \in \mathbb{R}$, there exists $\delta > 0$ such that $\|\mathcal{C}(t_2)x - \mathcal{C}(t_1)x\| < \varepsilon$ if $|t_2 - t_1| < \delta$. As for the operators $\{\mathcal{S}_E(t)\}_{t\ge0}$, $\{\chi_E(t)\}_{t\ge0}$, and $\{\mathcal{P}_E(t)\}_{t\ge0}$, we check them as follows. For any $t_1, t_2 \ge 0$ such that $|t_2 - t_1| < \delta$,

$$\bigl\|\mathcal{S}_E(t_2)x - \mathcal{S}_E(t_1)x\bigr\| \le \int_0^\infty \bigl\|E^{-1}\bigr\|\vartheta_z(\xi)\bigl\|\bigl(\mathcal{C}\bigl(t_2^z\xi\bigr) - \mathcal{C}\bigl(t_1^z\xi\bigr)\bigr)x\bigr\|\,d\xi < \varepsilon, \tag{3.17}$$

which yields $\|\mathcal{S}_E(t_2)x - \mathcal{S}_E(t_1)x\| \to 0$ as $\delta \to 0$. Simultaneously,

$$\bigl\|\chi_E(t_2)x - \chi_E(t_1)x\bigr\| = \int_{t_1}^{t_2}\bigl\|\mathcal{S}_E(\xi)x\bigr\|\,d\xi \le M\|x\|\bigl\|E^{-1}\bigr\||t_2 - t_1| \to 0 \quad \text{as } \delta \to 0, \tag{3.18}$$

and also

$$\begin{aligned}\|\mathcal{P}_E(t_2)x-\mathcal{P}_E(t_1)x\| &\le \int_0^\infty q\xi\,\|E^{-1}\|\vartheta_z(\xi)\|\big(\mathcal{S}(t_2^z\xi)-\mathcal{S}(t_1^z\xi)\big)x\|\,d\xi\\ &\le \frac{M}{\Gamma(2z)}\|x\|\,\|E^{-1}\|\,|t_2^z-t_1^z|\to 0 \quad \text{as } \delta\to 0.\end{aligned} \tag{3.19}$$

(iii) Referring to paper [22], $\mathcal{S}_E(t)$ and $\mathcal{P}_E(t)$ are obviously compact. Now we check the compactness of the operator $\chi_E(t)$. For each positive constant k, set $\{x\in U:\|x\|\le k\}$. Then U_k is clearly a bounded subset in U. We prove that $U(t):=\{\int_0^t\int_0^\infty E^{-1}\vartheta_z(\xi)\mathcal{C}(s^z\xi)x\,d\xi\,ds, x\in U_k\}$ is relatively compact in U for any positive constant k and $t\ge 0$. From (i) we know that $\chi_1(t):U\to U$ is also linear and bounded mapping U_k into a bounded subset of U. Then $U(t)=E^{-1}\chi_1(t)(U_k)$ is relatively compact in U for any $k>0$ and $t\ge 0$ due to the compactness of $E^{-1}:U\to X$. This completes the proof. □

Lemma 3.4 *Assume that system* (1.1) *is mildly solvable on* $\Omega:=[0,S]$ *w.r.t. u. Then there exists a constant* $M^*=M^*(u)>0$ *such that*

$$\|x(t)\|_\alpha\le M^*,\quad \forall t\in\Omega. \tag{3.20}$$

Proof If x is a mild solution of system (1.1) w.r.t. u on Ω, then x satisfies Eq. (3.13). Combining (H1), (H2), Lemma 3.3, and Hölder's inequality, we have

$$\begin{aligned}\|x(t)\|_\alpha &\le \|\mathcal{S}_E(t)E(x_0+g(x))\|+\|\chi_E(t)Ex_1\|+\int_0^t(t-s)^{z-1}\|\mathcal{P}_E(t-s)J(s,w(s))\|\,ds\\ &\quad+\int_0^t(t-s)^{z-1}\|\mathcal{P}_E(t-s)K(s)u(s)\|\,ds\\ &\le M\|x_0\|_\alpha+MC_{g1}\|x\|_\alpha+MC_{g2}+MS\|x_1\|_\alpha+\frac{C_JM\|E^{-1}\|S^{2z}}{\Gamma(2z+1)}\\ &\quad+\frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}S^{2z-\frac{1}{p}}\|u\|_U+\frac{C_JM\|E^{-1}\|}{\Gamma(2z)}\int_0^t(t-s)^{2z-1}\|x(s)\|_\alpha\,ds.\end{aligned} \tag{3.21}$$

Denoting

$$\begin{aligned}c&=M\|x_0\|_\alpha+MC_{g1}\|x\|_\alpha+MC_{g2}+MS\|x_1\|_\alpha\\ &\quad+\frac{C_JM\|E^{-1}\|S^{2z}}{\Gamma(2z+1)}+\frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}S^{2z-\frac{1}{p}}\|u\|_U,\end{aligned}$$

according to Lemma 2.3, there exists a positive constant M^* such that $\|x(t)\|_\alpha\le M^*(c)$ for $t\in\Omega$. Let $M^*=M^*(c)>0$. Then $\|x(t)\|_\alpha\le M^*$ for $t\in\Omega$. □

Theorem 3.5 *Assume that* (H1)–(H3) *hold. Then for each* $u\in P_{ad}$, *system* (1.1) *is mildly solvable on* $\Omega:=[0,S]$ *w.r.t. u, and the mild solution is unique.*

Proof Let $S_1\le S$, $\Omega_1:=[0,S_1]$, and $C_{0,S_1}:=C(\Omega_1,X^\alpha)$ equipped with the usual sup-norm, and let

$$B(1,S_1)=\Big\{h\in C_{0,S_1}:\max_{t\in[0,S_1]}\|h(t)-(x_0+g(x))-tx_1\|_\alpha\le 1\Big\}. \tag{3.22}$$

By definition $B(1,S_1) \subseteq C_{0,S_1}$ is a closed convex subset of C_{0,S_1}. According to (H1), we can deduce that $J(t,h(t))$ is a measurable function on $\Omega_1 := [0,S_1]$. For any $h \in B(1,S_1)$, there exists a constant $\lambda := \|x_0 + g(x)\|_\alpha + \|x_1\|_\alpha S_1 + 1$ satisfying the inequality

$$\|h\|_{0,S_1} \leq \lambda. \tag{3.23}$$

Applying (H1), for $t \in \Omega_1$, we can deduce

$$\big\|J\big(t,h(t)\big)\big\|_\alpha \leq C_J\big(1 + \big\|h(t)\big\|_\alpha\big) \leq C_J(1+\lambda). \tag{3.24}$$

Combining Lemma 3.3, Hölder's inequality, and (H1), we can obtain

$$\int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)J\big(s,h(s)\big)\big\|_\alpha\,ds \leq \frac{M\|E^{-1}\|C_J(1+\lambda)}{\Gamma(2z+1)}{S_1}^{2z}. \tag{3.25}$$

Thus $(t-s)^{z-1}\mathcal{P}_E(t-s)J(s,h(s))$ is Bochner integrable w.r.t. $s \in [0,t]$ for all $t \in \Omega_1$.

In addition, applying Lemma 3.3 and Hölder's inequality, we can deduce the following inequality:

$$\begin{aligned}
&\int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)K(s)u(s)\big\|_\alpha\,ds \\
&\quad\leq \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\big\|u(s)\big\|_U\,ds \\
&\quad\leq \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}\bigg(\int_0^t (t-s)^{\frac{p}{p-1}(2z-1)}\,ds\bigg)^{\frac{p-1}{p}}\bigg(\int_0^t \big\|u(s)\big\|_U^p\,ds\bigg)^{\frac{1}{p}} \\
&\quad\leq \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}S_1^{2z-\frac{1}{p}}\|u\|_U.
\end{aligned} \tag{3.26}$$

Also, $(t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u(s)$ is also Bochner integrable.

For $t \in \Omega_1$, let us define $\mathfrak{F} : B(1,S_1) \to C_{0,S_1}$ by

$$\begin{aligned}
(\mathfrak{F}h)(t) ={}& \mathcal{S}_E(t)E\big(x_0 + g(x)\big) + \chi_E(t)Ex_1 + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\big(s,h(s)\big)\,ds \\
&+ \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u(s)\,ds.
\end{aligned} \tag{3.27}$$

With the properties of the operators $\mathcal{S}_E(\cdot)$, $\chi_E(\cdot)$, $\mathcal{P}_E(\cdot)$ and the hypotheses (H1) and (H2), it is not difficult to verify that $\mathfrak{F}$ is a contraction map on $B(1,S_1)$ with suitably chosen $S_1 > 0$. In fact, for $t \in \Omega_1$, it comes from the inequalities

$$\begin{aligned}
&\big\|(\mathfrak{F}h)(t) - \big(x_0 + g(x)\big) - tx_1\big\|_\alpha \\
&\quad\leq \big\|\mathcal{S}_E(t)E\big(x_0 + g(x)\big) - \big(x_0 + g(x)\big)\big\|_\alpha + \big\|\chi_E(t)Ex_1 - tx_1\big\|_\alpha \\
&\qquad+ \int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)J\big(s,h(s)\big)\big\|_\alpha\,ds + \int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)K(s)u(s)\big\|_\alpha\,ds \\
&\quad\leq \big\|\mathcal{S}_E(t)E\big(x_0 + g(x)\big) - \big(x_0 + g(x)\big)\big\|_\alpha + \big\|\chi_E(t)Ex_1 - tx_1\big\|_\alpha \\
&\qquad+ \frac{M\|E^{-1}\|C_J(1+\lambda)}{\Gamma(2z+1)}t^{2z} + \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}\|u\|_U t^{2z-\frac{1}{p}}.
\end{aligned} \tag{3.28}$$

Because the operators $\{\mathcal{S}_E(t)\}_{t\geq 0}$ and $\{\chi_E(t)\}_{t\geq 0}$ are strongly continuous in X, we can choose S_1 small enough and $\varepsilon = \frac{1}{3}$ such that

$$\bigl\| \mathcal{S}_E(t)E\bigl(x_0 + g(x)\bigr) - \bigl(x_0 + g(x)\bigr) \bigr\| \leq \frac{1}{3} \quad \text{and} \quad \bigl\| \chi_E(t)Ex_1 - tx_1 \bigr\| \leq \frac{1}{3}. \tag{3.29}$$

Denote

$$S_{11} = \min\left\{ \frac{1}{3}, \left(\frac{\Gamma(2z+1)}{3M\|E^{-1}\|(C_J(1+\lambda)S_1^{\frac{1}{p}} + 2z\|K\|_\infty \|u\|_U)} \right)^{\frac{p}{2zp-1}} \right\}. \tag{3.30}$$

Furthermore, combining Eqs. (3.28) and (3.29), for all $t \in [0, S_{11}]$, we have

$$\bigl\| (\mathfrak{F}h)(t) - \bigl(x_0 + g(x)\bigr) - tx_1 \bigr\| \leq 1. \tag{3.31}$$

Therefore

$$B\bigl(B(1, S_1)\bigr) \subseteq B(1, S_1). \tag{3.32}$$

For any $h_1, h_2 \in B(1, S_1)$ with $\|h_1\|_{\Omega_1}^{\alpha}, \|h_2\|_{\Omega_1}^{\alpha} \leq \lambda$, applying Lemma 3.3 and (H1), we can obtain, for $t \in \Omega_1 := [0, S_1]$,

$$\begin{aligned} \bigl\| (\mathfrak{F}h_1)(t) - (\mathfrak{F}h_2)(t) \bigr\|_\alpha &\leq \int_0^t (t-s)^{z-1} \bigl\| \mathcal{P}_E(t-s)\bigl(J\bigl(s, h_1(s)\bigr) - J\bigl(s, h_2(s)\bigr)\bigr) \bigr\|_\alpha \, ds & (3.33) \\ &\leq \frac{ML_J(\lambda)\|E^{-1}\|}{\Gamma(2z)} \int_0^t (t-s)^{2z-1} \bigl\| h_1(s) - h_2(s) \bigr\|_\alpha \, ds, & (3.34) \end{aligned}$$

which yields

$$\bigl\| (\mathfrak{F}h_1)(t) - (\mathfrak{F}h_2)(t) \bigr\| \leq \frac{ML_J(\lambda)\|E^{-1}\|}{\Gamma(2z+1)} t^{2z} \|h_1 - h_2\|_{\Omega_1}^{\alpha}. \tag{3.35}$$

Moreover,

$$\|\mathfrak{F}h_1 - \mathfrak{F}h_2\|_{\Omega_1}^{\alpha} \leq \frac{ML_J(\lambda)\|E^{-1}\|}{\Gamma(2z+1)} S_1^{2z} \|h_1 - h_2\|_{\Omega_1}^{\alpha}. \tag{3.36}$$

Denote $S_{12} = \frac{1}{2}(\frac{\Gamma(2z+1)}{ML_J(\lambda)\|E^{-1}\|})^{\frac{1}{z}}$ and $S_1 = \min\{S_{11}, S_{12}\}$. Thus $\mathfrak{F}$ is a contraction map on $B(1, S_1)$. According to the contraction mapping principle, $\mathfrak{F}$ has a unique fixed point $h \in B(1, S_1)$. Consequently, on the interval Ω_1, h is a mild solution of system (1.1) w.r.t. u, and it is unique. Finally, denoting

$$S_{21} = S_1 + S_{11}, \qquad S_{22} = S_1 + S_{12}, \qquad \Delta S = \min\{S_{21} - S_1, S_{12}\} > 0 \tag{3.37}$$

and applying the method of above arguments, we can verify that system (1.1) has a unique mild solution on the interval $[0, \Delta S]$. For every interval $[\Delta S, 2\Delta S]$, $[2\Delta S, 3\Delta S]$, …, repeating the above procedures, we immediately obtain the unique mild solution for system (1.1). □

Remark 3.6 Let X and U be two separable reflexive Banach spaces. If we replace (H1) by the condition that $J : \Omega \times X \to X$ is Hölder's continuous w.r.t. t, that is, for any $\sigma > 0$, there exists a constant $L_J(\sigma) > 0$ such that

$$\|J(t,\zeta_1) - J(y,\zeta_2)\| \le L_J(\sigma)\big(|t-y|^\beta + |\zeta_1 - \zeta_2|\big), \tag{3.38}$$

where $\beta \in (0,1]$, provided that $|\zeta_1|, |\zeta_2| \le \sigma$; condition (H2) by

$$J \in \mathcal{L}\big(L^p(\Omega,U), L^p(\Omega,X)\big); \tag{3.39}$$

and condition (H3) by

$$U_{ad} = L^p(\Omega,U), \tag{3.40}$$

then we can use the same approach to derive the existence of mild solutions.

4 Continuous dependence

In this section, we study continuous dependence for the mild solution of system (1.1).

Theorem 4.1 *Assume that $x_0^1, x_0^2 \in \aleph$ where $\aleph$ is a bounded set. Let*

$$\begin{aligned} &x^1\big(t, x_0^1 + g(x^1), x_1^1, u\big) \\ &\quad = \mathcal{S}_E(t)E\big(x_0^1 + g(x^1)\big) + \chi_E(t)Ex_1^1 \\ &\qquad + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\big(s, w^1(s)\big)\,ds \\ &\qquad + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u(s)\,ds, \quad t \in \Omega, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &x^2\big(t, x_0^2 + g(x^2), x_1^2, v\big) \\ &\quad = \mathcal{S}_E(t)E\big(x_0^2 + g(x^2)\big) + \chi_E(t)Ex_1^2 \\ &\qquad + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\big(s, w^2(s)\big)\,ds \\ &\qquad + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)v(s)\,ds, \quad t \in \Omega. \end{aligned} \tag{4.2}$$

Then there exists a constant M^{**} such that

$$\begin{aligned} &\big\|x^1\big(t, x_0^1 + g(x^1), x_1^1, u\big) - x^2\big(t, x_0^2 + g(x^2), x_1^2, v\big)\big\| \\ &\quad \le M^{**}\big(\|x_0^1 - x_0^2\| + \|x^1 - x^2\| + \|x_1^1 - x_1^2\| + \|u - v\|_U\big), \quad t \in \Omega, \end{aligned} \tag{4.3}$$

where $M^{**} = \max\{MMS, MS, M, ML_g(\sigma), \frac{M\|K\|_\infty\|E^{-1}\|}{\Gamma(2z)}S^{2z-\frac{1}{p}}\} > 0$.

Proof Since $x_0^1, x_0^2 \in \aleph$, where $\aleph$ is a bounded set in X, by Lemma 3.4 there exists a constant $\sigma > 0$ such that $|x^1|, |x^2| \le \sigma$. By using Lemma 3.3, hypotheses (H1), (H2), and Hölder's inequality, for $t \in \Omega$, we have the following inequalities:

$$\begin{aligned}
&\left\|x^1\left(t, x_0^1 + g\left(x^1\right), x_1^1, u\right) - x^2\left(t, x_0^2 + g\left(x^2\right), x_1^2, v\right)\right\|_\alpha \\
&\quad \le \left\|\mathcal{S}_E(t)E\left(\left(x_0^1 + g\left(x^1\right)\right) - \left(x_0^2 + g\left(x^2\right)\right)\right)\right\|_\alpha \\
&\qquad + \left\|\chi_E(t)E\left(x_1^1 - x_1^2\right)\right\|_\alpha + \int_0^t (t-s)^{z-1}\left\|\mathcal{P}_E(t-s)\left(J\left(s, x^1(s)\right) - J\left(s, x^2(s)\right)\right)\right\|_\alpha ds \\
&\qquad + \int_0^t (t-s)^{z-1}\left\|\mathcal{P}_E(t-s)\left(K(s)u(s) - K(s)v(s)\right)\right\|_\alpha ds \qquad (4.4) \\
&\quad \le M\left\|E^{-1}\right\|\|E\|\left\|x_0^1 - x_0^2\right\| + M\left\|E^{-1}\right\|\|E\|\left\|g\left(x^1\right) - g\left(x^2\right)\right\| + MS\left\|E^{-1}\right\|\|E\|\left\|x_1^1 - x_1^2\right\| \\
&\qquad + \frac{L_J(\sigma)M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\left\|x^1(s) - x^2(s)\right\| ds \\
&\qquad + \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\left\|u(s) - v(s)\right\|_U ds \qquad (4.5) \\
&\quad \le M\left\|x_0^1 - x_0^2\right\| + ML_g(\sigma)\left\|x^1 - x^2\right\| + MS\left\|x_1^1 - x_1^2\right\| \\
&\qquad + \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)} t^{2z-\frac{1}{p}}\left(\int_0^t \left\|u(s) - v(s)\right\|_U^p ds\right)^{\frac{1}{p}} \\
&\qquad + \frac{L_J(\sigma)M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\left\|x^1(s) - x^2(s)\right\| ds \qquad (4.6) \\
&\quad \le M\left|x_0^1 - x_0^2\right| + ML_g(\sigma)\left|x^1 - x^2\right|_C + MS\left\|x_1^1 - x_1^2\right\| + \frac{\|K\|_\infty M\|E^{-1}\|}{\Gamma(2z)} S^{2z-\frac{1}{p}}\|u - v\|_U \\
&\qquad + \frac{L_J(\sigma)M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\left\|x^1(s) - x^2(s)\right\| ds. \qquad (4.7)
\end{aligned}$$

Using Lemma 2.3 again, we obtain

$$\begin{aligned}
&\left\|x^1\left(t, x_0^1 + g\left(x^1\right), x_1^1, u\right) - x^2\left(t, x_0^2 + g\left(x^2\right), x_1^2, v\right)\right\| \\
&\quad \le M^{**}\left(\left\|x_0^1 - x_0^2\right\| + \left\|x^1 - x^2\right\| + \left\|x_1^1 - x_1^2\right\| + \|u - v\|_U\right), \quad t \in \Omega. \qquad (4.8)
\end{aligned}$$

This completes the proof. □

5 Optimal control

In this section, we study the existence of optimal pairs for fractional control system (1.1). Firstly, we consider the following Lagrange problem:

(P) Find a control $u^\circ \in U_{ad}$ such that

$$\mathcal{J}\left(u^\circ\right) \le \mathcal{J}(u), \quad \forall u \in U_{ad}, \qquad (5.1)$$

where

$$\mathcal{J}(u) = \int_0^S \mathcal{I}\left(t, x^u(t), u(t)\right) dt, \qquad (5.2)$$

and x^u denotes the mild solution of system (1.1) corresponding to the control $u \in U_{ad}$.

For the existence of solution for problem (P), we introduce the following hypotheses:

(H4) (i) the functional $\mathcal{I} : \Omega \times X \times U \to R \cup \{\infty\}$ is Borel measurable;
(ii) $\mathcal{I}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times U$ for almost all $t \in \Omega$;
(iii) $\mathcal{I}(t, x, \cdot)$ is convex on ζ for each $x \in X^\alpha$ and almost all $t \in \Omega$;
(iv) there exist constants $e \geq 0$, $j > 0$, and nonnegative $\phi \in L^1(\Omega, \mathbb{R})$ such that

$$\mathcal{I}(t, x, u) \geq \phi(t) + e\|x\|_\Omega^\alpha + j\|u\|_U^p.$$

Now we can give the following result on the existence of fractional optimal controls for problem (P).

Theorem 5.1 *Under the assumptions in Theorem 3.5 and* (H4), *suppose that K is a strongly continuous operator. Then the optimal control problem (P) admits at least one optimal pair, that is, there exists an admissible control $u^\circ \in U_{ad}$ such that*

$$\mathcal{J}(u^\circ) = \int_0^S \mathcal{I}\big(t, x^\circ(t), u^\circ(t)\big)\, dt \leq \mathcal{J}(u) \quad \textit{for } u \in U_{ad}. \tag{5.3}$$

Proof If $\inf\{\mathcal{J}(u) : u \in U_{ad}\} = +\infty$, then there is nothing to prove. Assume that

$$\inf\big\{\mathcal{J}(u) : u \in U_{ad}\big\} = \varepsilon < +\infty. \tag{5.4}$$

Using assumption (H4), we have $\varepsilon > -\infty$. By the definition of infimum there exists a minimizing sequence of feasible pairs

$$\begin{aligned} &\big\{(x^r, u^r)\big\} \\ &\quad \subset A_{ad} \\ &\quad := \big\{(x, u) \mid x \text{ is a mild solution of system (1.1) corresponding to } u \in U_{ad}\big\}, \end{aligned} \tag{5.5}$$

such that $\mathcal{J}(x^r, u^r) \to \varepsilon$ as $r \to +\infty$. Since $\{u^r\} \subseteq U_{ad}$, $r = 1, 2, \ldots, \{u^r\}$ is bounded in $L^p(\Omega, U)$, and there exists a subsequence such that

$$u^r \to u^\circ \quad \text{in } L^p(\Omega, U). \tag{5.6}$$

Since U_{ad} is closed and convex, by the Marzur lemma we have $u^\circ \in U_{ad}$.

Let $x^r \in C_{0,S}$ denote the corresponding sequence of solutions of the integral equations

$$\begin{aligned} x^r(t) = {}& \mathcal{S}_E(t)E\big(x_0 + g(x^r)\big) + \chi_E(t)Ex_1 + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\big(s, x^r(s)\big)\, ds \\ &+ \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u^r(s)\, ds, \quad t \in \Omega. \end{aligned} \tag{5.7}$$

By Lemmas 2.3 and 3.4 we can verify that there exists $\sigma > 0$ such that

$$\|x^r\|_\Omega^\alpha \leq \sigma \quad \text{for } r = 0, 1, 2, \ldots.$$

Suppose x^r (x°) is the mild solution corresponding to u^r (u°), and x^r and (x°) satisfy the following integral equation:

$$x^\circ(t) = \mathcal{S}_E(t)E\big(x_0 + g\big(x^\circ\big)\big) + \chi_E(t)Ex_1 + \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)J\big(s, x^\circ(s)\big)\,ds$$
$$+ \int_0^t (t-s)^{z-1}\mathcal{P}_E(t-s)K(s)u^\circ(s)\,ds, \quad t \in \Omega. \tag{5.8}$$

For $t \in \Omega$, applying hypothesis (H1), Lemma 3.3, and Hölder's inequality, we obtain

$$\begin{aligned}
&\big\|x^r(t) - x^\circ(t)\big\|_\alpha \\
&\quad\leq \big\|\mathcal{S}_E(t)E(\big(g\big(x^r\big) - g\big(x^\circ\big)\big)\big\|_\alpha \\
&\qquad + \int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)\big(J\big(s, x^r(s)\big) - J\big(s, x^\circ(s)\big)\big)\big\|_\alpha\,ds \\
&\qquad + \int_0^t (t-s)^{z-1}\big\|\mathcal{P}_E(t-s)\big(K(s)u^r(s) - K(s)u^\circ(s)\big)\big\|_\alpha\,ds \qquad (5.9)\\
&\quad\leq M\big\|E^{-1}\big\|\|E\|L_g(\sigma)\big\|x^r - x^\circ\big\| + \frac{L_J(\sigma)M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\big\|x^r(s) - x^\circ(s)\big\|_\alpha\,ds \\
&\qquad + \frac{M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\big\|K(s)u^r(s) - K(s)u^\circ(s)\big\|\,ds \qquad (5.10)\\
&\quad\leq ML_g(\sigma)\big\|x^r - x^\circ\big\| + \frac{L_J(\sigma)M\|E^{-1}\|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\big\|x^r(s) - x^\circ(s)\big\|_\alpha\,ds \\
&\qquad + \frac{M\|E^{-1}\|}{\Gamma(2z)}S^{2z-\frac{1}{p}}\bigg(\int_0^S \big\|K(s)u^r(y) - K(s)u^\circ(s)\big\|^p\,ds\bigg)^{\frac{1}{p}} \qquad (5.11)\\
&\quad:= \eta_r^{(1)} + \eta_r^{(2)} + \eta_r^{(3)}. \qquad (5.12)
\end{aligned}$$

Since K is strongly continuous, $\|Ku^r - Ku^\circ\| \to 0$ as $r \to \infty$, and by Lemma 2.4 we have

$$\int_0^S \big\|K(s)u^r(s) - K(s)u^\circ(s)\big\|^p\,ds \to 0 \quad \text{as } r \to \infty,$$

which implies that $\eta_r^{(3)} \to 0$ as $r \to \infty$. Moreover, we have

$$\big\|x^r(t) - x^\circ(t)\big\|_\alpha \leq \big\|\eta_r^{(3)}\big\| + ML_g(\sigma)\big\|x^r - x^\circ\big\|$$
$$+ \frac{L_J(\sigma)M|E^{-1}|}{\Gamma(2z)}\int_0^t (t-s)^{2z-1}\big\|x^r(s) - x^\circ(s)\big\|_\alpha\,ds. \tag{5.13}$$

By Grönwall's inequality again, there exists a positive constant N^* such that

$$\big\|x^r(t) - x^\circ(t)\big\|_\alpha \leq N^*\big\|\eta_r^{(2)}\big\|_\alpha, \tag{5.14}$$

which yields that

$$x^r \to x^\circ \quad \text{in } C_{0,S} \text{ as } r \to \infty. \tag{5.15}$$

Note that assumption (H4) implies the Balder assumption. Hence by Balder's theorem we can conclude that

$$(x,u) \to \int_0^S \mathcal{I}\big(t,x(t),u(t)\big)\,dt \tag{5.16}$$

is sequentially lower semicontinuous in the weak topology of $L^p(\Omega,U) \subset L^1(\Omega,U)$ and the strong topology of $L^1(\Omega,X)$. Hence Ω is weakly lower semicontinuous on $L^p(\Omega,U)$, and since by (H5)(iv) $\Omega > -\infty$, Ω attains its infimum at $u\circ \in U_{ad}$, that is,

$$\varepsilon = \lim_{r\to\infty}\int_0^S \mathcal{I}\big(t,x^r(t),u^r(t)\big)\,dt \geq \int_0^S \mathcal{I}\big(t,x^\circ(t),u^\circ(t)\big)\,dt = \mathcal{J}\big(x^\circ,u^\circ\big) \geq \varepsilon. \tag{5.17}$$

The proof is completed. □

6 An illustrative example

In this section, we present an example illustrating the main results.

Example 6.1 Consider the following problem:

$$\begin{cases} {}_0^cD_t^{\frac{3}{2}}Ex(t,w) = \Delta x(t,w) + J_1(t,x(t,w)) + \int_0^t h(t-s)J_2(s,x(s,w))\,ds \\ \qquad\qquad + \int_\Sigma L(w,\zeta)u(\zeta,t)\,d\zeta, \quad w\in\Sigma, t\in\Omega, \\ x(t,w) = 0, \quad w\in\partial\Sigma, t\in\Omega, \\ x(0,w) - \sum_{i=1}^n \int_\Sigma m(\xi,w)x(t_i,\xi)\,d\xi = 0, \qquad x'(0,w) = 0, \quad w\in\Sigma, \end{cases} \tag{6.1}$$

where $\Sigma \subset \mathbb{R}^3$ is a bounded domain, $\partial\Sigma \in C^3$, Δ is the Laplace operator, $u \in L^2(\Omega\times\Sigma,\mathbb{R})$, $h \in L^1([0,S],\mathbb{R})$, $K : \bar{\Sigma}\times\bar{\Sigma} \to \mathbb{R}$ is continuous, and $m(\xi,w) : \Sigma\times\Sigma \to X$ is an L^2-Lebesgue-integrable function.

Assume that $J_1 : \Omega\times\mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\alpha_1,\beta_1 \geq 0$ such that

$$\begin{aligned} &\big\|J_1(t,\eta)\big\| \leq \alpha_1\big(1+|\eta|\big), \\ &\|J_1(t,\eta) - J_1(t,\bar{\eta}\| \leq \beta_1|\eta-\bar{\eta}|. \end{aligned}$$

Assume that $J_2 : [0,S]\times\mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\alpha_2,\beta_2 \geq 0$ such that

$$\begin{aligned} &\big\|J_2(t,\eta)\big\| \leq \alpha_2\big(1+|\eta|\big), \\ &\|J_2(t,\eta) - J_2(t,\bar{\eta}\| \leq \beta_2|\eta-\bar{\eta}|. \end{aligned}$$

Define

$$X = U = L^2(\Omega\times\Sigma,\mathbb{R}), \qquad D(G) = H^2(\Sigma)\cap H_0^1(\Sigma), \qquad Gx = -\left(\frac{\partial^2 x}{\partial w_1^2} + \frac{\partial^2 x}{\partial w_2^2} + \frac{\partial^2 x}{\partial w_3^2}\right)$$

for $x \in D(GE^{-1})$. Then GE^{-1} can generate a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\geq 0}$ on X. The controls functions $u : \mathcal{C}x(\Sigma) \to \mathbb{R}$ are such that $u \in L^2(\mathcal{C}x(\Sigma),\mathbb{R})$. We claim that

$t \to u(\cdot, t)$ going from Ω into U is measurable. We set

$$U(t) = \{u \in U : \|u\|_U \le \chi\},$$

where $\chi \in L^2(\Omega, \mathbb{R}^+)$. We restrict the admissible controls U_{ad} to all $u \in L^2(\mathcal{C}x(\Sigma), \mathbb{R})$ such that $\|u(\cdot, t)\|_{L^2\mathcal{C}x(\Sigma)} \le \chi$ almost everywhere.

Define $x(t)(w) = x(t, w)$,

$$K(t)u(t)(w) = \int_\Sigma L(w, \zeta)u(\zeta, t)\, d\zeta,$$

and

$$J\big(t, x(t)\big)(w) = F_1(t, x)(w) + \left(\int_0^t h(t-s)F_2\big(s, x(s)\big)\, ds\right)(w),$$

where $F_1(t, x)(w) = J_1(t, x(t, w))$ and $F_2(t, x)(w) = J_2(t, x(t, w))$.

Taking $\gamma = \frac{3}{2}$, we have $z = \frac{3}{4}$. Let $g : C(\Omega, X) \to X$ be given by $g(x)(w) = \sum_{i=0}^n L_g x(t_i)(w)$ with $L_g u(w) = \int_\Sigma m(\xi, w)u(\xi)\, d\xi$ for $u \in X$, $w \in \Sigma$ (noting that $L_g : X \to X$ is completely continuous). Thus assumption (H4) holds.

Thus problem (6.1) can be rewritten as follows:

$$\begin{cases} {}^c_0D^\gamma_t Ex(t) = Gx(t) + J(t, x(t)) + K(t)u(t), & t \in \Omega = [0, S], \\ x(0) = x_0 + g(x), \qquad x'(0) = x_1. \end{cases} \tag{6.2}$$

Consider the following cost function:

$$\mathcal{J}(u) = \int_0^S \mathcal{I}\big(t, x^u(t), u(t)\big)\, dt,$$

where $\mathcal{I} : \Omega \times C^{1,0}(\Omega \times \bar{\Sigma}, \mathbb{R}) \times L^2(\Omega \times \Sigma, \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ for $x \in C^{1,0}([0, S] \times \bar{\Sigma}, \mathbb{R})$, $u \in L^2(\Omega \times \Sigma, \mathbb{R})$. Then

$$\mathcal{I}\big(t, x^u(t), u(t)\big)(w) = \int_\Sigma \big|x^u(t, w)\big|^2\, dw + \int_\Sigma \big|u(w, t)\big|^2\, dw.$$

It is easy to verify that all the assumptions of Theorem 5.1 hold. Thus problem (6.1) admits at least one optimal pair.

7 Conclusions

In this paper, we mainly investigated the nonlocal fractional differential evolution equations of order (1,2) in Banach spaces. Applying the main tools from the fractional calculus, cosine family, measure of noncompactness method, fixed point theorem, Hölder's inequality, and Grönwall's inequality, we propose the definition of α-mild solutions and obtain the existence, uniqueness, and continuous dependence of the solution. Furthermore, we construct the Lagrange problem to analyze the optimal control of the systems. Finally, we provided an example to demonstrate the validity of our main results. For future research, it will be interesting and challenging to discuss similar problems for fractional evolution equations in the framework of Atangana–Baleanu derivatives.

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References

1. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
3. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
4. Zhou, Y.: Fractional Evolution Equations and Inclusions: Analysis and Control. Academic Press, San Diego (2016)
5. Li, L., Liu, J.G., Wang, L.: Cauchy problems for Keller–Segel type time-space fractional diffusion equation. J. Differ. Equ. **265**, 1044–1096 (2018)
6. Jajarmi, A., Baleanu, D.: A new fractional analysis on the interaction of HIV with $CD4^+$ T-cells. Chaos Solitons Fractals **113**, 221–229 (2018)
7. Afshari, H., Karapinar, E.: A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces. Adv. Differ. Equ. **616**, 1–11 (2020)
8. Yang, R., Liu, S., et al.: Consensus of fractional-order delayed multi-agent systems in Riemann–Liouville sense. Neurocomputing **396**, 123–129 (2020)
9. Shojaat, H., Afshari, H., Asgari, M.S.: A new class of mixed monotone operators with concavity and applications to fractional differential equations. TWMS J. Appl. Eng. Math. **11**(1), 122–133 (2021)
10. Hristov, J.: Linear viscoelastic responses and constitutive equations in terms of fractional operators with non-singular kernels: pragmatic approach, memory kernel correspondence requirement and analyses. Eur. Phys. J. Plus **134**, 1–31 (2019)
11. Machado, J.A., Ravichandran, C., Rivero, M., et al.: Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions. Fixed Point Theory Appl. **66**, 1–16 (2013)
12. Ravichandran, C., Logeswari, K., Fahd, J.: New results on existence in the framework of Atangana–Baleanu derivative for fractional integro-differential equations. Chaos Solitons Fractals **125**, 194–200 (2019)
13. Kumar, A., Chauhan, H.V.S., Ravichandran, C., et al.: Existence of solutions of non-autonomous fractional differential equations with integral impulse condition. Adv. Differ. Equ. **434**, 1–14 (2020)
14. Li, K., Peng, J., Jia, J.: Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives. J. Funct. Anal. **263**, 476–510 (2012)
15. Shu, X.B., Wang, Q.Q.: The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Comput. Math. Appl. **64**, 2100–2110 (2012)
16. Li, Y.: Regularity of mild solutions for fractional abstract Cauchy problem with order $\alpha \in (1,2)$. Z. Angew. Math. Phys. **66**, 3283–3298 (2015)
17. Kian, Y., Yamamoto, M.: On existence and uniqueness of solutions for semilinear fractional wave equations. Fract. Calc. Appl. Anal. **20**, 117–138 (2017)
18. Li, Y., Sun, H., Feng, Z.: Fractional abstract Cauchy problem with order $\alpha \in (1,2)$. Dyn. Partial Differ. Equ. **13**, 155–177 (2016)
19. Agarwal, R.P., Baleanu, D., et al.: A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. J. Comput. Appl. Math. **339**, 3–29 (2018)
20. Frederico, G., Torres, D.: Fractional conservation laws in optimal control theory. Nonlinear Dyn. **53**, 215–222 (2008)
21. Ozdemir, N., Karadeniz, D., Iskender, B.B.: Fractional optimal control problem of a distributed system in cylindrical coordinates. Phys. Lett. A **373**, 221–226 (2009)
22. Wang, J.R., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262–272 (2011)
23. Wang, J.R., Wei, W., Zhou, Y.: Fractional finite time delay evolution systems and optimal controls in infinite-dimensional spaces. J. Dyn. Control Syst. **17**, 515–535 (2011)

24. Li, K., Peng, J., Gao, J.: Controllability of nonlocal fractional differential systems of order $\alpha \in (1,2]$ in Banach spaces. Rep. Math. Phys. **71**, 33–43 (2013)
25. Chang, Y.K., Ponce, R.: Sobolve type time fractional differential equations and optimal controls with the order in $(1,2)$. Differ. Integral Equ. **32**, 517–540 (2019)
26. Chang, Y.K., Pei, Y., Ponce, R.: Existence and optimal controls for fractional stochastic evolution equations of Sobolev type via fractional resolvent operators. J. Optim. Theory Appl. **182**, 558–572 (2019)
27. Yan, Z.M., Jia, X.M.: Optimal controls of fractional impulsive partial neutral stochastic integro-differential systems with infinite delay in Hilbert spaces. Int. J. Control. Autom. Syst. **15**, 1051–1068 (2017)
28. Gorbatenko, Y.A.V.: Existence and uniqueness of mild solutions of second order semilinear differential equations in Banach space. Methods Funct. Anal. Topol. **17**(1), 1–9 (2011)
29. Xiang, X.L., Huawu, K.: Delay systems and optimal control. Acta Math. Appl. Sin. **16**, 27–35 (2000)
30. Adams, R.A., Fournier, J.: Sobolev Spaces: Pure and Applied Mathematics. Academic Press, Boston (2003)
31. Arendt, W., Batty, C.J.K., Hieber, M., Neubrander, F.: Vector-Valued Laplace Transforms and Cauchy Problems, 2nd edn. Birkhäuser, Basel (2011)
32. Goldstein, J.A.: Semigroups of Linear Operators and Applications. Oxford University Press, New York (1985)
33. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Hung. **32**, 75–96 (1978)
34. Zhou, Y., He, J.W.: New results on controllability of fractional evolution systems with order $\alpha \in (1,2)$. Evol. Equ. Control Theory, 1–19 (2020)