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Research on Sturm–Liouville boundary value problems of fractional p-Laplacian equation

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Abstract

In this work we investigate the following fractional p-Laplacian differential equation with Sturm–Liouville boundary value conditions:

$$\begin{cases} {}_tD_T^{\alpha}(\frac{1}{(h(t))^{p-2}}\phi_p(h(t){}_0^CD_t^{\alpha}u(t))) + a(t)\phi_p(u(t)) = \lambda f(t,u(t)), & \text{a.e. } t \in [0,T], \\ \alpha_1\phi_p(u(0)) - \alpha_2 {}_tD_T^{\alpha-1}(\phi_p({}_0^CD_t^{\alpha}u(0))) = 0, \\ \beta_1\phi_p(u(T)) + \beta_2 {}_tD_T^{\alpha-1}(\phi_p({}_0^CD_t^{\alpha}u(T))) = 0, \end{cases}$$

where ${}_0^CD_t^{\alpha}$, ${}_tD_T^{\alpha}$ are the left Caputo and right Riemann–Liouville fractional derivatives of order $\alpha \in (\frac{1}{2}, 1]$, respectively. By using variational methods and critical point theory, some new results on the multiplicity of solutions are obtained.

MSC: 34A08; 34B15

Keywords: Fractional p-Laplacian equation; Sturm–Liouville boundary value conditions; Multiplicity of solutions; Variational methods; Critical point theory

1 Introduction

Fractional differential equations have been extensively applied in mathematical modeling. Many scholars have developed a strong interest in this kind of problem and achieved some excellent results [1–8]. Especially, in the last several years, the investigations on the equations including both left and right fractional differential operators have got increasing attention. Left and right fractional differential operators are widely used in the physical phenomena of anomalous diffusion, such as fractional convection diffusion equation [9, 10]. In [11], Ervin and Roop first proposed a class of steady-state fractional convection-diffusion equations with variational structure

$$\begin{cases} -aD(p_0D_t^{-\beta} + q_tD_T^{-\beta})Du + b(t)Du + c(t)u = f, \\ u(0) = u(T) = 0, \end{cases}$$

where $0 \le \beta < 1$, D is the classical first derivative, ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann–Liouville fractional derivatives. The authors constructed a suitable fractional derivative space. The main research method is the Lax–Milgram theorem.

Jiao and Zhou [12] considered the Dirichlet problems

$$\begin{cases} \frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))) + \nabla F(t,u(t)) = 0, & \text{a.e. } t \in [0,T], 0 \leq \beta < 1, \\ u(0) = u(T) = 0. \end{cases}$$

The authors gave the variational structure of the problem. Under the Ambrosetti–Rabinowitz condition, the existence results were obtained by employing the mountain pass theorem and the minimization principle. The following year, the authors [13] further studied the following problems:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \frac{1}{2} < \alpha \leq 1, \\ u(0) = u(T) = 0. \end{cases}$$

Under the Ambrosetti–Rabinowitz condition, the existence of weak solution was obtained by using the mountain pass theorem. In addition, the authors also discussed the regularity of weak solution.

Bonanno et al. [14] and Rodríguez-López and Tersian [15] considered the Dirichlet problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), & t \neq t_j, \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), & j = 1,2,\ldots,n, \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (\frac{1}{2},1]$, $\lambda,\mu \in (0,+\infty)$, $f \in C([0,T] \times \mathbb{R},\mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$, $j = 1,2,\ldots,n$. $a \in C([0,T])$, and there exist a_1, a_2 such that $0 < a_1 \leq a(t) \leq a_2$. In addition,

$$\begin{aligned} &\Delta\bigl({}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)\bigr)(t_j) = {}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)\bigl(t_j^+\bigr) - {}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)\bigl(t_j^-\bigr), \\ &{}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)\bigl(t_j^+\bigr) = \lim_{t\to t_j^+}\bigl({}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)(t)\bigr), \\ &{}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)\bigl(t_j^-\bigr) = \lim_{t\to t_j^-}\bigl({}_tD_T^{\alpha-1}\bigl({}_0^CD_t^{\alpha}u\bigr)(t)\bigr). \end{aligned}$$

By employing variational methods and three critical points theorem, the existence results of solution were obtained.

Tian and Nieto [16] studied the Sturm–Liouville boundary value problems

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))) = \lambda f(u(t)), & \text{a.e. } t \in [0,T], \\ au(0) - b(\frac{1}{2}\,{}_0D_t^{-\beta}u'(0) + \frac{1}{2}\,{}_tD_T^{-\beta}u'(0)) = 0, \\ cu(T) + d(\frac{1}{2}\,{}_0D_t^{-\beta}u'(T) + \frac{1}{2}\,{}_tD_T^{-\beta}u'(T)) = 0, \end{cases}$$

where $0 \leq \beta < 1$, $a,c > 0$, $b,d \geq 0$, $\lambda > 0$. The variational structure of the problem was established and the existence result of the unbounded sequence of the solution was obtained by employing the critical point theory. Subsequently, Nyamoradi Nemat and Tersian Stepan

[17] further considered the Sturm–Liouville problems with p-Laplacian operators

$$\begin{cases} {}_tD_T^\alpha\big(\frac{1}{(h(t))^{p-2}}\phi_p(h(t){}_0^CD_t^\alpha u(t))\big) + a(t)\phi_p(u(t)) = \lambda f(t,u(t)), & \text{a.e. } t\in[0,T], \\ \alpha_1\phi_p(u(0)) - \alpha_{2t}D_T^{\alpha-1}(\phi_p({}_0^CD_t^\alpha u(0))) = 0, \\ \beta_1\phi_p(u(T)) + \beta_{2t}D_T^{\alpha-1}(\phi_p({}_0^CD_t^\alpha u(T))) = 0, \end{cases} \tag{1.1}$$

where $\alpha\in(\frac{1}{2},1]$, ${}_0^CD_t^\alpha$ is the left Caputo fractional derivative, ${}_tD_T^\alpha$ is the right Riemann–Liouville fractional derivative. $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$, $h(t)\in L^\infty([0,T],\mathbb{R})$ with $h_0 = \operatorname{ess\,inf}_{[0,T]} h(t) > 0$, $a\in C([0,T],\mathbb{R})$ with $a_0 = \operatorname{ess\,inf}_{[0,T]} a(t) > 0$, there exist a_1, a_2 such that $0 < a_1 \le a(t) \le a_2$, $\lambda > 0$, $f\in C([0,T]\times\mathbb{R},\mathbb{R})$, $\phi_p(x) = |x|^{p-2}x$ $(x\neq 0)$, $\phi_p(0) = 0$, $p > 1$. To illustrate the main results of [17], we first introduce the following hypothesis about f:

(F_1) There exists $\mu > p$ such that

$$0 < \mu F(t,\tau) \le \tau f(t,\tau), \quad \forall\tau\in\mathbb{R}, t\in[0,T],$$

where $F(t,\tau) = \int_0^\tau f(t,s)\,ds$;

(F_2) $c_{\inf} := \inf_{|\tau|=1} F(t,\tau) > 0$;

(F_3) There exist c_ν, $\nu > p-1$ such that

$$\big|f(t,u)\big| \le c_\nu|u|^\nu, \quad \forall(t,u)\in[0,T]\times\mathbb{R};$$

(F_4) $F(t,u) = o(|u|^p)$ as $|u|\to 0$ uniformly with respect to $\forall t\in[0,T]$.

Theorem 1.0 (see [17]) *Assume that* (F_1)–(F_4) *hold. Then* (1.1) *with* $\lambda = 1$ *has at least a solution.*

Based on the above work, this article further studies problem (1.1) with the concave-convex nonlinearity. In order to compare the results of this paper with Theorem 1.0, the main assumptions and conclusions of this paper are given below. In this paper, we study the case that the nonlinearity $f\in C([0,T]\times\mathbb{R},\mathbb{R})$ involves a combination of p-suplinear and p-sublinear terms. That is,

$$f(t,u) = f_1(t,u) + f_2(t,u), \tag{1.2}$$

where $f_1(t,u)$ is p-suplinear as $|u|\to\infty$ and $f_2(t,u)$ is p-sublinear growth at infinity. Here we give some reasonable assumptions on f_1 and f_2 as follows:

(H_1) $f_1(t,x) = o(|x|^{p-1})$ as $(|x|\to 0)$ uniformly with respect to $\forall t\in[0,T]$;

(H_2) There exist $d_1 > 0$, $d_\infty > 0$, $\theta > p$ such that

$$xf_1(t,x) - \theta F_1(t,x) \ge -d_1|x|^p, \quad \forall t\in[0,T], |x|\ge d_\infty,$$

where $F_1(t,x) = \int_0^x f_1(t,s)\,ds$;

(H_3) $\lim_{|x|\to\infty}\frac{F_1(t,x)}{|x|^\theta} = \infty$ uniformly with respect to $\forall t\in[0,T]$;

(H_4) There exist $1 < r < p$, $b\in C([0,T],\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$ such that

$$F_2(t,x) \ge b(t)|x|^r, \quad \forall(t,x)\in[0,T]\times\mathbb{R},$$

where $F_2(t,x) = \int_0^x f_2(t,s)\,ds$;

(H_5) There exist $b_1 \in L^1([0,T],\mathbb{R}^+)$ such that

$$\left|f_2(t,x)\right| \le b_1(t)|x|^{r-1}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Theorem 1.1 *Assume that* (H_1)–(H_5) *hold. Then problem* (1.1) *with* $\lambda = 1$ *has at least two nontrivial weak solutions.*

Remark 1.1 Clearly, conditions (H_2) and (H_3) are weaker than condition (F_1) of Theorem 1.0. In addition, the nonlinear function f studied in this paper is more general, it contains both p-suplinear and p-sublinear terms. Consequently, our conclusion generalizes Theorem 1.0 in [17].

Moreover, we also consider that the nonlinear function f satisfies p-sublinear growth. The specific assumptions are as follows:

(H_6) There exist $L > 0$, $0 < \beta \le p$ such that

$$F(t,x) \le L\left(1 + |x|^{\beta}\right), \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{1.3}$$

where $F(t,x) = \int_0^x f(t,s)\,ds$.

(H_7) There exist $1 < r_1 < p$, $b \in L^1([0,T],\mathbb{R}^+)$ such that

$$\left|f(t,x)\right| \le r_1 b(t)|x|^{r_1-1}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

(H_8) There exist an open interval $\Pi \subset [0,T]$ and constants η, $\delta > 0$, $1 < r_2 < p$ such that

$$F(t,x) \ge \eta |x|^{r_2}, \quad \forall (t,x) \in \Pi \times [-\delta,\delta].$$

Theorem 1.2 *Suppose that assumption* (H_6) *holds. Additionally, we assume also that*

(H_9) *there exist* $r > 0$, $\omega \in E^{\alpha,p}$ *such that*

$$\|\omega\|_a^p + \frac{\beta_1 h(T)}{\beta_2}\left|\omega(T)\right|^p + \frac{\alpha_1 h(0)}{\alpha_2}\left|\omega(0)\right|^p > pr, \qquad \int_0^T F\big(t,\omega(t)\big)\,dt > 0 \tag{1.4}$$

and

$$\begin{aligned} \frac{1}{A_r} &:= \frac{\int_0^T \max_{|x|\le M(rp/\Lambda)^{1/p}} F(t,x)\,dt}{r} < \frac{1}{A_l} \\ &:= \frac{p\int_0^T F(t,\omega(t))\,dt}{\|\omega\|_a^p + \frac{\beta_1 h(T)}{\beta_2}|\omega(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|\omega(0)|^p} \end{aligned} \tag{1.5}$$

hold, where $\Lambda = \min\{a_0, h_0\}$,

$$M := \left(\max\left\{\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}, 1\right\} + \left[\frac{2^{p-1}}{T}\max\left\{1, \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^p\right\}\right]^{\frac{1}{p}}\right),$$
$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for every λ *in* $\Lambda_r = (A_l, A_r)$, *problem* (1.1) *has at least three weak solutions.*

Remark 1.2 Assumption (H_6) studies both $0 < \beta < p$ and $\beta = p$. Obviously, when $p = 2$, assumption (H_6) contains the condition $0 < \beta < 2$ in [14, 15]. Thus, our conclusion extends the existing results.

Theorem 1.3 *Suppose that assumptions* (H_7)–(H_8) *hold. Assume also that*
(H_{10}) $f(t,x) = -f(t,-x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$.
Then problem (1.1) *with* $\lambda = 1$ *has infinitely many nontrivial weak solutions.*

2 Preliminaries

For the convenience of readers, this section firstly introduces some basic definitions and lemmas of fractional calculus theory.

Definition 2.1 (Left and right Riemann–Liouville fractional derivatives, [18]) Let u be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional derivatives of order $0 \leq \gamma < 1$ for function u denoted by ${}_aD_t^\gamma u(t)$ and ${}_tD_b^\gamma u(t)$, respectively, are defined by

$$
\begin{aligned}
{}_aD_t^\gamma u(t) &= \frac{d}{dt}\, {}_aD_t^{\gamma-1} u(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\gamma} u(s)\, ds \right), \\
{}_tD_b^\gamma u(t) &= -\frac{d}{dt}\, {}_tD_b^{\gamma-1} u(t) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \left(\int_t^b (s-t)^{-\gamma} u(s)\, ds \right),
\end{aligned}
$$

where $t \in [a,b]$.

Let $AC([a,b])$ be the space of absolutely continuous functions within $[a,b]$ (see [16]).

Definition 2.2 (Left and right Caputo fractional derivatives, [18]) Let $0 < \gamma < 1$ and $u \in AC([a,b])$, then the left and right Caputo fractional derivatives of order γ for function u denoted by ${}_a^CD_t^\gamma u(t)$ and ${}_t^CD_b^\gamma u(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^CD_t^\gamma u(t)$ and ${}_t^CD_b^\gamma u(t)$ are represented by

$$
\begin{aligned}
{}_a^CD_t^\gamma u(t) &= {}_aD_t^{\gamma-1} u'(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} u'(s)\, ds, \\
{}_t^CD_b^\gamma u(t) &= -{}_tD_b^{\gamma-1} u'(t) = -\frac{1}{\Gamma(1-\gamma)} \int_t^b (s-t)^{-\gamma} u'(s)\, ds,
\end{aligned}
$$

where $t \in [a,b]$.

Let us recall that, for any fixed $t \in [0,T]$ and $1 \leq r < \infty$,

$$
\begin{aligned}
&\|u\|_{L^r([0,t])} = \left(\int_0^t \left|u(\xi)\right|^r d\xi \right)^{\frac{1}{r}}, \qquad \|u\|_{L^r} = \left(\int_0^T \left|u(\xi)\right|^r d\xi \right)^{\frac{1}{r}}, \\
&\|u\|_\infty = \max_{t\in[0,T]} \left|u(t)\right|.
\end{aligned}
$$

Definition 2.3 ([16]) Let $\alpha \in (\frac{1}{2}, 1]$, $p \in [1,\infty)$. The fractional derivative space

$$
E^{\alpha,p} = \left\{ u \mid u \in AC\big([0,T],\mathbb{R}\big), {}_0^CD_t^\alpha u \in L^p\big([0,T],\mathbb{R}\big) \right\}
$$

is defined by closure of $C^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T \left[|u(t)|^p + \left|{}_0^C D_t^\alpha u(t)\right|^p\right] dt\right)^{\frac{1}{p}}. \tag{2.1}$$

Lemma 2.1 ([12]) *Let $0<\alpha\le 1$, $1\le p<\infty$. For $\forall f \in L^p([0,T],\mathbb{R})$, one has*

$$\left\|{}_0D_\xi^{-\alpha} f\right\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])}, \quad \forall \xi\in[0,t], t\in[0,T].$$

Lemma 2.2 ([16]) *Let $0<\alpha\le 1$, $1\le p<\infty$. For $\forall f \in L^p([0,T],\mathbb{R})$, one has*

$$\left\|{}_\xi D_T^{-\alpha} f\right\|_{L^p([t,T])} \le \frac{(T-t)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([t,T])}, \quad \forall \xi\in[t,T], t\in[0,T].$$

Lemma 2.3 ([18]) *Let $n\in\mathbb{N}$, $n-1<\alpha\le n$. If $f\in AC^n([a,b],\mathbb{R})$ or $f\in C^n([a,b],\mathbb{R})$, then*

$${}_aD_t^{-\alpha}\left({}_a^C D_t^\alpha f(t)\right) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(t-a)^j, \quad \forall t\in[a,b],$$

$${}_tD_b^{-\alpha}\left({}_t^C D_b^\alpha f(t)\right) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(b)}{j!}(b-t)^j, \quad \forall t\in[a,b].$$

In particular, if $0<\alpha<1$, $f\in AC([a,b],\mathbb{R})$ or $f\in C^1([a,b],\mathbb{R})$, then

$${}_aD_t^{-\alpha}\left({}_a^C D_t^\alpha f(t)\right) = f(t)-f(a), \qquad {}_tD_b^{-\alpha}\left({}_t^C D_b^\alpha f(t)\right) = f(t)-f(b).$$

Lemma 2.4 ([17]) *Let $\frac{1}{2}<\alpha\le 1$, $1\le p<\infty$. If $u\in E^{\alpha,p}$, then*

$$\|u\|_\infty \le M\|u\|_{\alpha,p},$$

where

$$M := \left(\max\left\{\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q-q+1)^{\frac{1}{q}}}, 1\right\} + \left[\frac{2^{p-1}}{T}\max\left\{1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p\right\}\right]^{\frac{1}{p}}\right), \quad \frac{1}{p}+\frac{1}{q}=1.$$

Lemma 2.5 ([17]) *Let $1/p<\alpha\le 1$, $1<p<\infty$, if $a\in C([0,T],\mathbb{R})$ and $0<a_1\le a(t)\le a_2$, $h(t)\in L^\infty([0,T],\mathbb{R})$, then by Lemma 2.4 one has*

$$\|u\|_\infty \le \frac{M}{(\min\{a_0,h_0\})^{\frac{1}{p}}}\left(\int_0^T a(t)|u(t)|^p\,dt + \int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}},$$

where $h_0 = \operatorname{ess\,inf}_{[0,T]} h(t) > 0$, $a_0 = \operatorname{ess\,inf}_{[0,T]} a(t) > 0$.

Remark 2.1 It is also easy to check that, if $a\in C([0,T],\mathbb{R})$ and $0<a_1\le a(t)\le a_2$, $h(t)\in L^\infty([0,T],\mathbb{R})$ with $h_0 = \operatorname{ess\,inf}_{[0,T]} h(t) > 0$, then an equivalent norm in $E^{\alpha,p}$ is the following:

$$\|u\|_a = \left(\int_0^T a(t)|u(t)|^p\,dt + \int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}}. \tag{2.2}$$

By combining Lemma 2.5, we can see that, for $\forall u \in E^{\alpha,p}$, if $1/p < \alpha \leq 1$, then

$$\|u\|_\infty \leq \frac{M}{\Lambda^{1/p}}\|u\|_a, \tag{2.3}$$

where $\Lambda = \min\{a_0, h_0\}$.

Lemma 2.6 ([16]) *Let $0 < \alpha \leq 1$, $1 < p < \infty$. The fractional derivative space $E^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2.7 ([16]) *Let $1/p < \alpha \leq 1$, $1 < p < \infty$. Assume that the sequence $\{u_k\}$ converges weakly to u in $E^{\alpha,p}$, i.e., $u_k \rightharpoonup u$, then $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e.,*

$$\|u_k - u\|_\infty \to 0, \quad k \to \infty.$$

Lemma 2.8 ([17]) *Assume that $1/p < \alpha \leq 1$, $1 < p < \infty$, then $E^{\alpha,p}$ is compactly embedded in $C([0,T],\mathbb{R})$.*

Lemma 2.9 ([18]) *Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, $1/p + 1/q < 1 + \alpha$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \alpha$. If $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$, then*

$$\int_a^b \left[{}_aD_t^{-\alpha}u(t)\right]v(t)\,dt = \int_a^b u(t)\left[{}_tD_b^{-\alpha}v(t)\right]dt. \tag{2.4}$$

By multiplying the equation in problem (1.1) by any $v \in E^{\alpha,p}$ and integrating on $[0,T]$, one has

$$\begin{aligned} &\int_0^T {}_tD_T^\alpha\left(\frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big)\right)\cdot v(t)\,dt + \int_0^T a(t)\phi_p\big(u(t)\big)v(t)\,dt \\ &\quad = \lambda\int_0^T f\big(t,u(t)\big)v(t)\,dt. \end{aligned} \tag{2.5}$$

From Definitions 2.1, 2.2 and Lemma 2.9, we can get

$$\begin{aligned} &\int_0^T {}_tD_T^\alpha\left(\frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big)\right)\cdot v(t)\,dt \\ &\quad = -\int_0^T \frac{d}{dt}\left[{}_tD_T^{\alpha-1}\left(\frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big)\right)\right]\cdot v(t)\,dt \\ &\quad = \frac{\beta_1 h(T)}{\beta_2}\phi_p\big(u(T)\big)v(T) + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\big(u(0)\big)v(0) \\ &\qquad + \int_0^T \left[{}_tD_T^{\alpha-1}\left(\frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big)\right)\right]\cdot v'(t)\,dt \\ &\quad = \frac{\beta_1 h(T)}{\beta_2}\phi_p\big(u(T)\big)v(T) + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\big(u(0)\big)v(0) \\ &\qquad + \int_0^T \frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big){}_0D_t^{\alpha-1}v'(t)\,dt \\ &\quad = \frac{\beta_1 h(T)}{\beta_2}\phi_p\big(u(T)\big)v(T) + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\big(u(0)\big)v(0) \\ &\qquad + \int_0^T \frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^CD_t^\alpha u(t)\big){}_0^CD_t^\alpha v(t)\,dt. \end{aligned} \tag{2.6}$$

Getting the similar result for the second part of equation (1.1), we can give the definition of weak solution for problem (1.1).

Definition 2.4 The function $u \in E^{\alpha,p}$ is a weak solution of problem (1.1) if the identity

$$\begin{aligned}
&\frac{\beta_1 h(T)}{\beta_2}\phi_p\big(u(T)\big)v(T) + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\big(u(0)\big)v(0) \\
&\quad + \int_0^T \frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^C D_t^\alpha u(t)\big){}_0^C D_t^\alpha v(t)\,dt \\
&\quad + \int_0^T a(t)\phi_p\big(u(t)\big)v(t)\,dt = \lambda\int_0^T f\big(t,u(t)\big)v(t)\,dt,
\end{aligned}$$

holds for any $v \in E^{\alpha,p}$.

Define the functional $I : E^{\alpha,p} \to \mathbb{R}$ as follows:

$$\begin{aligned}
I(u) &= \frac{1}{p}\int_0^T h(t)\big|{}_0^C D_t^\alpha u(t)\big|^p\,dt + \frac{1}{p}\int_0^T a(t)\big|u(t)\big|^p\,dt \\
&\quad + \frac{\beta_1 h(T)}{p\beta_2}\big|u(T)\big|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\big|u(0)\big|^p - \lambda\int_0^T F\big(t,u(t)\big)\,dt \\
&= \frac{1}{p}\|u\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\big|u(T)\big|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\big|u(0)\big|^p - \lambda\int_0^T F\big(t,u(t)\big)\,dt.
\end{aligned} \tag{2.7}$$

According to the continuity of f, it is easy to prove $I \in C^1(E^{\alpha,p},\mathbb{R})$. For $\forall v \in E^{\alpha,p}$, one has

$$\begin{aligned}
\langle I'(u), v\rangle &= \int_0^T \frac{1}{(h(t))^{p-2}}\phi_p\big(h(t){}_0^C D_t^\alpha u(t)\big){}_0^C D_t^\alpha v(t)\,dt \\
&\quad + \int_0^T a(t)\big|u(t)\big|^{p-2}u(t)v(t)\,dt + \frac{\beta_1 h(T)}{\beta_2}\phi_p\big(u(T)\big)v(T) \\
&\quad + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\big(u(0)\big)v(0) - \lambda\int_0^T f\big(t,u(t)\big)v(t)\,dt.
\end{aligned} \tag{2.8}$$

Then

$$\langle I'(u), u\rangle = \|u\|_a^p + \frac{\beta_1 h(T)}{\beta_2}\big|u(T)\big|^p + \frac{\alpha_1 h(0)}{\alpha_2}\big|u(0)\big|^p - \lambda\int_0^T f\big(t,u(t)\big)u(t)\,dt. \tag{2.9}$$

Therefore, the critical point of functional I corresponds to the weak solution of problem (1.1).

To prove our main results, we introduce the following tools.

Definition 2.5 ([19]) Let X be a real Banach space, $I \in C^1(X,\mathbb{R})$. $I(u)$ satisfies the (PS) condition if a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ which satisfies the conditions $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 2.10 ([19]) *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and*

(i) *there exist constants $\rho, \eta > 0$ such that $I|_{\partial B_\rho} \geq \eta$;*
(ii) *there exists $e \in X/\overline{B_\rho}$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \eta$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\bigl(g(s)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], X\bigr) : g(0) = 0, g(1) = e\bigr\}.$$

Definition 2.6 ([19]) Let X be a real Banach space. Let

$$\Sigma = \bigl\{A \subset X - \{0\} | A \text{ is closed in } X \text{ and symmetric with respect to } 0\bigr\}.$$

Let $A \in \sum$, if there is an odd mapping $G \in C(A, \mathbb{R}^n \backslash \{0\})$ and n is the smallest integer with this property, then we say that the deficit of A is n, and $\gamma(A) = n$.

Lemma 2.11 ([19]) *Let $I \in C^1(X, \mathbb{R})$ be an even functional on X and I satisfy the (PS) condition. For any $n \in \mathbb{N}$, let*

$$\Sigma_n = \bigl\{A \in \Sigma | \gamma(A) \geq n\bigr\}, \qquad c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I(u), \qquad K_c = \bigl\{u \in X | I(u) = c, I'(u) = 0\bigr\}.$$

(1) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is the critical value of I.*
(2) *If there exists a constant $l \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+l} = c \in \mathbb{R}$, and $c \neq I(0)$, then $\gamma(K_c) \geq l + 1$.*

Remark 2.2 According to Remark 7.3 in [19], if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then K_c contains an infinite number of different points. That is, I has an infinite number of different critical points on X.

Lemma 2.12 ([20]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$, $\overline{x} \in X$ with $r < \Phi(\overline{x})$ such that
(i) $\sup\{\Psi(x) : \Phi(x) \leq r\} < r\frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,
(ii) *for each*

$$\lambda \in \Lambda_r = \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\{\Psi(x) : \Phi(x) \leq r\}}\right),$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3 Main results

In order to prove the theorems, the following lemma plays an essential role.

Lemma 3.1 *Under the assumption given in Theorem 1.1, I satisfies the (PS) condition.*

Proof Assuming that $\{u_k\}_{k\in\mathbb{N}} \subset E^{\alpha,p}$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$, then there exists $D > 0$ such that

$$\left|I(u_k)\right| \le D, \qquad \left\|I'(u_k)\right\|_{(E^{\alpha,p})^*} \le D \tag{3.1}$$

for $k \in \mathbb{N}$, where $(E^{\alpha,p})^*$ is the conjugate space of $E^{\alpha,p}$.

The first step, we prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $E^{\alpha,p}$. If not, we assume that $\|u_k\|_a \to +\infty$ as $k \to \infty$. Let $z_k = \frac{u_k}{\|u_k\|_a}$, then $\|z_k\|_a = 1$. Since $E^{\alpha,p}$ is a reflexive Banach space, there exists a subsequence of $\{z_k\}$ (still denoted as $\{z_k\}$) such that $z_k \rightharpoonup z_0$ $(k \to \infty)$ in $E^{\alpha,p}$, then $z_k \to z_0$ in $C([0,T],\mathbb{R})$. By (H_4) and (H_5), one has

$$\left|f_2(t,u)\cdot u\right| \le b_1(t)|u|^r, \qquad \left|F_2(t,u)\right| \le \frac{1}{r} b_1(t)|u|^r. \tag{3.2}$$

The following two cases are discussed.

Case 1: $z_0 \ne 0$. Let $\Omega = \{t \in [0,T] \,|\, |z_0(t)| > 0\}$, then $\operatorname{meas}(\Omega) > 0$. Because $\|u_k\|_a \to +\infty$ $(k \to \infty)$ and $|u_k(t)| = |z_k(t)| \cdot \|u_k\|_a$, so for $t \in \Omega$, one has $|u_k(t)| \to +\infty$ $(k \to \infty)$. On the one hand, by (2.3), (2.7), (3.1), (3.2), we have

$$\begin{aligned}
&\int_0^T F_1(t,u_k)\,dt \\
&\quad= \frac{1}{p}\|u_k\|_a^p + \left[\frac{\beta_1 h(T)}{p\beta_2}\left|u_k(T)\right|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\left|u_k(0)\right|^p\right] - \int_0^T F_2(t,u_k)\,dt - I(u_k) \\
&\quad\le \frac{1}{p}\|u_k\|_a^p + \frac{1}{p}\left[\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right]\|u_k\|_\infty^p + \frac{1}{r}\int_0^T b_1(t)|u|^r\,dt + D \\
&\quad\le \frac{1}{p}\|u_k\|_a^p\left[1 + \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\frac{M^p}{\Lambda}\right] + \frac{1}{r}\|b_1\|_{L^1}\|u\|_\infty^r + D, \\
&\quad\le \frac{1}{p}\|u_k\|_a^p\left[1 + \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\frac{M^p}{\Lambda}\right] + \frac{M^r}{r\Lambda^{r/p}}\|b_1\|_{L^1}\|u\|_a^r + D.
\end{aligned}$$

Since $\theta > p > r > 1$, so

$$\int_0^T \frac{F_1(t,u_k)}{\|u_k\|_a^\theta}\,dt \le o(1), \quad k \to \infty. \tag{3.3}$$

On the other hand, according to Fatou's lemma, the properties of Ω and (H_3), one has

$$\lim_{k\to\infty}\int_0^T \frac{F_1(t,u_k)}{\|u_k\|_a^\theta}\,dt \ge \lim_{k\to\infty}\int_\Omega \frac{F_1(t,u_k)}{\|u_k\|_a^\theta}\,dt = \lim_{k\to\infty}\int_\Omega \frac{F_1(t,u_k)}{|u_k(t)|^\theta}\left|z_k(t)\right|^\theta\,dt = +\infty.$$

This is a contradiction to (3.3).

Case 2: $z_0 \equiv 0$. According to the continuity of f, there exists $d_0 > 0$ such that

$$\left|uf_1(t,u) - \theta F_1(t,u)\right| \le d_0, \quad \forall |u| \le d_\infty, t \in [0,T].$$

Combined with condition (H_2), we get

$$uf_1(t,u) - \theta F_1(t,u) \geq -d_1|u|^p - d_0, \quad \forall |u| \in \mathbb{R}, t \in [0,T]. \tag{3.4}$$

According to (1.2), (2.3), (2.7), (2.9), (3.1), (3.2), (3.4) and Hölder's inequality, we have

$$\begin{aligned}
o(1) = \frac{\theta D + D\|u_k\|_a}{\|u_k\|_a^p} &\geq \frac{\theta I(u_k) - \langle I'(u_k), u_k\rangle}{\|u_k\|_a^p} \\
&\geq \left(\frac{\theta}{p} - 1\right) + \frac{1}{\|u_k\|_a^p}\int_0^T \left[u_k f_1(t,u_k) - \theta F_1(t,u_k)\right] dt \\
&\quad + \frac{1}{\|u_k\|_a^p}\int_0^T \left[u_k f_2(t,u_k) - \theta F_2(t,u_k)\right] dt \\
&\geq \left(\frac{\theta}{p} - 1\right) - \frac{1}{\|u_k\|_a^p}\int_0^T \left(d_1|u_k|^p + d_0\right) dt - \frac{1}{\|u_k\|_a^p}\left(\frac{\theta}{r} + 1\right)\int_0^T b_1(t)|u_k|^r \, dt \\
&\geq \left(\frac{\theta}{p} - 1\right) - d_1\int_0^T \frac{|u_k|^p}{\|u_k\|_a^p} dt - \frac{Td_0}{\|u_k\|_a^p} - \frac{1}{\|u_k\|_a^p}\left(\frac{\theta}{r} + 1\right)\|b_1\|_{L^1}\|u_k\|_\infty^r \\
&\geq \left(\frac{\theta}{p} - 1\right) - d_1\int_0^T |z_k|^p \, dt - \frac{Td_0}{\|u_k\|_a^p} - \left(\frac{\theta}{r} + 1\right)\|b_1\|_{L^1}\frac{M^r}{\Lambda^{r/p}} \cdot \frac{\|u_k\|_a^r}{\|u_k\|_a^p} \\
&\geq \left(\frac{\theta}{p} - 1\right), \quad k \to \infty.
\end{aligned}$$

It is a contradiction. Therefore, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $E^{\alpha,p}$. Suppose that the sequence $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, still denoted as $\{u_k\}_{k\in\mathbb{N}}$, there exists $u \in E^{\alpha,p}$ such that $u_k \rightharpoonup u$ in $E^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. So

$$\begin{cases}
\langle I'(u_k) - I'(u), u_k - u\rangle \to 0, \quad k \to \infty, \\
\int_0^T [f(t,u_k(t)) - f(t,u(t))][u_k(t) - u(t)] \, dt \to 0, \quad k \to \infty, \\
|u_k(T) - u(T)|^p \to 0, \quad k \to \infty, \\
|u_k(0) - u(0)|^p \to 0, \quad k \to \infty.
\end{cases}$$

Since

$$\begin{aligned}
\|u_k - u\|_a^p = \langle I'(u_k) - I'(u), u_k - u\rangle + \int_0^T \left[f\left(t,u_k(t)\right) - f\left(t,u(t)\right)\right]\left[u_k(t) - u(t)\right] dt \\
- \frac{\beta_1 h(T)}{\beta_2}\left|u_k(T) - u(T)\right|^p - \frac{\alpha_1 h(0)}{\alpha_2}\left|u_k(0) - u(0)\right|^p,
\end{aligned}$$

so $\|u_k - u\|_a \to 0$ $(k \to \infty)$. □

The proof process of Theorem 1.1 is given below, which is structured into four steps.

Proof of Theorem 1.1 *Step* 1. Obviously, $I(0) = 0$. According to Lemma 3.1, $I \in C^1(E^{\alpha,p},\mathbb{R})$ satisfies the (PS) condition.

Step 2. We will prove that condition (i) in Lemma 2.10 holds. By (H_1), for $\forall\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$F_1(t,u) \leq \varepsilon|u|^p, \quad \forall t \in [0,T], |u| \leq \delta. \tag{3.5}$$

For $\forall u \in E^{\alpha,p}$, by (2.2), (2.3), (2.7), (3.2), (3.5), one has

$$\begin{aligned} I(u) &\geq \frac{1}{p}\|u\|_a^p - \int_0^T F\bigl(t,u(t)\bigr)\,dt \geq \frac{1}{p}\|u\|_a^p - \varepsilon\int_0^T |u|^p\,dt - \frac{1}{r}\int_0^T b_1(t)|u|^r\,dt \\ &\geq \frac{1}{p}\|u\|_a^p - \varepsilon\cdot\frac{1}{a_0}\int_0^T a(t)|u|^p\,dt - \frac{1}{r}\|b_1\|_{L^1}\|u\|_\infty^r \\ &\geq \frac{1}{p}\|u\|_a^p - \frac{\varepsilon}{a_0}\|u\|_a^p - \frac{M^r}{r\Lambda^{r/p}}\|b_1\|_{L^1}\|u\|_a^r \\ &= \biggl[\biggl(\frac{1}{p} - \frac{\varepsilon}{a_0}\biggr) - \frac{M^r}{r\Lambda^{r/p}}\|b_1\|_{L^1}\|u\|_a^{r-p}\biggr]\|u\|_a^p. \end{aligned} \tag{3.6}$$

Choose $\varepsilon = \frac{a_0}{2p}$, we obtain

$$I(u) \geq \biggl[\frac{1}{2p} - \frac{M^r}{r\Lambda^{r/p}}\|b_1\|_{L^1}\|u\|_a^{r-p}\biggr]\|u\|_a^p.$$

Let $\rho = (\frac{r\Lambda^{r/p}}{4pM^r\|b_1\|_{L^1}})^{\frac{1}{r-p}}$, $\eta = \frac{1}{4p}\rho^p$, then for $u \in \partial B_\rho$ one has $I(u) \geq \eta > 0$.

Step 3. We will prove that there exist $e \in E^{\alpha,p}$ and $\|e\|_a > \rho$ such that $I(e) < 0$, where ρ is defined in Step 2. According to (H_3), there exist two constants $d_2, d_3 > 0$ such that

$$F_1(t,u) \geq d_2|u|^\theta - d_3, \quad \forall t \in [0,T], u \in \mathbb{R}. \tag{3.7}$$

So, for $\forall u \in E^{\alpha,p}\backslash\{0\}$, $\xi \in \mathbb{R}^+$, by (2.3), (2.7), (3.7) and Hölder's inequality, we get

$$\begin{aligned} I(\xi u) &\leq \frac{\xi^p}{p}\|u\|_a^p + \frac{\xi^p}{p}\|u\|_\infty^p\biggl(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\biggr) - d_2\xi^\theta\int_0^T |u|^\theta\,dt + d_3T \\ &\leq \frac{\xi^p}{p}\|u\|_a^p + \frac{\xi^p}{p}\frac{M^p}{\Lambda}\|u\|_a^p\biggl(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\biggr) \\ &\quad - d_2\xi^\theta\biggl(T^{\frac{p-\theta}{\theta}}\int_0^T \bigl|u(t)\bigr|^p\,dt\biggr)^{\frac{\theta}{p}} + d_3T \\ &\leq \frac{1}{p}\xi^p\biggl[1 + \frac{M^p}{\Lambda}\biggl(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\biggr)\biggr]\|u\|_a^p - d_2\xi^\theta T^{\frac{p-\theta}{p}}\|u\|_{L^p}^\theta + d_3T. \end{aligned}$$

Since $\theta > p$, the above formula implies that when ξ_0 is sufficiently large, $I(\xi_0 u) \to -\infty$. Let $e = \xi_0 u$, one has $I(e) < 0$, so condition (ii) in Lemma 2.10 holds. From Lemma 2.10, we know that I has one critical value $c^{(1)} \geq \eta > 0$ as follows:

$$c^{(1)} = \inf_{g\in\Gamma}\max_{s\in[0,1]} I\bigl(g(s)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], E^{\alpha,p}\bigr) : g(0) = 0, g(1) = e\bigr\}.$$

Therefore, there exists $0 \neq u^{(1)} \in E^{\alpha,p}$ such that

$$I\bigl(u^{(1)}\bigr) = c^{(1)} \geq \eta > 0, \qquad I'\bigl(u^{(1)}\bigr) = 0. \tag{3.8}$$

That is, the first nontrivial weak solution of (1.1) exists.

Step 4. It is known from (3.6) that I is bounded below in $\overline{B_\rho}$. Choose $\varphi \in E^{\alpha,p}$ such that $\varphi(t) \not\equiv 0$ in $[0,T]$. For $\forall l \in (0,+\infty)$, by (2.7), (H_3), and (H_4), we have

$$\begin{aligned} I(l\varphi) &\le \frac{l^p}{p}\|\varphi\|_a^p\left[1+\frac{M^p}{\Lambda}\left(\frac{\beta_1 h(T)}{\beta_2}+\frac{\alpha_1 h(0)}{\alpha_2}\right)\right]-\int_0^T F_2\bigl(t,l\varphi(t)\bigr)\,dt \\ &\le \frac{l^p}{p}\|\varphi\|_a^p\left[1+\frac{M^p}{\Lambda}\left(\frac{\beta_1 h(T)}{\beta_2}+\frac{\alpha_1 h(0)}{\alpha_2}\right)\right]-l^r\int_0^T b(t)\bigl|\varphi(t)\bigr|^r\,dt. \end{aligned} \tag{3.9}$$

Thus, from $1 < r < p$, we know that, for small enough l_0 satisfying $\|l_0\varphi\|_a \le \rho$, one has $I(l_0\varphi) < 0$. Let $u = l_0\varphi$, one has

$$c^{(2)} = \inf I(u) < 0, \qquad \|u\|_a \le \rho,$$

where ρ is defined in Step 2. Then, according to the Ekeland variational principle, there exists a minimization sequence $\{v_k\}_{k\in\mathbb{N}} \subset \overline{B_\rho}$ such that

$$I(v_k) \to c^{(2)}, \qquad I'(v_k) \to 0, \quad k \to \infty.$$

That is, $\{v_k\}_{k\in\mathbb{N}}$ is a (PS) sequence. According to Lemma 3.1, I satisfies the (PS) condition. Therefore, $c^{(2)} < 0$ is another critical value of I. So there exists $0 \not\equiv u^{(2)} \in E^{\alpha,p}$ such that

$$I\bigl(u^{(2)}\bigr) = c^{(2)} < 0, \qquad \bigl\|u^{(2)}\bigr\|_a < \rho. \qquad \square$$

The proof of Theorem 1.2 is given below.

Proof of Theorem 1.2 The functionals $\Phi : E^{\alpha,p} \to \mathbb{R}$ and $\Psi : E^{\alpha,p} \to \mathbb{R}$ are defined as follows:

$$\Phi(u) = \frac{1}{p}\|u\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\bigl|u(T)\bigr|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\bigl|u(0)\bigr|^p, \qquad \Psi(u) = \int_0^T F\bigl(t,u(t)\bigr)\,dt,$$

then $I(u) = \Phi(u) - \lambda\Psi(u)$. Through simple calculation, we get

$$\inf_{u\in E^{\alpha,p}} \Phi(u) = \Phi(0) = 0, \qquad \Psi(0) = \int_0^T F(t,0)\,dt = 0.$$

Furthermore, Φ and Ψ are continuous Gâteaux differential and

$$\begin{aligned} \bigl\langle \Phi'(u),v\bigr\rangle &= \int_0^T h(t)\phi_p\bigl({}_0^C D_t^\alpha u(t)\bigr){}_0^C D_t^\alpha v(t)\,dt + \int_0^T a(t)\phi_p\bigl(u(t)\bigr)v(t)\,dt \\ &\quad + \frac{\beta_1 h(T)}{\beta_2}\phi_p\bigl(u(T)\bigr)v(T) + \frac{\alpha_1 h(0)}{\alpha_2}\phi_p\bigl(u(0)\bigr)v(0), \quad \forall u,v \in E^{\alpha,p}, \end{aligned} \tag{3.10}$$

$$\bigl\langle \Psi'(u),v\bigr\rangle = \int_0^T f\bigl(t,u(t)\bigr)v(t)\,dt, \quad \forall u,v \in E^{\alpha,p}. \tag{3.11}$$

In addition, $\Phi' : E^{\alpha,p} \to (E^{\alpha,p})^*$ is continuous. Next, we prove that $\Psi' : E^{\alpha,p} \to (E^{\alpha,p})^*$ is a continuous compact operator. Assuming that $\{u_n\} \subset E^{\alpha,p}$, $u_n \rightharpoonup u$ $(n \to \infty)$, then $\{u_n\}$ uniformly converges to u on $C([0,T])$. Because $f \in C([0,T]\times\mathbb{R},\mathbb{R})$, so $f(t,u_n) \to f(t,u)$

$(n \to \infty)$. Thus $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Then, Ψ' is strongly continuous. From Proposition 26.2 in [21], Ψ' is a compact operator. And then we show that Φ is weakly semicontinuous. Assuming that $\{u_n\} \subset E^{\alpha,p}$, $\{u_n\} \rightharpoonup u$, then $\{u_n\}$ uniformly converges to u on $C([0,T])$, and $\liminf_{n\to\infty} \|u_n\|_a \geq \|u\|_a$. So,

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &= \liminf_{n\to\infty} \left(\frac{1}{p}\|u_n\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\big|u_n(T)\big|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\big|u_n(0)\big|^p \right) \\ &\geq \frac{1}{p}\|u\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\big|u(T)\big|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\big|u(0)\big|^p = \Phi(u).\end{aligned}$$

Thus Φ is weakly semicontinuous. In addition, we will show that Φ' is coercive and has a continuous inverse on $(E^{\alpha,p})^*$. For $u \in E^{\alpha,p}\backslash\{0\}$, by (3.10), one has

$$\lim_{\|u\|_a\to+\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|_a} = \lim_{\|u\|_a\to+\infty} \frac{\|u\|_a^p + \frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p}{\|u\|_a} = +\infty,$$

then Φ' is coercive. For $\forall u, v \in E^{\alpha,p}$, by (3.10), we obtain

$$\begin{aligned}&\big\langle \Phi'(u) - \Phi'(v), u - v\big\rangle \\ &\quad= \int_0^T h(t)\big(\phi_p\big({}_0^CD_t^\alpha u(t)\big) - \phi_p\big({}_0^CD_t^\alpha v(t)\big)\big)\big({}_0^CD_t^\alpha u(t) - {}_0^CD_t^\alpha v(t)\big)\,dt \\ &\qquad+ \int_0^T a(t)\big(\phi_p\big(u(t)\big) - \phi_p\big(v(t)\big)\big)\big(u(t) - v(t)\big)\,dt \\ &\qquad+ \frac{\beta_1 h(T)}{\beta_2}\big(\phi_p\big(u(T)\big) - \phi_p\big(v(T)\big)\big)\big(u(T) - v(T)\big) \\ &\qquad+ \frac{\alpha_1 h(0)}{\alpha_2}\big(\phi_p\big(u(0)\big) - \phi_p\big(v(0)\big)\big)\big(v(T) - v(0)\big) \\ &\quad\geq \int_0^T h(t)\big(\phi_p\big({}_0^CD_t^\alpha u(t)\big) - \phi_p\big({}_0^CD_t^\alpha v(t)\big)\big)\big({}_0^CD_t^\alpha u(t) - {}_0^CD_t^\alpha v(t)\big)\,dt \\ &\qquad+ \int_0^T a(t)\big(\phi_p\big(u(t)\big) - \phi_p\big(v(t)\big)\big)\big(u(t) - v(t)\big)\,dt.\end{aligned}$$

From [22], we can see that there exist constants $c_p, d_p > 0$ such that

$$\begin{aligned}&\big\langle \Phi'(u) - \Phi'(v), u - v\big\rangle \\ &\quad\geq \begin{cases} c_p \int_0^T h(t)|{}_0^CD_t^\alpha u(t) - {}_0^CD_t^\alpha v(t)|^p + a(t)|u(t) - v(t)|^p\,dt, & p \geq 2; \\ d_p \int_0^T \frac{h(t)|{}_0^CD_t^\alpha u(t) - {}_0^CD_t^\alpha v(t)|^2}{(|{}_0^CD_t^\alpha u(t)| + |{}_0^CD_t^\alpha v(t)|)^{2-p}} + \frac{a(t)|u(t)-v(t)|^2}{(|u(t)|+|v(t)|)^{2-p}}\,dt, & 1 < p < 2. \end{cases}\end{aligned} \tag{3.12}$$

If $p \geq 2$, then

$$\big\langle \Phi'(u) - \Phi'(v), u - v\big\rangle \geq c_p\|u - v\|_a^p.$$

Consequently, Φ' is uniformly monotonous. If $1 < p < 2$, by Hölder's inequality, one has

$$\begin{aligned}
&\int_0^T \left|{}_0^C D_t^\alpha u(t) - {}_0^C D_t^\alpha v(t)\right|^p dt \\
&\quad \leq c\left(\int_0^T \frac{|{}_0^C D_t^\alpha u(t) - {}_0^C D_t^\alpha v(t)|^2}{(|{}_0^C D_t^\alpha u(t)| + |{}_0^C D_t^\alpha v(t)|)^{2-p}}\, dt\right)^{\frac{p}{2}} \left(\|u\|_a + \|v\|_a\right)^{\frac{p(2-p)}{2}},
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^T \left(\phi_p\left({}_0^C D_t^\alpha u(t)\right) - \phi_p\left({}_0^C D_t^\alpha v(t)\right)\right)\left({}_0^C D_t^\alpha u(t) - {}_0^C D_t^\alpha v(t)\right) dt \\
&\quad \geq \frac{c}{(\|u\|_a + \|v\|_a)^{2-p}} \left(\int_0^T \left|{}_0^C D_t^\alpha u(t) - {}_0^C D_t^\alpha v(t)\right|^p dt\right)^{\frac{2}{p}}.
\end{aligned} \tag{3.13}$$

Combined with (3.12) and (3.13), we obtain

$$\left\langle \Phi'(u) - \Phi'(v), u - v\right\rangle \geq \frac{c\|u - v\|_a^2}{(\|u\|_a + \|v\|_a)^{2-p}}.$$

Thus, Φ' is strictly monotonous. From Theorem 26.A(d) in [21], $(\Phi')^{-1}$ exists and is continuous.

The second step is to verify that condition (i) in Lemma 2.12 holds. If $x \in E^{\alpha,p}$ satisfies $\Phi(x) \leq r$, then by (2.3) one has

$$\Phi(x) = \frac{1}{p}\|x\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\left|x(T)\right|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\left|x(0)\right|^p \geq \frac{1}{p}\|x\|_a^p \geq \frac{\Lambda}{pM^p}\|x\|_\infty^p$$

and

$$\begin{aligned}
\left\{x \in E^{\alpha,p} : \Phi(x) \leq r\right\} &\subseteq \left\{x : \frac{\Lambda}{pM^p}\|x\|_\infty^p \leq r\right\} = \left\{x : \|x\|_\infty^p \leq \frac{rpM^p}{\Lambda}\right\} \\
&= \left\{x : \|x\|_\infty \leq M\left(\frac{rp}{\Lambda}\right)^{\frac{1}{p}}\right\}.
\end{aligned}$$

Thus,

$$\sup\left\{\Psi(x) : \Phi(x) \leq r\right\} = \sup\left\{\int_0^T F\left(t, x(t)\right) dt : \Phi(x) \leq r\right\} \leq \int_0^T \max_{|x| \leq M(rp/\Lambda)^{1/p}} F(t,x)\, dt,$$

combined with (1.5), we get

$$\begin{aligned}
\frac{\sup\{\Psi(x) : \Phi(x) \leq r\}}{r} &= \frac{\sup\{\int_0^T F(t,x(t))\, dt : \Phi(x) \leq r\}}{r} \\
&\leq \frac{\int_0^T \max_{|x| \leq M(rp/\Lambda)^{1/p}} F(t,x)\, dt}{r} \\
&< \frac{p\int_0^T F(t,\omega(t))\, dt}{\|\omega\|_a^p + \frac{\beta_1 h(T)}{\beta_2}|\omega(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|\omega(0)|^p} = \frac{\Psi(\omega)}{\Phi(\omega)},
\end{aligned}$$

which implies that condition (i) of Lemma 2.12 holds.

The third step is to prove that, for any $\lambda \in \Lambda_r = (A_l, A_r)$, the functional $\Phi - \lambda\Psi$ is coercive. For $x \in E^{\alpha,p}$, by (1.3), (2.3) we have

$$\int_0^T F\bigl(t, x(t)\bigr)\,dt \le L \int_0^T \bigl(1 + \bigl|x(t)\bigr|^\beta\bigr)\,dt \le LT + LT\|x\|_\infty^\beta \le LT + \frac{LTM^\beta}{\Lambda^{\beta/p}}\|x\|_a^\beta. \tag{3.14}$$

For $x \in E^{\alpha,p}$, $\lambda \in \Lambda_r$, by (3.14) we get

$$\begin{aligned}\Phi(x) - \lambda\Psi(x) \ge{}& \frac{1}{p}\|x\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\bigl|x(T)\bigr|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\bigl|x(0)\bigr|^p \\ &- \lambda\left(LT + \frac{LTM^\beta}{\Lambda^{\beta/p}}\|x\|_a^\beta\right).\end{aligned}$$

If $0 < \beta < p$, for all $\lambda > 0$, one has

$$\lim_{\|x\|_a \to +\infty} \bigl(\Phi(x) - \lambda\Psi(x)\bigr) = +\infty.$$

Obviously, the functional $\Phi - \lambda\Psi$ is coercive. If $\beta = p$, we obtain

$$\Phi(x) - \lambda\Psi(x) \ge \left(\frac{1}{p} - \frac{\lambda LTM^p}{\Lambda}\right)\|x\|_a^p - \lambda LT.$$

Choose

$$L < \frac{\Lambda \int_0^T \max_{|x| \le M(rp/\Lambda)^{1/p}} F(t,x)\,dt}{prTM^p},$$

for $\lambda < A_r$, one has $\frac{1}{p} - \frac{\lambda LTM^p}{\Lambda} > 0$. Thus,

$$\lim_{\|x\|_a \to +\infty} \bigl(\Phi(x) - \lambda\Psi(x)\bigr) = +\infty.$$

So $\Phi - \lambda\Psi$ is coercive. Therefore, the conditions in Lemma 2.12 are all true. By Lemma 2.12, we get that, for each $\lambda \in \Lambda_r$, the functional $I = \Phi - \lambda\Psi$ has at least three different critical points in $E^{\alpha,p}$. □

Finally, the proof process of Theorem 1.3 is given.

Proof of Theorem 1.3 In the first step, $I \in C^1(E^{\alpha,p}, \mathbb{R})$ is bounded below. By (H_7), one has

$$\bigl|F(t,u)\bigr| \le b(t)|u|^{r_1}, \quad \forall (t,u) \in [0,T] \times \mathbb{R}.$$

Combining (2.3) and (2.7), we can get

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\bigl|u(T)\bigr|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\bigl|u(0)\bigr|^p - \int_0^T F\bigl(t, u(t)\bigr)\,dt \\ &\ge \frac{1}{p}\|u\|_a^p - \int_0^T F\bigl(t, u(t)\bigr)\,dt \ge \frac{1}{p}\|u\|_a^p - \int_0^T b(t)|u|^{r_1}\,dt \\ &\ge \frac{1}{p}\|u\|_a^p - \|b\|_{L^1}\|u\|_\infty^{r_1} \ge \frac{1}{p}\|u\|_a^p - \frac{M^{r_1}}{\Lambda^{r_1/p}}\|b\|_{L^1}\|u\|_a^{r_1}. \end{aligned} \tag{3.15}$$

Since $1 < r_1 < p$, (3.15) indicates that $I(u) \to \infty$ as $\|u\|_a \to \infty$, so I is bounded below.

In the second step, I satisfies the (PS) condition on $E^{\alpha,p}$. Assume that $\{u_k\} \subset E^{\alpha,p}$ is a sequence such that

$$\left|I(u_k)\right| \le D, \qquad I'(u_k) \to 0 \quad (k \to \infty),$$

where $D > 0$ is a constant. Then (3.15) shows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded on $E^{\alpha,p}$. Suppose that the sequence $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, still recorded as $\{u_k\}_{k\in\mathbb{N}}$, there exists $u \in E^{\alpha,p}$ such that $u_k \rightharpoonup u$ in $E^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. So

$$\begin{cases} \langle I'(u_k) - I'(u), u_k - u\rangle \to 0, \quad k \to \infty, \\ \int_0^T [f(t,u_k(t)) - f(t,u(t))][u_k(t) - u(t)]\,dt \to 0, \quad k \to \infty, \\ |u_k(T) - u(T)|^p \to 0, \quad k \to \infty, \\ |u_k(0) - u(0)|^p \to 0, \quad k \to \infty. \end{cases}$$

Since

$$\begin{aligned} \|u_k - u\|_a^p = {} & \left\langle I'(u_k) - I'(u), u_k - u\right\rangle + \int_0^T \left[f\left(t,u_k(t)\right) - f\left(t,u(t)\right)\right]\left[u_k(t) - u(t)\right]dt \\ & - \frac{\beta_1 h(T)}{\beta_2}\left|u_k(T) - u(T)\right|^p - \frac{\alpha_1 h(0)}{\alpha_2}\left|u_k(0) - u(0)\right|^p, \end{aligned}$$

so $\|u_k - u\|_a \to 0$ $(k \to \infty)$. This means that $u_k \to u$ in $E^{\alpha,p}$. That is, I satisfies the (PS) condition on $E^{\alpha,p}$. In addition, (2.7) and (H_{10}) indicate that I is an even functional and $I(0) = 0$.

Fix $n \in \mathbb{N}$, then take n disjoint open intervals Π_i such that $\cup_{i=1}^n \Pi_i \subset \Pi$. Let $u_i \in (W^{1,2}(\Pi_i,\mathbb{R}) \cap E^{\alpha,p})\backslash\{0\}$ satisfy $\|u_i\|_a = 1$, and remember

$$E_n = \operatorname{span}\{u_1, u_2, \dots, u_n\}, \qquad S_n = \left\{u \in E_n \,|\, \|u\|_a = 1\right\}.$$

Therefore, for $u \in E_n$, there exists $\lambda_i \in \mathbb{R}$ such that

$$u = \sum_{i=1}^n \lambda_i u_i, \quad \forall t \in [0,T], \tag{3.16}$$

then

$$\begin{aligned} \|u\|_a^p &= \int_0^T \left[a(t)\left|u(t)\right|^p + h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p\right]dt \\ &= \sum_{i=1}^n |\lambda_i|^p \int_{\Pi_i} \left[a(t)\left|u_i(t)\right|^p + h(t)\left|{}_0^C D_t^\alpha u_i(t)\right|^p\right]dt \\ &= \sum_{i=1}^n |\lambda_i|^p \|u_i\|_a^p = \sum_{i=1}^n |\lambda_i|^p, \quad \forall u \in E_n. \end{aligned} \tag{3.17}$$

For $u \in S_n$, by (2.2), (2.3), (3.16), and (H_8), one has

$$\begin{aligned} I(su) &= \frac{1}{p}\|su\|_a^p + \frac{\beta_1 h(T)}{p\beta_2}\big|su(T)\big|^p + \frac{\alpha_1 h(0)}{p\alpha_2}\big|su(0)\big|^p - \int_0^T F\big(t, su(t)\big)\,dt \\ &\leq \frac{|s|^p}{p}\|u\|_a^p + \frac{|s|^p}{p}\left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\|u\|_\infty^p - \sum_{i=1}^n \int_{\Pi_i} F(t, s\lambda_i u_i)\,dt \\ &\leq \frac{|s|^p}{p}\left[1 + \frac{M^p}{\Lambda}\left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\right] \\ &\quad - \eta|s|^{r_2}\sum_{i=1}^n |\lambda_i|^{r_2}\int_{\Pi_i}|u_i|^{r_2}\,dt, \quad 0 < s \leq \frac{\delta\Lambda^{1/p}}{M\lambda^*}, \end{aligned} \tag{3.18}$$

where $\lambda^* = \max_{i\in\{1,2,\ldots,n\}}|\lambda_i| > 0$ is a constant. Because $1 < r_2 < p$, (3.18) shows that there exist $\epsilon, \sigma > 0$ such that

$$I(\sigma u) < -\epsilon, \quad \forall u \in S_n. \tag{3.19}$$

Let

$$S_n^\sigma = \{\sigma u | u \in S_n\}, \qquad \Delta = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n |\lambda_i|^p < \sigma^p\right\},$$

then by (3.19) we obtain

$$I(u) < -\epsilon, \quad \forall u \in S_n^\sigma.$$

Combining I is an even functional and $I(0) = 0$, we get

$$S_n^\sigma \subset I^{-\epsilon} \in \Sigma.$$

In addition, it can be seen from (3.17) that the mapping $(\lambda_1, \lambda_2, \ldots, \lambda_n) \to \sum_{i=1}^n \lambda_i u_i$ from $\partial\Delta$ to S_n^σ is odd and homeomorphic. Thus, according to some properties of the genus (Propositions 7.5 and 7.7 in [19]), one has

$$\gamma\big(I^{-\epsilon}\big) \geq \gamma\big(S_n^\sigma\big) = n.$$

Therefore $I^{-\epsilon} \in \Sigma_n$, so $\Sigma_n \neq \phi$. Let

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I(u).$$

Then, since I is bounded below, we can get $-\infty < c_n \leq -\epsilon < 0$. That is, for $\forall n \in \mathbb{N}$, c_n is a negative real number. Therefore, according to Lemma 2.11, I has infinitely many nontrivial critical points, that is, problem (1.1) has infinitely many nontrivial weak solutions. □

4 Conclusions

This paper mainly explores the multiplicity of solutions for a fractional p-Laplacian differential equation with Sturm–Liouville boundary value conditions. By employing variational methods, the multiplicity results of weak solutions are obtained under the conditions of p-suplinear growth, p-sublinear growth, and the combination of p-suplinear growth and p-sublinear growth. Compared with the existing related work, the research results of this paper weaken the existing related conditions and improve and enrich the related results to a certain extent.

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