Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Solution of the system of nonlinear PDEs characterizing CES property under quasi-homogeneity conditions

Haila Alodan[1], Bang-Yen Chen[2], Sharief Deshmukh[1] and Gabriel-Eduard Vîlcu[3,4*]

*Correspondence:
gvilcu@upg-ploiesti.ro
[3]Research Center in Geometry, Topology and Algebra, University of Bucharest, Str. Academiei 14, Bucharest 70109, Romania
[4]Department of Cybernetics, Economic Informatics, Finance and Accountancy, Petroleum-Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti 100680, Romania
Full list of author information is available at the end of the article

Abstract

The constant elasticity of substitution (CES for short) is a basic property widely used in some areas of economics that involves a system of second-order nonlinear partial differential equations. One of the most remarkable results in mathematical economics states that under homogeneity condition i.e. the production function is a homogeneous function of a certain degree, there are no other production models with the CES property apart from the famous Cobb–Douglas and Arrow–Chenery–Minhas–Solow production functions. In this paper we generalize this classification result to a much wider framework of production functions under quasi-homogeneity conditions, showing in particular the existence of three new classes of production models with the CES property.

MSC: Primary 35G50; secondary 91B38; 91B02; 91B15

Keywords: System of nonlinear PDEs; Production function; Elasticity of substitution; CES property

1 Introduction

A fundamental concept used in the modeling of a production process $\mathcal{P}$ is that of production function. Let us denote by n the number of inputs involved in the production process ($n \geq 2$), by $x_1, \dots, x_n$ the factors of production (i.e. inputs - whatever is used in the production process $\mathcal{P}$, like natural resources, labor, capital, and entrepreneur), and by f the resulting output of the process $\mathcal{P}$. If $\mathbb{R}_+$ is the set of all real positive numbers and $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1, \dots, x_n > 0\}$, then a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ with non-vanishing first derivatives, defined by $f = f(x_1, \dots, x_n)$, is said to be the production function associated with the production process $\mathcal{P}$.

One of the most important economic indicators used in the analysis of changes in the income shares of inputs is the Hicks elasticity of substitution (HES) independently introduced by Hicks [1] and Robinson [2]. For two distinct inputs x_i and x_j ($i, j \in \{1, \dots, n\}$), this economic indicator, usually denoted by H_{ij}, is defined for all combinations of inputs

$(x_1,\dots,x_n)\in\mathbb{R}^n_+$ by

$$H_{ij}(x_1,\dots,x_n)=\frac{\frac{1}{x_i f_{x_i}}+\frac{1}{x_j f_{x_j}}}{-\frac{f_{x_i x_i}}{f_{x_i}^2}+\frac{2f_{x_i x_j}}{f_{x_i}f_{x_j}}-\frac{f_{x_j x_j}}{f_{x_j}^2}},\tag{1}$$

where $f_{x_i}, f_{x_i x_j},\dots$, etc. denote the partial derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j},\dots$, etc. If the relation

$$H_{ij}(x_1,\dots,x_n)=\sigma\tag{2}$$

holds for all combinations of inputs $(x_1,\dots,x_n)\in\mathbb{R}^n_+$ and for all $i,j\in\{1,\dots,n\}$, $i\neq j$, where σ is a nonzero real constant, then f is said to have the CES property.

Notice that there are two important production models exhibiting the CES property widely utilized in economics (see for instance the recent works [3–7]). The first one is the Cobb–Douglas (CD) production function introduced in [8] for two inputs (labor and capital). In the general case of n inputs, the CD production function is defined by [9–11]

$$f(x_1,\dots,x_n)=A\cdot\prod_{i=1}^{n}x_i^{\alpha_i},$$

where $A>0$ and $\alpha_1,\dots,\alpha_n\neq 0$.

A second production function having the CES property is the Arrow–Chenery–Minhas–Solow (ACMS) production function originally introduced in [12] in order to generalize the CD production function. In the case of n inputs, the ACMS production function is given by [13–15]

$$f(x_1,\dots,x_n)=A\left(\sum_{i=1}^{n}k_i^{\rho}x_i^{\rho}\right)^{\frac{\gamma}{\rho}},$$

where $A,k_1,\dots,k_n,\gamma>0$, $\rho<1$, $\rho\neq 0$. We recall that the CD production function can be recovered from the ACMS production function as a limit case (see [16]).

It is known that for a CD production function we have $H_{ij}(x_1,\dots,x_n)=1$, while for an ACMS production function we have $H_{ij}(x_1,\dots,x_n)=\frac{1}{1-\rho}\neq 1$.

Notice that both CD and ACMS production models are homogeneous functions. There is a remarkable result in economic theory stating that, under homogeneity condition i.e. the production function is a homogeneous function of some degree, there are no other two-factor production models exhibiting the CES property apart from CD and ACMS production functions [12]. A complete proof of this result can be found in Losonczi (see [17, Theorem 10]), the precise statement being the following.

Theorem 1.1 ([17]) *Let $f:\mathbb{R}^2_+\to\mathbb{R}_+$ be a twice differentiable production function with two inputs, homogeneous of degree $q\neq 0$. If f satisfies the constant elasticity of substitution property* (2) *for a nonzero real constant σ, then*

$$f(x,y)=\begin{cases}Cx^{\alpha}y^{q-\alpha} & \text{if } \sigma=1,\\ (\beta_1 x^{\frac{q}{\beta}}+\beta_2 y^{\frac{q}{\beta}})^{\beta} & \text{if } \sigma\neq 1,\end{cases}$$

where α is any nonzero real constant with $q - \alpha \neq 0$, C, β_1, β_2 are positive constants, and $\beta = \frac{q\sigma}{\sigma - 1}$.

We remark that the condition $q \neq 0$ in the above theorem is a natural one, since in the case $q = 0$ the Hicks elasticity of substitution is indeterminate (see [17, Remark 10]). The generalization of Theorem 1.1 for an arbitrary number of production factors was obtained by the second author of the present work in [18, Theorem 1]. It is important to note that this interesting result is no longer true for other classes of production models. For example, it has recently been demonstrated that in the class of composite production functions, also known as quasi-product production models [19, 20], there are four different production functions with the CES property (see [21, Theorem 4.1]).

By weakening the property of homogeneity to quasi-homogeneity, we arrive at some more general production models, known as quasi-homogeneous (in short QH) models. It is worth mentioning that this broader property for production models was first proposed by Eichhorn and Oettli [22], and the importance of QH models has been further highlighted in various works (see e.g. [23, Sect. 6.2], [24, Chap. 12], [25–27]). Recently, in [28, 29], the authors studied such models with two inputs, deriving their analytical expression in case of unit elasticity of substitution. Moreover, such models with n inputs ($n \geq 2$) were investigated in [30]; the authors classified QH models with proportional marginal rate of substitution property and also those that exhibit a constant elasticity of production with respect to a settled factor of production. We recall that a production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called a *weight-homogeneous* (shortly WH) or a QH production function having degree q and weight vector $g = (g_1, \dots, g_n) \in \mathbb{R}^n$, where $g_1^2 + \cdots + g_n^2 \neq 0$, if it satisfies

$$f\left(\lambda^{g_1} x_1, \dots, \lambda^{g_n} x_n\right) = \lambda^q f(x_1, \dots, x_n) \tag{3}$$

for all points $(x_1, \dots, x_n) \in \mathbb{R}^n_+$ and all $\lambda > 0$. It is obvious that in the particular case when the weight vector is $(1, \dots, 1)$, a QH function having degree q reduces to a q-homogeneous function. More generally, a QH function having degree q and equal weights $(g, \dots, g)$ is again a homogeneous function, but now the degree of homogeneity is $\frac{q}{g}$. Obviously, the class of QH functions is considerably larger than that of homogeneous functions. For instance, the function f defined by

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n-1} \alpha_i x_i^{\frac{1}{i}} x_{i+1}^{\frac{1}{i+1}},$$

where $\alpha_1, \dots, \alpha_{n-1}$ are arbitrary positive constants, provides us a very simple example of QH production model with $q = 2$ and $g = (1, 2, \dots, n)$, which clearly is not homogeneous.

We note that property (3) mathematically models a precise economic situation encountered in a production process when a multiplication of the inputs with different powers of an identical factor leads to a multiplication of the output by a power of the same factor. This situation can occur when it is not possible to identically multiply all the factors of production due to the lack of one or more physical inputs.

It is also worth mentioning that a differentiable function f depending on the variables $x_1, x_2, \dots, x_n$, $n \geq 2$, is quasi-homogeneous having degree q and weights $(g_1, \dots, g_n)$ if and only if the following identity is satisfied [31, 32]:

$$\sum_{i=1}^{n} g_i x_i f_{x_i} = qf. \tag{4}$$

Notice that (4) is known as the *generalized Euler identity* and its general solution is [30]

$$f(x_1,\dots,x_n) = x_i^{\frac{q}{g_i}} h\left(\left(\frac{x_1^{g_i}}{x_i^{g_1}}\right),\dots,\left(\frac{x_{i-1}^{g_i}}{x_i^{g_{i-1}}}\right),\left(\frac{x_{i+1}^{g_i}}{x_i^{g_{i+1}}}\right),\dots,\left(\frac{x_n^{g_i}}{x_i^{g_n}}\right)\right), \tag{5}$$

where i is any index from the set $\{1,\dots,n\}$ such that $g_i \neq 0$, and h is any differentiable function that depends on $(n-1)$ variables.

The aim of this work is to establish the next result that generalizes the well-known classification of homogeneous production models exhibiting the CES property to the much wider class of weight-homogeneous production functions.

Theorem 1.2 *Suppose that f is a twice differentiable QH production function having degree $q \neq 0$ and weights $(g_1,\dots,g_n)$. Then:*

(i) *f exhibits unitary elasticity of substitution, that is, f meets condition (2) for $\sigma = 1$, if and only if the function f reduces to a CD production model expressed as*

$$f(x_1,\dots,x_n) = Ax_1^{\alpha_1}x_2^{\alpha_2}\cdot\dots\cdot x_n^{\alpha_n}, \tag{6}$$

where A and $\alpha_i \neq 0$ are real constants such that $A > 0$, $\alpha_i \neq 0$, $i = 1,\dots,n$, and $\sum_{i=1}^{n} \alpha_i g_i = q$.

(ii) *If $n = 2$, then f satisfies the constant elasticity of substitution property for a nonzero real constant $\sigma \neq 1$ if and only if one of the next situations occurs:*

a. *f reduces to a production model expressed by*

$$f(x_1,x_2) = \left(\frac{a_1^{\frac{\sigma-1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}}}{a_2^{\frac{\sigma-1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + 1}\right)^{\frac{\sigma q}{(\sigma-1)g_1}}, \tag{7}$$

where a_1, a_2 are positive constants, provided that $g_2 = 0$.

b. *f reduces to a production model given by*

$$f(x_1,x_2) = \left(\frac{a_2^{\frac{\sigma-1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}}}{a_1^{\frac{\sigma-1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + 1}\right)^{\frac{\sigma q}{(\sigma-1)g_2}}, \tag{8}$$

where a_1, a_2 are positive constants, provided that $g_1 = 0$.

c. *f reduces to a two-input ACMS production model expressed by*

$$f(x_1,x_2) = \left(a_1^{\frac{\sigma-1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + a_2^{\frac{\sigma-1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma q}{(\sigma-1)g_1}}, \tag{9}$$

where a_1, a_2 are positive constants, provided that $g_1 = g_2$.

d. *f reduces to a production model expressed by*

$$f(x_1,x_2) = Ax_2^{\frac{q}{g_2}} e^{V\left(\frac{x_1^{g_2}}{x_2^{g_1}}\right)}, \tag{10}$$

where A is a positive constant and V is an antiderivative of the function v of variable $u = \frac{x_1^{g_2}}{x_2^{g_1}}$ defined implicitly by the identity

$$\left[1 - \frac{q}{g_2(g_1 - g_2)} \cdot \frac{1}{uv(u)}\right]^{g_1 - g_2} = Bu^{\frac{\sigma-1}{\sigma}}\left[1 - \frac{q}{g_1 g_2} \cdot \frac{1}{uv(u)}\right]^{g_1}, \tag{11}$$

for some positive constant B, provided that $g_1 g_2 \neq 0$ and $g_1 \neq g_2$.

2 Proof of Theorem 1.2

Suppose that f is a QH production function having degree q and weights $(g_1, \dots, g_n)$. Then the generalized Euler identity (4) holds. Differentiating now this identity with respect to each variable x_i, $i = 1, \dots, n$, due to the fact that f is twice differentiable, we derive

$$g_i x_i f_{x_i x_i} + \sum_{j \neq i} g_j x_j f_{x_i x_j} = (q - g_i) f_{x_i}, \quad i = 1, \dots, n. \tag{12}$$

(i) We assume first that f satisfies the CES property for $\sigma = 1$. Then we obtain from (1) and (2) that

$$x_j f_{x_i x_j} = \frac{1}{2}\left(f_{x_i} + \frac{x_j}{x_i} f_{x_j}\right) + \frac{x_j}{2}\left(f_{x_j} \frac{f_{x_i x_i}}{f_{x_i}} + f_{x_i} \frac{f_{x_j x_j}}{f_{x_j}}\right) \tag{13}$$

for $1 \leq i < j \leq n$.

Then, substituting (13) in (12) and using (4), we find

$$\frac{f_{x_i x_i}}{f_{x_i}} \cdot \frac{qf}{2f_{x_i}} + \frac{1}{2}\sum_{j=1}^{n} g_j x_j \frac{f_{x_j x_j}}{f_{x_j}} = q - \frac{1}{2}\sum_{j=1}^{n} g_j - \frac{qf}{2x_i f_{x_i}} \tag{14}$$

for $i = 1, \dots, n$.

Considering now (14) as a system of n equations with n unknowns $\frac{f_{x_1 x_1}}{f_{x_1}}, \dots, \frac{f_{x_n x_n}}{f_{x_n}}$, we obtain

$$\frac{f_{x_i x_i}}{f_{x_i}} = \frac{f_{x_i}}{f} - \frac{1}{x_i}, \quad i = 1, \dots, n \tag{15}$$

and replacing (15) in (13), we get

$$f_{x_i x_j} = \frac{f_{x_i} f_{x_j}}{f}, \quad 1 \leq i < j \leq n. \tag{16}$$

Following the proof of [18, Theorem 1 (Case (a))], we derive that the solution of (15) and (16) is

$$f(x_1, \dots, x_n) = A x_1^{\alpha_1} x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n},$$

where $\alpha_i \neq 0$, $i = 1, \dots, n$, and $A > 0$. Now, taking into account that the function f given above is a QH production function having the degree q and weights $(g_1, \dots, g_n)$, it follows immediately that the constants $\alpha_1, \dots, \alpha_n$ satisfy the relation $\sum_{i=1}^{n} \alpha_i g_i = q$. Hence we conclude that indeed f is the CD model expressed by (6).

Conversely, if f is a CD production function expressed by (6), then it is well known that f has unit elasticity of substitution.

(ii) If $n = 2$, then taking $i = 1$ and $i = 2$ in (12), we derive

$$g_1x_1f_{x_1x_1} + g_2x_2f_{x_1x_2} = (q - g_1)f_{x_1} \tag{17}$$

and

$$g_2x_2f_{x_2x_2} + g_1x_1f_{x_1x_2} = (q - g_2)f_{x_2}. \tag{18}$$

Suppose that f satisfies the CES property for $\sigma \neq 1$. Then we obtain from (1) and (2) that

$$f_{x_1x_2} = \frac{1}{2\sigma}\left(\frac{f_{x_1}}{x_2} + \frac{f_{x_2}}{x_1}\right) + \frac{1}{2}\left(f_{x_2}\frac{f_{x_1x_1}}{f_{x_1}} + f_{x_1}\frac{f_{x_2x_2}}{f_{x_2}}\right). \tag{19}$$

Replacing now (19) in (17) and (18), and using also the generalized Euler identity, we find

$$\frac{f_{x_ix_i}}{f_{x_i}} \cdot \frac{qf}{2f_{x_i}} + \frac{1}{2}\sum_{j=1}^{2} g_jx_j\frac{f_{x_jx_j}}{f_{x_j}} = q - g_i - \frac{1}{2\sigma}\sum_{j=1}^{2} g_j + \frac{g_i}{\sigma} - \frac{qf}{2\sigma x_if_{x_i}} \tag{20}$$

for $i = 1, 2$.

From (20), we obtain after some straightforward computation that $\frac{f_{x_1x_1}}{f_{x_1}}$ and $\frac{f_{x_2x_2}}{f_{x_2}}$ can be expressed as

$$\frac{f_{x_ix_i}}{f_{x_i}} = \alpha_ix_i\left(\frac{f_{x_i}}{f}\right)^2 + \beta_i\left(\frac{f_{x_i}}{f}\right) - \frac{1}{\sigma x_i}, \quad i = 1, 2, \tag{21}$$

where

$$\alpha_1 = \frac{g_1(g_1 - g_2)}{q^2} \cdot \frac{\sigma - 1}{\sigma}, \qquad \alpha_2 = \frac{g_2(g_2 - g_1)}{q^2} \cdot \frac{\sigma - 1}{\sigma}, \tag{22}$$

$$\beta_1 = 1 + \frac{g_2 - 2g_1}{q} \cdot \frac{\sigma - 1}{\sigma}, \qquad \beta_2 = 1 + \frac{g_1 - 2g_2}{q} \cdot \frac{\sigma - 1}{\sigma}. \tag{23}$$

Now, it is easy to see that (21) can be written as

$$\frac{f_{x_ix_i}}{f} = \alpha_ix_i\left(\frac{f_{x_i}}{f}\right)^3 + \beta_i\left(\frac{f_{x_i}}{f}\right)^2 - \frac{1}{\sigma x_i}\frac{f_{x_i}}{f}, \quad i = 1, 2, \tag{24}$$

and inserting (24) in (19) we derive

$$\frac{2f \cdot f_{x_1x_2}}{f_{x_1}f_{x_2}} = \beta_1 + \beta_2 + \alpha_1\frac{x_1f_{x_1}}{f} + \alpha_2\frac{x_2f_{x_2}}{f}. \tag{25}$$

We can split now the proof into two cases, as follows.

Case 1: $g_1 \cdot g_2 = 0$. As the weights g_1 and g_2 cannot be simultaneously 0, it follows in this case that either $g_1 \neq 0$ and $g_2 = 0$, or $g_2 \neq 0$ and $g_1 = 0$.

Suppose first that $g_1 \neq 0$ and $g_2 = 0$. Then it is clear from (22) and (23) that

$$\alpha_1 = \frac{g_1^2(\sigma - 1)}{\sigma q^2}, \qquad \alpha_2 = 0, \qquad \beta_1 = 1 - \frac{2g_1}{q}, \qquad \beta_2 = 1 + \frac{g_1(\sigma - 1)}{\sigma q}, \tag{26}$$

and, in view of (5), we derive that f can be written as

$$f(x_1, x_2) = x_1^{\frac{q}{g_1}} h\bigl(x_2^{g_1}\bigr)$$

for a twice differentiable function h, or equivalently

$$f(x_1, x_2) = x_1^{\frac{q}{g_1}} H(x_2), \tag{27}$$

where $H(x_2) := h(x_2^{g_1})$.

We first remark that, due to (26), the function f given by (27) automatically satisfies (24) for $i = 1$, regardless of the function H. Taking now $i = 2$ in (24), in view of (26), we obtain

$$\frac{f_{x_2x_2}}{f} = \left[1 + \frac{g_1(\sigma - 1)}{\sigma q}\right]\left(\frac{f_{x_2}}{f}\right)^2 - \frac{1}{\sigma x_2}\frac{f_{x_2}}{f}. \tag{28}$$

Inserting (27) in (28), we derive

$$\frac{H''(x_2)}{H(x_2)} = \left[1 + \frac{g_1(\sigma - 1)}{\sigma q}\right]\left[\frac{H'(x_2)}{H(x_2)}\right]^2 - \frac{1}{\sigma x_2}\frac{H'(x_2)}{H(x_2)}, \tag{29}$$

where the symbol "$'$" stands for the derivative with respect to x_2.

Next, with the help of the substitution

$$Z(x_2) = \frac{H'(x_2)}{H(x_2)}, \tag{30}$$

we get that (29) reduces to a first-order differential equation, namely

$$Z'(x_2) = \frac{g_1(\sigma - 1)}{\sigma q} Z^2(x_2) - \frac{1}{\sigma x_2} Z(x_2). \tag{31}$$

As the above equation is generalized homogeneous, we can use the substitution

$$Z(x_2) = \frac{W(x_2)}{x_2} \tag{32}$$

in order to reduce (31) to a separable first-order differential equation:

$$x_2 W'(x_2) = \frac{g_1(\sigma - 1)}{\sigma q}\left[W^2(x_2) + \frac{q}{g_1} W(x_2)\right]. \tag{33}$$

We can easily solve (33), obtaining the solution

$$W(x_2) = \frac{q}{g_1} \frac{C x_2^{\frac{\sigma-1}{\sigma}}}{1 - C x_2^{\frac{\sigma-1}{\sigma}}}, \tag{34}$$

where C is a positive constant.

Now, from (30), (32), and (34), we derive the solution of (29):

$$H(x_2) = D\left(1 - Cx_2^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma q}{(\sigma-1)g_1}}, \tag{35}$$

where D is a positive constant.

Next, using (27) and (35), we obtain the solution of (28):

$$f(x_1, x_2) = Dx_1^{\frac{q}{g_1}}\left[1 - Cx_2^{\frac{\sigma-1}{\sigma}}\right]^{-\frac{\sigma q}{(\sigma-1)g_1}}.$$

If we denote

$$A = -C \cdot D^{-\frac{(\sigma-1)g_1}{\sigma q}}, \qquad B = D^{-\frac{(\sigma-1)g_1}{\sigma q}},$$

then we can write f as

$$f(x_1, x_2) = \left(Bx_1^{-\frac{\sigma-1}{\sigma}} + Ax_1^{-\frac{\sigma-1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma q}{(\sigma-1)g_1}},$$

and it is easy to check that the production function f obtained above satisfies also (25). Hence we conclude that in this case f can be expressed by (7). Conversely, if f is a production model expressed by (7), then a direct computation shows that f has the elasticity of substitution σ.

If we suppose now that $g_1 = 0$ and $g_2 \neq 0$, in a similar way we conclude that f can be expressed by (8). Conversely, if the production model f is given by (8), then we can check by a straightforward computation that f has the elasticity of substitution σ.

Case 2: $g_1 \cdot g_2 \neq 0$. We can distinguish now two sub-subcases, according to whether α_1 is 0 or not.

Subcase 2.1: $\alpha_1 = 0$. Then it follows that $g_1 = g_2 \neq 0$, and from (22) and (23) we obtain

$$\alpha_i = 0, \qquad \beta_i = 1 + \frac{g_1(1-\sigma)}{\sigma q}, \quad i = 1, 2.$$

By taking $i = 1$ in (24) and making the substitution

$$z(x_1, x_2) = \frac{f_{x_1}(x_1, x_2)}{f(x_1, x_2)}, \tag{36}$$

one arrives at the following first-order partial differential equation:

$$z_{x_1} = \frac{g_1(1-\sigma)}{\sigma q} z^2 - \frac{1}{\sigma x_1} z.$$

The above equation is generalized homogeneous with respect to x_1 and the substitution

$$z(x_1, x_2) = \frac{w(x_1, x_2)}{x_1} \tag{37}$$

leads to the next simpler form:

$$x_1 w_{x_1} = \frac{g_1(1-\sigma)}{\sigma q} w^2 + \left(1 - \frac{1}{\sigma}\right) w. \tag{38}$$

Using the method of characteristics, we find that the solution of (38) is

$$w(x_1,x_2) = -\frac{q}{g_1}\cdot\frac{\mathcal{C}(x_2)x_1^{\frac{\sigma-1}{\sigma}}}{1-\mathcal{C}(x_2)x_1^{\frac{\sigma-1}{\sigma}}}, \tag{39}$$

where $\mathcal{C}$ is a function of variable x_2.

Hence, using (36), (37), and (39), we find that the solution of (24) for $i = 1$ is

$$f(x_1,x_2) = \mathcal{D}(x_2)\bigl[1-\mathcal{C}(x_2)x_1^{\frac{\sigma-1}{\sigma}}\bigr]^{\frac{\sigma q}{(\sigma-1)g_1}},$$

where $\mathcal{D}$ is a function of variable x_2. Next, using the notations

$$\mathcal{A}(x_2) = -\mathcal{C}(x_2)\mathcal{D}(x_2)^{\frac{(\sigma-1)g_1}{\sigma q}}, \qquad \mathcal{B}(x_2) = \mathcal{D}(x_2)^{\frac{(\sigma-1)g_1}{\sigma q}},$$

we can write f in the form

$$f(x_1,x_2) = \bigl[\mathcal{A}(x_2)x_1^{\frac{\sigma-1}{\sigma}} + \mathcal{B}(x_2)\bigr]^{\frac{\sigma q}{(\sigma-1)g_1}}. \tag{40}$$

Taking now into account that in this subcase f has the property

$$f\bigl(\lambda^g x_1, \lambda^g x_2\bigr) = \lambda^q f(x_1,x_2),$$

for all $(x_1,x_2)\in\mathbb{R}^2_+$ and $\lambda > 0$, we derive from (40) that

$$\mathcal{A}(x_2) = A, \qquad \mathcal{B}(x_2) = Bx_2^{\frac{\sigma-1}{\sigma}},$$

where A and B are nonzero real constants. Therefore we get

$$f(x_1,x_2) = \bigl(Ax_1^{\frac{\sigma-1}{\sigma}} + Bx_2^{\frac{\sigma-1}{\sigma}}\bigr)^{\frac{\sigma q}{(\sigma-1)g_1}},$$

and it is easy to check that the production function f obtained above also satisfies (24) for $i = 2$, as well as (25).

Hence we conclude that in this subcase f is an ACMS production function expressed by (9). Conversely, if f is an ACMS production function expressed by (9), then it is well known that f has the elasticity of substitution σ.

Subcase 2.2: $\alpha_1 \neq 0$. Then we have $g_1 \neq g_2$, and since $g_2 \neq 0$, it follows from (5) that f can be written as

$$f(x_1,x_2) = x_2^{\frac{q}{g_2}}h(u) \tag{41}$$

for a function h of variable $u = \frac{x_1^{g_2}}{x_2^{g_1}}$, which is twice differentiable. Next we denote by the prime symbol "′" the derivative taken with respect to u. Then from (41) we obtain

$$f_{x_1} = g_2\frac{x_2^{\frac{q}{g_2}}}{x_1}uh', \tag{42}$$

$$f_{x_2} = x_2^{\frac{q}{g_2}-1}\left(\frac{q}{g_2}h - g_1 u h'\right), \tag{43}$$

$$f_{x_1x_2} = g_2 \frac{x_2^{\frac{q}{g_2}-1}}{x_1}\left[\left(\frac{q}{g_2} - g_1\right)uh' - g_1 u^2 h''\right], \tag{44}$$

and

$$f_{x_2x_2} = x_2^{\frac{q}{g_2}-2}\left[\frac{q}{g_2}\left(\frac{q}{g_2} - 1\right)h + g_1\left(g_1 - \frac{2q}{g_2} + 1\right)uh' + g_1^2 u^2 h''\right]. \tag{45}$$

By replacing now (41), (42), (43), (44), (45) in (24) and (25), and taking account of (22) and (23), after some long and tedious computations we arrive in all cases at the same second-order differential equation:

$$u\left(\frac{h''}{h}\right) = \frac{g_1g_2(g_1 - g_2)(\sigma - 1)}{\sigma q^2}u^2\left(\frac{h'}{h}\right)^3 + \frac{\sigma q - (\sigma - 1)(2g_1 - g_2)}{\sigma q}u\left(\frac{h'}{h}\right)^2 + \frac{\sigma - 1 - \sigma g_2}{\sigma g_2}\left(\frac{h'}{h}\right). \tag{46}$$

Using the substitution

$$v(u) = \frac{h'(u)}{h(u)}, \tag{47}$$

one obtains that (46) reduces to the next first-order differential equation:

$$v' = \frac{g_1g_2(g_1 - g_2)(\sigma - 1)u}{\sigma q^2}v^3 - \frac{(\sigma - 1)(2g_1 - g_2)}{\sigma q}v^2 + \frac{\sigma - 1 - \sigma g_2}{\sigma g_2 u}v. \tag{48}$$

We remark that (48) is a particular type of Abel equation of the first kind [33, 34] investigated in [35] by employing a transformation originally introduced by Kamke [36]. Next, with the help of the substitution

$$w(u) = u \cdot v(u), \tag{49}$$

we derive that (48) reduces to a separable first-order differential equation:

$$uw' = \frac{g_1g_2(g_1 - g_2)(\sigma - 1)}{\sigma q^2}w\left(w - \frac{q}{g_2(g_1 - g_2)}\right)\left(w - \frac{q}{g_1g_2}\right). \tag{50}$$

Now we can easily obtain the solution of (50) in the implicit form

$$w^{g_2}\left(w - \frac{q}{g_2(g_1 - g_2)}\right)^{g_1 - g_2}\left(w - \frac{q}{g_1g_2}\right)^{-g_1} = Bu^{\frac{\sigma - 1}{\sigma}}, \tag{51}$$

where B represents any positive constant.

Next, using (47), (49), and (51), we deduce that

$$h(u) = A \cdot e^{\int v(u)\,du} \tag{52}$$

for a positive constant A, where v satisfies the following functional identity:

$$\left(1-\frac{q}{g_2(g_1-g_2)}\cdot\frac{1}{uv}\right)^{g_1-g_2}\left(1-\frac{q}{g_1g_2}\cdot\frac{1}{uv}\right)^{-g_1}=Bu^{\frac{\sigma-1}{\sigma}}. \tag{53}$$

Finally, from (41), (52), and (53), we get that the solution of (24) and (25) is

$$f(x_1,x_2)=Ax_2^{\frac{q}{g_2}}e^{\int v(u)\,du},$$

which is a production model expressed by (10), where v is a function of the variable $u=\frac{x_1^{g_2}}{x_2^{g_1}}$ satisfying (11). Conversely, if f is given by (10) such that the relation (11) is satisfied, then a direct computation shows that f has the elasticity of substitution σ.

3 Closing remarks

There is a fundamental result in economic theory stating that there are only two homogeneous production models with the CES property, namely CD and ACMS production functions. This work deals with weight-homogeneous production models, proving the existence of three new production functions exhibiting the CES property and therefore generalizing the main results of [12, 17, 18, 28, 29]. The new classification obtained in the present work will certainly have implications in the further development and use of production models in theoretical and applied economics.

We note that the proof of assertion (i) in Theorem 1.2 concerning the classification of quasi-homogeneous production functions with n inputs ($n\geq 2$) and unit elasticity of substitution follows the arguments from [18, Theorem 1], but the methods developed in [18] cannot be applied if the elasticity of substitution is a nonzero constant different from 1, even in the particular setting of two inputs. For this reason, in the proof of assertion (ii) we used an interplay of standard and non-standard techniques in order to manipulate the original system of second-order nonlinear partial differential equations with the help of generalized Euler equation. After some very long and tedious calculations involving a series of substitutions, we finally arrived at some basic differential equations and discussed the validity of obtained solutions in accordance with the quasi-homogeneity hypothesis on the production model. Finally, it is important to point out that our method of proof in Theorem 1.2(ii) does not work if the number of inputs is $n\geq 3$. Consequently, an open and very challenging problem is the generalization of Theorem 1.2(ii) to the case of more than two production factors.

Acknowledgements
This research project was supported by a grant from the "Research Center of the Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors are in unison for the publication of this manuscript.

Authors' contributions
All authors jointly worked on the results and read and approved the final manuscript.

Author details
[1]Department of Mathematics, King Saud University, Riyadh 11495, Saudi Arabia. [2]Department of Mathematics, Michigan State University, East Lansing, Michigan 48824–1027, USA. [3]Research Center in Geometry, Topology and Algebra, University of Bucharest, Str. Academiei 14, Bucharest 70109, Romania. [4]Department of Cybernetics, Economic Informatics, Finance and Accountancy, Petroleum-Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti 100680, Romania.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 December 2020 Accepted: 9 May 2021 Published online: 17 May 2021

References
1. Hicks, J.R.: Theory of Wages. Macmillan, London (1932)
2. Robinson, J.V.: The Economics of Imperfect Competition. Macmillan, London (1933)
3. Cheng, M., Han, Y.: Application of a modified CES production function model based on improved PSO algorithm. Appl. Math. Comput. **387**, 125178 (2020)
4. Cheng, M., Xiang, M.: Application of a combination production function model. Appl. Math. Comput. **236**, 33–40 (2014)
5. de-Córdoba, G.F., Galiano, G.: An economic cross-diffusion mutualistic model for cities emergence. Comput. Math. Appl. **79**(3), 643–655 (2020)
6. Reynes, F.: The Cobb-Douglas function as a flexible function. A new perspective on homogeneous functions through the lens of output elasticities. Math. Soc. Sci. **97**, 11–17 (2019)
7. Vîlcu, G.-E.: On a generalization of a class of production functions. Appl. Econ. Lett. **25**(2), 106–110 (2018)
8. Cobb, C.W., Douglas, P.H.: A theory of production. Am. Econ. Rev. **18**, 139–165 (1928)
9. Vîlcu, A.D., Vîlcu, G.-E.: Some characterizations of the quasi-sum production models with proportional marginal rate of substitution. C. R. Math. Acad. Sci. Paris **353**, 1129–1133 (2015)
10. Vîlcu, G.-E.: A geometric perspective on the generalized Cobb-Douglas production functions. Appl. Math. Lett. **24**(5), 777–783 (2011)
11. Wang, X.: A geometric characterization of homogeneous production models in economics. Filomat **30**(13), 3465–3471 (2016)
12. Arrow, K.J., Chenery, H.B., Minhas, B.S., Solow, R.M.: Capital-labor substitution and economic efficiency. Rev. Econ. Stat. **43**, 225–250 (1961)
13. Chen, B.-Y.: On some geometric properties of quasi-sum production models. J. Math. Anal. Appl. **392**(2), 192–199 (2012)
14. Chen, B.-Y.: Solutions to homogeneous Monge-Ampère equations of homothetic functions and their applications to production models in economics. J. Math. Anal. Appl. **411**, 223–229 (2014)
15. Vîlcu, A.D., Vîlcu, G.-E.: On some geometric properties of the generalized CES production functions. Appl. Math. Comput. **218**(1), 124–129 (2011)
16. Chen, B.-Y., Vîlcu, G.-E.: Geometric classifications of homogeneous production functions. Appl. Math. Comput. **225**, 345–351 (2013)
17. Losonczi, L.: Production functions having the CES property. Acta Math. Acad. Paedagog. Nyházi. **26**(1), 113–125 (2010)
18. Chen, B.-Y.: Classification of *h*-homogeneous production functions with constant elasticity of substitution. Tamkang J. Math. **43**(2), 321–328 (2012)
19. Aydin, M.E., Ergüt, M.: Composite functions with Allen determinants and their applications to production models in economics. Tamkang J. Math. **45**(4), 427–435 (2014)
20. Fu, Y., Wang, W.G.: Geometric characterizations of quasi-product production models in economics. Filomat **31**(6), 1601–1609 (2017)
21. Alodan, H., Chen, B.-Y., Deshmukh, S., Vîlcu, G.-E.: On some geometric properties of quasi-product production models. J. Math. Anal. Appl. **474**(1), 693–711 (2019)
22. Eichhorn, W., Oettli, W.: Mehrproduktunternehmungen mit linearen expansionswegen. Oper.-Res.-Verfahren **6**, 101–117 (1969)
23. Eichhorn, W.: Theorie der homogenen Produktionsfunktion. Springer, Berlin (1970)
24. Jensen, B.: The Dynamic Systems of Basic Economic Growth Models. Mathematics and Its Applications. Springer, Dordrecht (1994)
25. Färe, R.: Ray-homothetic production functions. Econometrica **45**, 133–146 (1977)
26. Mak, K.-T.: General homothetic production correspondences. In: Dogramaci, A., Färe, R. (eds.) Applications of Modern Production Theory: Efficiency and Productivity. Springer, Dordrecht (1988)
27. Shephard, R.: Some remarks on the theory of homogeneous production functions. Z. Nationalökon. **31**, 251–256 (1971)
28. Khatskevich, G.A., Pranevich, A.F.: On quasi-homogeneous production functions with constant elasticity of factors substitution. J. Belarus. State Univ. Econ. **1**, 46–50 (2017)
29. Khatskevich, G.A., Pranevich, A.F.: Quasi-homogeneous production functions with unit elasticity of factors substitution by Hicks. Econ. Simul. Forecast. **11**, 135–140 (2017)
30. Vîlcu, A.D., Vîlcu, G.-E.: On quasi-homogeneous production functions. Symmetry **11**(8), 976 (2019)

31. Anosov, D.V., Aranson, S.K., Arnold, V.I., Bronshtein, I.U., Grines, V.Z., Il'yashenko, Y.S.: Ordinary Differential Equations and Smooth Dynamical Systems. Springer, Berlin (1997)
32. Goriely, A.: Integrability and Nonintegrability of Dynamical Systems. Advanced Series in Nonlinear Dynamics, vol. 19. World Scientific, Singapore (2001)
33. Panayotounakos, D.E.: Exact analytic solutions of unsolvable classes of first and second order nonlinear ODEs I. Abel's equations. Appl. Math. Lett. **18**(2), 155–162 (2005)
34. Polyanin, A.D., Zaitsev, V.F.: Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems, 2nd edn. Chapman & Hall, Boca Raton (2018)
35. Markakis, M.P.: Closed-form solutions of certain Abel equations of the first kind. Appl. Math. Lett. **22**(9), 1401–1405 (2009)
36. Kamke, E.: Losungmethoden und Losungen. Teubner, Stuttgart (1983)