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Existence of solutions for fourth-order nonlinear boundary value problems

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Abstract

In this paper, we discuss the existence and approximation of solutions for a fourth-order nonlinear boundary value problem by using a quasilinearization technique. In the presence of a lower solution α and an upper solution β in the reverse order $\alpha \geq \beta$, we show the existence of (extreme) solution.

Keywords: Boundary value problem; Quasilinearization method; Upper solution and lower solution; Extreme solution

1 Introduction

In this paper, we are concerned with the existence and approximation of solutions for the fourth-order nonlinear boundary value problem

$$\begin{cases} x^{(4)}(t) = f(t, x(t)), & t \in I, \\ x'(0) = A, & x'(1) = B, \\ x'''(0) = C, & x'''(1) = D, \end{cases} \tag{1.1}$$

where $I = [0,1]$, $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, and $A, B, C, D \in \mathbb{R}$.

The quasilinearization method is one of important tools to deal with nonlinear boundary value problems, see [1–5] and the references therein. In [6], Khan studied the second order nonlinear Neumann problem

$$\begin{cases} -x''(t) = f(t, x(t)), & t \in I, \\ x'(0) = A, & x'(1) = B, \end{cases} \tag{1.2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and $A, B \in \mathbb{R}$. By using the quasilinearization technique, the author obtained the existence and approximation of solutions of (1.2) in the presence of a lower solution α and an upper solution β in the reverse order $\alpha \geq \beta$. For the case that a lower solution α is not greater than an upper solution β, we also refer the reader to the papers [7–10].

There are a few papers which studied fourth-order boundary value problems with the help of the quasilinearization technique, see [11–14]. In [13], Ma, Zhang, and Fu discussed a fourth-order boundary value problem

$$\begin{cases} x^{(4)}(t) = g(t, x(t), x''(t)), & t \in I, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{1.3}$$

where $g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. They showed the existence of solutions between a lower solution α and an upper solution β without any growth restriction on g by means of the monotonicity method. Li [14] obtained the existence and uniqueness result for (1.3) by the method of lower and upper solutions in the presence of a lower solution α and an upper solution β with $\alpha \leq \beta$.

Inspired by [6, 14], in this paper, we study the existence of solution for (1.1) in the presence of a lower solution α and an upper solution β in the reverse order $\alpha \geq \beta$.

The paper is organized as follows. In Sect. 2, we establish a comparison principle related to problem (1.1). In Sect. 3, the concept of a lower and upper solution of (1.1) is introduced and the method of a lower and upper solution is mentioned. In Sect. 4, using the approach of quasilinearization, we obtain the existence result of (extreme) solution for (1.1), and we also discuss the quadratic convergence of the approximate sequence.

2 Comparison principle

Consider the linear problems

$$\begin{cases} -x''(t) + Mx(t) = \sigma(t), & t \in I, \\ x'(0) = a, \qquad x'(1) = b \end{cases} \tag{2.1}$$

and

$$\begin{cases} x^{(4)}(t) - \lambda^4 x(t) = h(t), & t \in I, \\ x'(0) = A, \qquad x'(1) = B, \\ x'''(0) = C, \qquad x'''(1) = D, \end{cases} \tag{2.2}$$

where $M, \lambda, a, b, A, B, C, D \in R$, $\sigma, h \in C(I)$.

From [6], if $M \neq -n^2\pi^2$, (2.1) has a unique solution of the form

$$\overline{x}(t) = P^M_{(a,b)}(t) + \int_0^1 G_M(t,s)\sigma(s)\,ds, \tag{2.3}$$

where

$$P^M_{(a,b)}(t) = \begin{cases} \frac{1}{\lambda \sin\lambda}(a\cos\lambda(1-t) - b\cos\lambda t), & M = -\lambda^2, \lambda > 0, \\ \frac{1}{\lambda \sinh\lambda}(b\cosh\lambda t - a\cosh\lambda(1-t)), & M = \lambda^2, \lambda > 0, \end{cases}$$

$$\text{if } M = -\lambda^2 < 0, \quad G_M(t,s) = -\frac{1}{\lambda\sin\lambda}\begin{cases} \cos\lambda(1-s)\cos\lambda t, & 0 \leq t \leq s \leq 1, \\ \cos\lambda(1-t)\cos\lambda s, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$\text{if } M = \lambda^2 > 0, \quad G_M(t,s) = \frac{1}{\lambda\sinh\lambda}\begin{cases} \cosh\lambda(1-s)\cosh\lambda t, & 0 \leq t \leq s \leq 1, \\ \cosh\lambda(1-t)\cosh\lambda s, & 0 \leq s \leq t \leq 1. \end{cases}$$

Lemma 2.1 ([6]) *(i) Let $-\frac{\pi^2}{4} \leq M < 0$. If $a \leq 0 \leq b$ and $\sigma(t) \geq 0$, then $\overline{x}(t) \leq 0$ for all $t \in I$. If $b \leq 0 \leq a$ and $\sigma(t) \leq 0$, then $\overline{x}(t) \geq 0$ for all $t \in I$.*

(ii) Let $M > 0$. If $a \leq 0 \leq b$ and $\sigma(t) \geq 0$, then $\overline{x}(t) \geq 0$ for all $t \in I$. If $b \leq 0 \leq a$ and $\sigma(t) \leq 0$, then $\overline{x}(t) \leq 0$ for all $t \in I$.

Lemma 2.2 *Let $0 < \lambda \leq \frac{\pi}{2}$. Then (2.2) has a unique solution $\widetilde{x}(t)$. Moreover, $\widetilde{x}(t) \geq 0$ for all $t \in I$ if $B \leq 0 \leq A$, $C + \lambda^2 A \leq 0 \leq D + \lambda^2 B$ and $h(t) \leq 0$; $\widetilde{x}(t) \leq 0$ for all $t \in I$ if $A \leq 0 \leq B$, $D + \lambda^2 B \leq 0 \leq C + \lambda^2 A$, $h(t) \geq 0$.*

Proof Letting $y(t) = -x''(t) - \lambda^2 x(t)$, we get

$$\begin{cases} -y''(t) + \lambda^2 y(t) = h(t), & t \in I, \\ y'(0) = -C - \lambda^2 A, & y'(1) = -D - \lambda^2 B \end{cases} \tag{2.4}$$

and

$$\begin{cases} -x''(t) - \lambda^2 x(t) = y(t), & t \in I, \\ x'(0) = A, & x'(1) = B. \end{cases} \tag{2.5}$$

Hence

$$\widetilde{x}(t) = P_{(A,B)}^{-\lambda^2}(t) + \int_0^1 G_{-\lambda^2}(t,s) y(s)\, ds$$

is a solution of (2.2), where

$$y(t) = P_{(-C-\lambda^2 A, -D-\lambda^2 B)}^{\lambda^2}(t) + \int_0^1 G_{\lambda^2}(t,s) h(s)\, ds.$$

Clearly, the solution of (2.2) is unique since the solution of (2.4) or (2.5) is unique.

If $C + \lambda^2 A \leq 0 \leq D + \lambda^2 B$ and $h(t) \leq 0$, using (ii) of Lemma 2.1, we get that $y(t) \leq 0$. Together with $B \leq 0 \leq A$, we obtain that $\widetilde{x}(t) \geq 0$ by (i) of Lemma 2.1. Similarly, $\widetilde{x}(t) \leq 0$ if $A \leq 0 \leq B$, $D + \lambda^2 B \leq 0 \leq C + \lambda^2 A$, $h(t) \geq 0$. This completes the proof. □

3 Lower and upper solutions

Definition 3.1 Function $\alpha \in C^4(I)$ is called a lower solution of (1.1) if

$$\begin{cases} \alpha^{(4)}(t) \leq f(t, \alpha(t)), & t \in I, \\ \alpha'(0) = A, & \alpha'(1) = B, \\ \alpha'''(0) \leq C, & \alpha'''(1) \geq D. \end{cases}$$

An upper solution $\beta \in C^4(I)$ of (1.1) is defined similarly by reversing the inequalities.

Theorem 3.1 *Let $0 < \lambda \leq \frac{\pi}{2}$. Suppose that α and β are respectively lower and upper solutions of (1.1) such that $\alpha(t) \geq \beta(t)$, $t \in I$. If $f(t,x) - \lambda^4 x$ is nonincreasing in x, then there exists a solution $x \in C^4(I)$ of (1.1) such that*

$$\beta(t) \leq x(t) \leq \alpha(t), \quad t \in I.$$

Proof Define $p(\alpha(t),x,\beta(t)) = \min\{\alpha(t),\max\{x,\beta(t)\}\}$. Then $p(\alpha(t),x,\beta(t))$ is continuous and satisfies $\beta(t) \le p(\alpha(t),x,\beta(t)) \le \alpha(t)$ for all $x \in \mathbb{R}$, $t \in I$. Consider the boundary value problem

$$\begin{cases} x^{(4)}(t) - \lambda^4 x(t) = \psi(t,x(t)), & t \in I, \\ x'(0) = A, \qquad x'(1) = B, \\ x'''(0) = C, \qquad x'''(1) = D, \end{cases} \tag{3.1}$$

where $\psi(t,x) = f(t,p(\alpha(t),x,\beta(t))) - \lambda^4 p(\alpha(t),x,\beta(t))$. Problem (3.1) is equivalent to the integral equation

$$\begin{aligned} &x = Tx, \\ &(Tx)(t) := P^{-\lambda^2}_{(A,B)}(t) + \int_0^1 G_{-\lambda^2}(t,\tau)\biggl\{P^{\lambda^2}_{(-C-\lambda^2 A,-D-\lambda^2 B)}(\tau) \\ &\qquad + \int_0^1 G_{\lambda^2}(\tau,s)\psi\bigl(s,x(s)\bigr)\,ds\biggr\}\,d\tau. \end{aligned}$$

Since α, β, ψ, $P^{-\lambda^2}_{(A,B)}$, $P^{\lambda^2}_{(-C-\lambda^2 A,-D-\lambda^2 B)}$, $G_{-\lambda^2}$ and G_{λ^2} are continuous, there exist constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} &\bigl|\psi(t,x)\bigr| \le c_1, \quad t \in I, x \in \mathbb{R}, \\ &\bigl|P^{-\lambda^2}_{(A,B)}(t)\bigr| \le c_2, \qquad \bigl|P^{\lambda^2}_{(-C-\lambda^2 A,-D-\lambda^2 B)}(t)\bigr| \le c_2, \quad t \in I, \\ &\bigl|G_{-\lambda^2}(t,s)\bigr| \le c_3, \qquad \bigl|G_{\lambda^2}(t,s)\bigr| \le c_3, \quad t,s \in I. \end{aligned}$$

Let $\|u\| = \max_{t\in I}|u(t)|$ and $\Omega = \{u \in C(I) : \|u\| \le c_2 + c_3(c_2 + c_3 c_1)\}$. It is easy to show that $T : \Omega \to \Omega$ is continuous and compact. Hence, T has a fixed point $x \in \Omega$ by the Schauder fixed point theorem. Moreover, $x \in C^4(I)$ is a solution of (3.1).

Let $v(t) = \alpha(t) - x(t)$, $t \in I$. Then $v'(0) = 0 = v'(1)$ and $v'''(0) + \lambda^2 v'(0) \le 0 \le v'''(1) + \lambda^2 v'(1)$. Since $f(t,x) - \lambda^4 x$ is nonincreasing in x and $p(\alpha(t),x,\beta(t)) \le \alpha(t)$, $t \in I$, we can see that

$$\begin{aligned} &v^{(4)}(t) - \lambda^4 v(t) \\ &\quad = \bigl(\alpha^{(4)}(t) - \lambda^4\alpha(t)\bigr) - \bigl(x^{(4)}(t) - \lambda^4 x(t)\bigr) \\ &\quad \le \bigl(f\bigl(t,\alpha(t)\bigr) - \lambda^4\alpha(t)\bigr) - \bigl(f\bigl(t,p\bigl(\alpha(t),x(t),\beta(t)\bigr)\bigr) - \lambda^4 p\bigl(\alpha(t),x(t),\beta(t)\bigr)\bigr) \\ &\quad \le 0, \end{aligned}$$

which implies $\alpha(t) \ge x(t)$, $t \in I$. Similarly, $x(t) \ge \beta(t)$, $t \in I$. Hence,

$$\begin{aligned} x^{(4)}(t) - \lambda^4 x(t) &= f\bigl(t,p\bigl(\alpha(t),x(t),\beta(t)\bigr)\bigr) - \lambda^4 p\bigl(\alpha(t),x(t),\beta(t)\bigr) \\ &= f\bigl(t,x(t)\bigr) - \lambda^4 x(t), \quad t \in I, \end{aligned}$$

that is, x is a solution of (1.1). This completes the proof. □

Theorem 3.2 *Assume that α and β are respectively lower and upper solutions of problem* (1.1). *If $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and for some $0 < \lambda \leq \frac{\pi}{2}$,*

$$f\big(t,\alpha(t)\big) - \lambda^4\alpha(t) \leq f\big(t,\beta(t)\big) - \lambda^4\beta(t),$$

then $\alpha(t) \geq \beta(t)$, $t \in I$.

Proof Let $m(t) = \alpha(t) - \beta(t)$, $t \in I$. Then $m(t) \in C^4(I)$, $m'(0) = 0 = m'(1)$, $m'''(0) \leq 0 \leq m'''(1)$, and $m'''(0) + \lambda^2 m'(0) \leq 0 \leq m'''(1) + \lambda^2 m'(1)$. Using the definition of lower and upper solutions, we have

$$\begin{aligned} m^{(4)}(t) - \lambda^4 m(t) &= \big(\alpha^{(4)}(t) - \lambda^4\alpha(t)\big) - \big(\beta^{(4)}(t) - \lambda^4\beta(t)\big) \\ &\leq \big(f\big(t,\alpha(t)\big) - \lambda^4\alpha(t)\big) - \big(f\big(t,\beta(t)\big) - \lambda^4\beta(t)\big) \\ &\leq 0, \end{aligned}$$

which implies that $m(t) \geq 0$, $t \in I$ by Lemma 2.2. This completes the proof. □

4 Main results

To prove the main theorem, we need the following assumptions:

(H_1) The functions $\alpha, \beta \in C^4(I)$ are respectively lower and upper solutions of (1.1), and $\alpha(t) \geq \beta(t)$, $t \in I$.

(H_2) $f \in C^2(I \times \mathbb{R}, \mathbb{R})$ and $0 < f_x(t,x) \leq (\frac{\pi}{2})^4, f_{xx}(t,x) \leq 0$ for $(t,x) \in I \times [\min\beta(t), \max\alpha(t)]$.

(H_3) There exists a constant $k \in (0, \frac{\pi}{2}]$ such that

$$f(t,x_1) - f(t,x_2) \leq k^4(x_1 - x_2)$$

for $\beta(t) \leq x_2 \leq x_1 \leq \alpha(t)$, $t \in I$.

Theorem 4.1 *Let (H_1) and (H_2) hold. Then there exists a monotone sequence $\{\omega_n\}$ converging uniformly and quadratically to a solution of* (1.1).

Proof Taylor's theorem and condition (H_2) imply that

$$f(t,x) = f(t,y) + f_x(t,y)(x-y) + \frac{f_{xx}(t,\zeta)}{2!}(x-y)^2 \leq f(t,y) + f_x(t,y)(x-y),$$

for $(t,x), (t,y) \in I \times [\min\beta(t), \max\alpha(t)]$, where $\zeta \in (\min\{x,y\}, \max\{x,y\})$. Define

$$F(t,x,y) = f(t,y) + f_x(t,y)(x-y), \quad x,y \in \mathbb{R}, t \in I,$$

then $F(t,x,y)$ is continuous and satisfies the relations

$$f(t,x) \leq F(t,x,y), \qquad f(t,x) = F(t,x,x)$$

for $(t,x), (t,y) \in I \times [\min\beta(t), \max\alpha(t)]$.

Let $\lambda > 0$ and $\lambda^4 = \max\{f_x(t,x) : (t,x) \in I \times [\min \beta(t), \max \alpha(t)]\}$. Then $0 < \lambda \leq \frac{\pi}{2}$. Put $\varphi \in [\beta, \alpha] = \{x \in C^4(I) : \beta(t) \leq x(t) \leq \alpha(t), t \in I\}$ and consider the problem

$$\begin{cases} x^{(4)}(t) = H_\varphi(t,x) := \lambda^4 x + F(t, p(\alpha, x, \varphi), \varphi) - \lambda^4 p(\alpha, x, \varphi), \quad t \in I, \\ x'(0) = A, \quad x'(1) = B, \quad x'''(0) = C, \quad x'''(1) = D. \end{cases} \tag{4.1}$$

Clearly,

$$\begin{cases} \alpha^{(4)}(t) \leq f(t,\alpha) \leq F(t,\alpha,\varphi) = \lambda^4 \alpha + F(t, p(\alpha,\alpha,\varphi), \varphi) - \lambda^4 p(\alpha,\alpha,\varphi), \\ \alpha'(0) = A, \quad \alpha'(1) = B, \quad \alpha'''(0) \leq C, \quad \alpha'''(1) \geq D, \end{cases}$$

$$\begin{cases} \beta^{(4)}(t) \geq f(t,\beta) \geq f(t,\varphi) + \lambda^4(\beta - \varphi) = F(t,\varphi,\varphi) + \lambda^4 \beta - \lambda^4 \varphi \\ \qquad = \lambda^4 \beta(t) + F(t, p(\alpha,\beta,\varphi), \varphi) - \lambda^4 p(\alpha,\beta,\varphi), \\ \beta'(0) = A, \quad \beta'(1) = B, \quad \beta'''(0) \geq C, \quad \beta'''(1) \leq D, \end{cases}$$

that is, α and β are respectively lower and upper solutions of (4.1).

On the other hand, the function

$$H_\varphi(t,x) - \lambda^4 x = f(t,\varphi) - f_x(t,\varphi)\varphi + \bigl(f_x(t,\varphi) - \lambda^4\bigr) p(\alpha, x, \varphi)$$

is nonincreasing in x. From Theorem 3.1, (4.1) has a solution $\omega_1 \in [\beta, \alpha]$. Moreover, ω_1 is an upper solution of (4.1), which implies that

$$\begin{cases} x^{(4)}(t) = H_{\omega_1}(t,x), \\ x'(0) = A, \quad x'(1) = B, \quad x'''(0) = C, \quad x'''(1) = D \end{cases} \tag{4.2}$$

has a solution $\omega_2 \in [\omega_1, \alpha]$. Repeating the process, we obtain a sequence $\{\omega_n\}$ satisfying

$$\begin{aligned} &\beta(t) \leq \omega_1(t) \leq \cdots \leq \omega_n(t) \leq \alpha(t), \\ &\omega_n^{(4)} = H_{\omega_{n-1}}(t, \omega_n) = F(t, \omega_n, \omega_{n-1}), \\ &\omega_n'(0) = A, \quad \omega_n'(1) = B, \quad \omega_n'''(0) = C, \quad \omega_n'''(1) = D, \end{aligned}$$

and $\{\omega_n\}$ is uniformly convergent. Let $\lim_{n\to\infty} \omega_n(t) = x$. Since F is continuous, we have

$$\lim_{n\to\infty} F(t, \omega_n, \omega_{n-1}) = F(t, x, x) = f(t, x),$$

which implies that x is a solution of problem (1.1).

To show that the convergence of the sequence $\{\omega_n\}$ is quadratic, we begin by writing $e_n(t) = x(t) - \omega_n(t)$, $t \in I$, $n \in \mathbb{N}^+$, where x is a solution of (1.1). It is clear that $e_n \geq 0$ on I and $e_n'(0) = e_n'(1) = e_n'''(0) = e_n'''(1) = 0$. Let $\rho^4 = \min\{f_x(t,x) : (t,x) \in I \times [\min \beta(t), \max \alpha(t)]\}$.

Then $0 < \rho \le \frac{\pi}{2}$. In view of Taylor's theorem, we obtain

$$\begin{aligned} e_n^{(4)} &= x^{(4)} - \omega_n^{(4)} = f(t,x) - F(t,\omega_n,\omega_{n-1}) \\ &= f(t,\omega_{n-1}) + f_x(t,\omega_{n-1})(x-\omega_{n-1}) + \frac{f_{xx}(t,\xi)}{2!}(x-\omega_{n-1})^2 \\ &\quad - \bigl[f(t,\omega_{n-1}) + f_x(t,\omega_{n-1})(\omega_n - \omega_{n-1})\bigr] \\ &= f_x(t,\omega_{n-1})e_n + \frac{f_{xx}(t,\xi)}{2!}e_{n-1}^2 \\ &\ge \rho^4 e_n + \frac{f_{xx}(t,\xi)}{2!}\|e_{n-1}\|^2, \quad t \in I, \end{aligned}$$

where $\omega_{n-1}(t) < \xi(t) < x(t)$, $t \in I$. Let $\gamma(t)$ be the unique solution of the boundary value problem

$$\begin{cases} \gamma^{(4)} - \rho^4\gamma = \frac{f_{xx}(t,\xi(t))}{2!}\|e_{n-1}\|^2, & t \in I, \\ \gamma'(0) = 0, & \gamma'(1) = 0, \\ \gamma'''(0) = 0, & \gamma'''(1) = 0. \end{cases}$$

Similar to (3.1), γ satisfies

$$\gamma(t) = \int_0^1 G_{-\rho^2}(t,\tau)\left\{\int_0^1 G_{\rho^2}(\tau,s)\frac{f_{xx}(s,\xi(s))}{2!}\|e_{n-1}\|^2\,ds\right\}d\tau \le \delta\|e_{n-1}\|^2,$$

where

$$\delta = \frac{1}{2}\max\left\{\left|\int_0^1 G_{-\rho^2}(t,\tau)\int_0^1 G_{\rho^2}(\tau,s)f_{xx}(s,x)\,ds\,d\tau\right| : (t,x) \in I \times [\min\beta, \max\alpha]\right\}.$$

Setting $K_n(t) = e_n(t) - \gamma(t)$, $t \in I$, we get $K_n'(0) = K_n'(1) = K_n'''(1) = K_n'''(0) = 0$ and $K_n^{(4)} - \rho^4 K_n \ge 0$ on I. By Lemma 2.2, we easily obtain $e_n(t) \le \gamma(t)$, $t \in I$. Thus $\|e_n\| \le \delta\|e_{n-1}\|^2$ and we conclude that the convergence of the sequence $\{\omega_n\}$ is quadratic. This completes the proof. □

Remark 4.1 In (H_2), if the assumption $f_{xx}(t,x) \le 0$ is replaced by $f_{xx}(t,x) \ge 0$ for $(t,x) \in I \times [\min\beta(t), \max\alpha(t)]$, and we let the other assumptions in Theorem 4.1 hold, then

$$f(t,x) \ge F(t,x,y), \qquad f(t,x) = F(t,x,x)$$

for $x, y \in [\min\beta(t), \max\alpha(t)]$, $t \in I$. One can obtain a monotonically nonincreasing sequence $\{\omega_n\}$ of solutions of (4.1) with

$$\beta(t) \le \omega_n(t) \le \cdots \le \omega_1(t) \le \alpha(t), \quad t \in I,$$

which converges uniformly and quadratically to a solution of (1.1).

Theorem 4.2 *Let (H_1) and (H_3) hold. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty}\alpha_n(t) = u(t)$, $\lim_{n\to\infty}\beta_n(t) = r(t)$ uniformly on I, and r, u*

are the minimal and maximal solutions of (1.1), *respectively, such that*

$$\beta_0 \le \beta_1 \le \beta_2 \le \cdots \beta_n \le r \le x \le u \le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1 \le \alpha_0$$

on I, *where* x *is a solution of* (1.1) *such that* $\beta(t) \le x(t) \le \alpha(t)$, $t \in I$.

Proof For any $\phi \in [\beta, \alpha]$, we consider the equation

$$\begin{cases} x^{(4)}(t) - k^4 x(t) = f(t,\phi(t)) - k^4\phi(t), & t \in I, \\ x'(0) = A, \qquad x'(1) = B, \\ x'''(0) = C, \qquad x'''(1) = D. \end{cases} \tag{4.3}$$

From (H_3), we have

$$\begin{cases} \alpha^{(4)} - k^4\alpha \le f(t,\alpha) - k^4\alpha \le f(t,\phi) - k^4\phi, & t \in I, \\ \alpha'(0) = A, \qquad \alpha'(1) = B, \qquad \alpha'''(0) \le C, \qquad \alpha'''(1) \ge D, \end{cases}$$

$$\begin{cases} \beta^{(4)} - k^4\beta \ge f(t,\beta) - k^4\beta \ge f(t,\phi) - k^4\phi, & t \in I, \\ \beta'(0) = A, \qquad \beta'(1) = B, \qquad \beta'''(0) \ge C, \qquad \beta'''(1) \le D. \end{cases}$$

Hence, α, β are respectively lower and upper solution of (4.3). By Lemma 2.2 and Theorem 3.1, (4.3) has a unique solution $x \in [\beta, \alpha]$. We can define an operator Q by $x = Q\phi$ and Q is an operator from $[\beta, \alpha]$ to $[\beta, \alpha]$. Thus $Q\alpha \le \alpha$, $\beta \le Q\beta$.

Now, we prove that Q is nondecreasing in $[\beta, \alpha]$. Let $\beta \le \mu_1 \le \mu_2 \le \alpha$ and $\eta = Q\mu_1 - Q\mu_2$. Then, by (H_3), we have

$$\eta^{(4)} - k^4\eta = f(t,\mu_1) - k^4\mu_1 - f(t,\mu_2) + k^4\mu_2 \ge 0,$$
$$\eta'(0) = \eta'(1) = \eta'''(0) = \eta'''(1) = 0.$$

By Lemma 2.2, $\eta(t) \le 0$, which implies $Q\mu_1 \le Q\mu_2$.

Define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = Q\alpha_n$, $\beta_{n+1} = Q\beta_n$ for $n = 0, 1, 2, \ldots$ From the fact that $Q\alpha \le \alpha$, $\beta \le Q\beta$ and the monotonicity of Q, we have

$$\beta_0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_n \le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1 \le \alpha_0$$

on I and

$$\begin{aligned} \alpha_n(t) = P^{-k^2}_{(A,B)}(t) + \int_0^1 G_{-k^2}(t,\tau)\bigg\{ & P^{k^2}_{(-C-k^2A,-D-k^2B)}(\tau) \\ & + \int_0^1 G_{k^2}(\tau,s)\big[f\big(s,\alpha_{n-1}(s)\big) - k^4\alpha_{n-1}(s)\big]\,ds\bigg\}\,d\tau, \\ \beta_n(t) = P^{-k^2}_{(A,B)}(t) + \int_0^1 G_{-k^2}(t,\tau)\bigg\{ & P^{k^2}_{(-C-k^2A,-D-k^2B)}(\tau) \\ & + \int_0^1 G_{k^2}(\tau,s)\big[f\big(s,\beta_{n-1}(s)\big) - k^4\beta_{n-1}(s)\big]\,ds\bigg\}\,d\tau. \end{aligned}$$

Therefore $\lim_{n\to\infty}\alpha_n(t) = u(t)$, $\lim_{n\to\infty}\beta_n(t) = r(t)$ uniformly on I. Clearly, u, r are solutions of (1.1).

Finally, we prove that if $x \in [\beta,\alpha]$ is a solution of (1.1), then $r \le x \le u$ on I. To this end, we assume that, without loss of generality, $\beta_n(t) \le x(t) \le \alpha_n(t)$ for some n. We also know that $\beta_{n+1} \le x \le \alpha_{n+1}$ on I from the monotonicity of Q. Since $\beta_0 \le x \le \alpha_0$ on I, we can conclude that

$$\beta_n(t) \le x(t) \le \alpha_n(t), \quad t \in I, \text{ for all } n.$$

Passing to the limit as $n \to \infty$, we obtain that $r \le x \le u$. This completes the proof. □

Example 4.1 Consider the following problem:

$$\begin{cases} x^{(4)}(t) = f(t,x(t)) := \frac{1}{10}\left(x^2(t) - \sqrt[3]{\frac{t+1}{3} + x(t)}\right), & t \in I, \\ x'(0) = x'(1) = x'''(0) = x'''(1) = 0. \end{cases} \tag{4.4}$$

Let $\alpha = \sqrt{2}$ and $\beta = \frac{1}{2}$. It is easy to check that α, β are respectively lower and upper solutions of (4.4). Moreover, for

$$f_x(t,u) = \frac{1}{5}u - \frac{1}{30}\left(\frac{t+1}{3} + u\right)^{-\frac{2}{3}}, \qquad f_{xx}(t,u) = \frac{1}{5} + \frac{1}{45}\left(\frac{t+1}{3} + u\right)^{-\frac{5}{3}},$$

we can easily check that $0 < f_x(t,u) \le (\frac{\pi}{2})^4$ and $f_{xx}(t,u) \ge 0$ for $(t,u) \in I \times [\frac{1}{2},\sqrt{2}]$. By Remark 4.1, there exists a monotonically nonincreasing sequence $\{\omega_n\}$ which converges uniformly and quadratically to a solution of (4.4).

Example 4.2 Consider the following problem:

$$\begin{cases} x^{(4)}(t) = f(t,x(t)) := c(4 + x^2(t))\left[\frac{1+t}{4} - \arctan\frac{1}{(x(t)-t)^2+c}\right], & t \in I, \\ x'(0) = x'(1) = 1, \qquad x'''(0) = x'''(1) = 0, \end{cases} \tag{4.5}$$

where $c > 0$ is sufficiently small.

Let $\alpha(t) = A + t$ and $\beta(t) = t - c\frac{(t-t^2)^2}{24}$, $t \in I$, here $A > \sqrt{(\tan 0.25)^{-1}}$. We have $\alpha'(0) = \alpha'(1) = 1$, $\alpha'''(0) = \alpha'''(1) = 0$,

$$f(t,\alpha) = c\left(4 + \alpha^2\right)\left(\frac{1+t}{4} - \arctan\frac{1}{A^2 + c}\right) \ge c\left(4 + \alpha^2\right)\left(\frac{1}{4} - \arctan\frac{1}{A^2}\right) > 0 = \alpha^{(4)}(t),$$

which means that α is a lower solution of (4.5).

Moreover, $\beta'(0) = \beta'(1) = 1$, $\beta'''(0) = \frac{c}{2} > 0$, $\beta'''(1) = -\frac{c}{2} < 0$, and

$$f(t,\beta) = c\left(4 + \beta^2\right)\left\{\frac{1+t}{4} - \arctan\frac{1}{c[c(\frac{t-t^2}{24})^2 + 1]}\right\}.$$

There exist a constant $c^* > 0$ such that $\arctan\frac{1}{c} > \frac{\pi}{4}$ if $c \in (0,c^*)$. For such $c \in (0,c^*)$,

$$f(t,\beta) < 4c\left(\frac{1}{2} - \frac{\pi}{4}\right) = c(2 - \pi) < -c = \beta^{(4)}(t),$$

hence β is an upper solution of (4.5) and (H_1) is satisfied.

On the other hand, for $\forall(t,u) \in I \times [\min \beta(t), \max \alpha(t)]$, we have

$$\left|f_x(t,u)\right| \leq \frac{1}{2}\sqrt{c}\left(A^2 + 4A + 7\right).$$

Hence (H_3) is satisfied for $c > 0$ sufficiently small. By Theorem 4.2, (4.5) has the maximal and minimal solutions on $[\beta, \alpha]$.

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References

1. Mohamed, E.G., Donal, O.R.: A quasilinearization method for a class of second order singular nonlinear differential equations with nonlinear boundary conditions. Nonlinear Anal., Real World Appl. **8**(1), 174–186 (2007)
2. Korman, P.: A maximum principle for fourth order ordinary differential equations. Appl. Anal. **33**(3–4), 267–273 (1989)
3. Sreedhar, C.V., Devi, J.V.: Generalized quasilinearization using coupled lower and upper solutions for periodic boundary value problem of an integro differential equation. Eur. J. Pure Appl. Math. **12**(4), 1662–1675 (2019)
4. Lakshmikantham, V., Vatsala, A.S.: Generalized Quasilinearization for Nonlinear Problems. Springer, New York (1998)
5. Wang, W.B., Shen, J.H., Luo, Z.G.: Multi-point boundary value problems for second-order functional differential equations. Comput. Math. Appl. **56**(8), 2065–2072 (2008)
6. Khan, R.A.: Existence and approximation of solutions of second order nonlinear Neumann problems. Electron. J. Differ. Equ. **2005**, 3 (2005)
7. Ahmad, B., Khan, R.A., Eloe, P.W.: Generalized quasilinearization method for a second order three point boundary-value problem with nonlinear boundary conditions. Electron. J. Differ. Equ. **2002**, 90 (2002)
8. Ahmad, B., Khan, R.A., Sivasundaram, S.: Generalized quasilinearization method for nonlinear functional differential equations. J. Appl. Math. Stoch. Anal. **16**(1), 33–43 (2003)
9. Nieto, J.J.: Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions. Proc. Am. Math. Soc. **125**(9), 2599–2604 (1997)
10. Wang, W.L., Tian, J.F.: Generalized monotone iterative method for nonlinear boundary value problems with causal operators. Bound. Value Probl. **2014**(1), 192 (2014)
11. Li, F.F., Sun, J.T., Jia, M.: Monotone iterative method for the second-order three-point boundary value problem with upper and lower solutions in the reversed order. Appl. Math. Comput. **217**(9), 4840–4847 (2011)
12. Yang, J., Song, N.N., Jin, Y.: Upper and lower solution method for a fourth-order four-point boundary value problem on time scales. Math. Pract. Theory **43**(21), 205–211 (2013) (Chinese)
13. Ma, R.Y., Zhang, J.H., Fu, S.M.: The method of lower and upper solutions for fourth-order two-point boundary value problems. J. Math. Anal. Appl. **215**(2), 415–422 (1997)
14. Li, Y.X.: Existence and method of lower and upper solutions for fourth-order nonlinear boundary value problems. Acta Math. Sci. Ser. A Chin. Ed. **23**(2), 245–252 (2003)