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Complete and incomplete Bell polynomials associated with Lah–Bell numbers and polynomials

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Abstract

The nth r-extended Lah–Bell number is defined as the number of ways a set with $n+r$ elements can be partitioned into ordered blocks such that r distinguished elements have to be in distinct ordered blocks. The aim of this paper is to introduce incomplete r-extended Lah–Bell polynomials and complete r-extended Lah–Bell polynomials respectively as multivariate versions of r-Lah numbers and the r-extended Lah–Bell numbers and to investigate some properties and identities for these polynomials. From these investigations we obtain some expressions for the r-Lah numbers and the r-extended Lah–Bell numbers as finite sums.

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1 Introduction

It is well known that the unsigned Lah number $L(n,k)$ $(n \geq k \geq 0)$ counts the number of ways a set with n elements can be partitioned into k nonempty linearly ordered subsets (see [4, 7, 8]). The nth Lah–Bell number B_n^L $(n \geq 0)$ is the number of ways a set with n elements can be partitioned into nonempty linearly ordered subsets. Thus

$$B_n^L = \sum_{k=0}^{n} L(n,k) \quad (n \geq 0) \text{ (see [7, 8]).} \tag{1}$$

From (1) it follows that the generating function of Lah–Bell numbers is given by

$$e^{\frac{t}{1-t}} = \sum_{n=0}^{\infty} B_n^L \frac{t^n}{n!} \quad \text{(see [7, 8]),} \tag{2}$$

where

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!} \quad (k \geq 0) \text{ (see [7, 13, 15, 17]).} \tag{3}$$

Explicitly, we see from (3) that the Lah numbers are given by

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1} \quad (n \geq k \geq 0) \text{ (see [7–10, 17, 19])}.$$

Let n, k, r be nonnegative integers with $n \geq k$. Then the r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks (see [17]). The r-extended Lah–Bell number $B_{n,r}^L$ is defined as the number of ways a set with $n+r$ elements can be partitioned into ordered blocks such that r distinguished elements have to be in distinct ordered blocks (see [8]). By the definitions of r-Lah numbers and r-extended Lah–Bell numbers we have

$$B_{n,r}^L = \sum_{k=0}^{n} L_r(n,k) \quad (n \geq 0) \text{ (see [8])}. \tag{4}$$

From (4) we see that the generating function of r-extended Lah-Bell numbers is given by

$$e^{\frac{t}{1-t}}\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty} B_{n,r}^L \frac{t^n}{n!} \quad \text{(see [8, 15])}, \tag{5}$$

where

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^{k}\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=k}^{\infty} L_r(n,k)\frac{t^n}{n!} \quad \text{(see [8, 17])} \tag{6}$$

for nonnegative integers k.

Explicitly, the r-Lah numbers are given by

$$L_r(n,k) = \frac{n!}{k!}\binom{n+2r-1}{k+2r-1} \quad (n \geq k \geq 0) \text{ (see [7–10, 17, 19])}.$$

In [8] the r-extended Lah–Bell polynomials are defined by

$$e^{x(\frac{t}{1-t})}\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty} B_{n,r}^L(x)\frac{t^n}{n!}. \tag{7}$$

It is well known that the complete Bell polynomials are defined by

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(x_1,x_2,\ldots,x_n)\frac{t^n}{n!} \quad \text{(see [2–4, 6, 11, 14])}. \tag{8}$$

Then it can be shown that the complete Bell polynomials are given by

$$B_n(x_1,x_2,\ldots,x_n) = \sum_{j_1+2j_2+\cdots+nj_n=n} \frac{n!}{j_1!j_2!\cdots j_n!}\left(\frac{x_1}{1}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_n}{n!}\right)^{j_n}, \tag{9}$$

where the sum runs over all nonnegative integers $j_1, j_2, \ldots, j_n$ satisfying $j_1 + 2j_2 + \cdots + nj_n = n$. The incomplete Bell polynomials are given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})\frac{t^n}{n!} \quad (n \geq 0) \text{ (see [6, 11, 14]).} \tag{10}$$

Thus

$$\begin{aligned} &B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \\ &\quad = \sum_{\pi(n,k)} \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned} \tag{11}$$

where the sum runs over the set $\pi(n,k)$ of all nonnegative integers $(j_i)_{i\geq 1}$ satisfying $j_1 + j_2 + \cdots + j_{n-k+1} = k$ and $1j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.

The complete and incomplete Bell polynomials are related by

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \quad (n \geq 1).$$

Let f be a C^∞-function, that is, f is a function that has continuous derivatives of all orders on $(-\infty, \infty)$. Then by (8) we have

$$\begin{aligned} e^{f(x+t)} &= \exp\left(\sum_{j=0}^{\infty} f^{(j)}(x)\frac{t^j}{j!}\right) \\ &= \exp\left(f(x) + \sum_{j=1}^{\infty} f^{(j)}(x)\frac{t^j}{j!}\right) \\ &= e^{f(x)}\left(1 + \sum_{n=1}^{\infty} B_n\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)\right)\frac{t^n}{n!}\right), \end{aligned} \tag{12}$$

where $f^{(j)}(x)$ is the jth derivative of $f(x)$, and $\exp(t) = e^t$.

We observe that

$$\begin{aligned} \frac{d^m}{dx^m} e^{f(x)} &= \frac{\partial^m}{\partial x^m} e^{f(x+t)}\bigg|_{t=0} = \frac{\partial^m}{\partial t^m} e^{f(x+t)}\bigg|_{t=0} \\ &= e^{f(x)} B_m\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x)\right). \end{aligned} \tag{13}$$

From (12) and (13) we obtain the Kölbig–Coeffey equation

$$\frac{d^m}{dx^m} e^{f(x)} = e^{f(x)} B_m\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x)\right) \quad (m \geq 1) \text{ (see [5, 12]).} \tag{14}$$

The exponential incomplete r-Bell polynomials are defined by the generating function

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} a_j \frac{t^j}{j!}\right)^k \left(\sum_{i=0}^{\infty} b_{i+1}\frac{t^i}{i!}\right)^r = \sum_{n=k}^{\infty} B^{(r)}_{n+r,k+r}(a_1, a_2, \ldots : b_1, b_2, \ldots)\frac{t^n}{n!}. \tag{15}$$

From (15) we note that

$$\begin{aligned} &B^{(r)}_{n+r,k+r}(a_1,a_2,\ldots:b_1,b_2,\ldots) \\ &\quad= \sum_{\Lambda(n,k,r)} \left[\frac{n!}{k_1!k_2!k_3!\cdots}\left(\frac{a_1}{1!}\right)^{k_1}\left(\frac{a_2}{2!}\right)^{k_2}\left(\frac{a_3}{3!}\right)^{k_3}\cdots\right] \\ &\qquad\times \left[\frac{r!}{r_0!r_1!r_2!\cdots}\left(\frac{b_1}{0!}\right)^{r_0}\left(\frac{b_2}{1!}\right)^{r_1}\left(\frac{b_3}{2!}\right)^{r_2}\cdots\right], \end{aligned} \tag{16}$$

where $\Lambda(n,k,r)$ denotes the set of all nonnegative integers $(k_i)_{i\ge1}$ and $(r_i)_{i\ge0}$ such that

$$\sum_{i\ge1} k_i = k, \qquad \sum_{i\ge0} r_i = r, \quad \text{and} \quad \sum_{i\ge1} i(k_i + r_i) = n \quad \text{(see [4, 6, 14])}.$$

Let $(a_i)_{i\ge1}$ and $(b_i)_{i\ge1}$ are sequences of positive integers. Then the number $B^{(r)}_{n+r,k+r}(a_1,a_2,\ldots;b_1,b_2,\ldots)$ counts the number of partitions of an $(n+r)$-set into $(k+r)$ blocks satisfying:

- The first r elements belong to different blocks;
- Any block of size i containing no elements from the first r elements can be colored with a_i colors;
- Any block of size i containing one element from the first r elements can be colored with b_i colors.

The complete r-Bell polynomials are given by

$$\exp\left(\sum_{i=1}^{\infty} a_i \frac{t^i}{i!}\right)\left(\sum_{j=0}^{\infty} b_{j+1}\frac{t^j}{j!}\right)^r = \sum_{n=0}^{\infty} B^{(r)}_n(a_1,a_2,\ldots;b_1,b_2,\ldots)\frac{t^n}{n!} \tag{17}$$

(see [4, 6, 11, 14]).

By (16) and (17) we get

$$B^{(r)}_n(a_1,a_2,\ldots:b_1,b_2,\ldots) = \sum_{k=0}^{n} B^{(r)}_{n+r,k+r}(a_1,a_2,\ldots:b_1,b_2,\ldots) \quad \text{(see [6])}. \tag{18}$$

The incomplete and complete Bell polynomials have applications to such diverse areas as combinatorics, probability, algebra, and analysis. The number of monomials appearing in the incomplete Bell polynomial $B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$ is the number of partitioning n into k parts, and the coefficient of each monomial is the number of partitioning n as the corresponding k parts. Also, the incomplete Bell polynomials $B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$ appear in the Faà di Bruno formula concerning higher-order derivatives of composite functions (see [6]). In addition, the incomplete Bell polynomials can be used in constructing sequences of binomial type (see [16]), and there are certain connections between incomplete Bell polynomials and combinatorial Hopf algebras such as the Hopf algebra of word symmetric functions, the Hopf algebra of symmetric functions, and the Fa di Bruno algebra (see [1]). The complete Bell polynomials $B_n(x_1,x_2,\ldots,x_n)$ have applications to probability theory (see [6, 12, 18]). Indeed, the nth moment $\mu_n = E[X^n]$ of the random variable X is the nth complete Bell polynomial in the first n cumulants $\mu_n = B_n(\kappa_1,\kappa_2,\ldots,\kappa_n)$. The reader can refer to the Ph.D. thesis of Port [18] for many applications to probability theory and combinatorics. Many special numbers, like Stirling numbers of both kinds, Lah numbers, and idempotent numbers, appear in many combinatorial and number-theoretic identities involving complete and incomplete Bell polynomials. We refer the reader to the Introduction in [11] for further details.

The incomplete Lah–Bell polynomials (see (22)) and the complete Lah–Bell polynomials (see (25)) are respectively multivariate versions of the unsigned Lah numbers and the Lah–Bell numbers. Note here that the incomplete Bell polynomials (see (10)) and the incomplete Lah–Bell polynomials are related as given in (23), whereas the complete Bell polynomials (see (8)) and the complete Lah–Bell polynomials are related as given in (26). The incomplete r-extended Lah–Bell polynomials (see (30)) and the complete r-extended Lah–Bell polynomials (see (32)) are respectively extended versions of the incomplete Lah–Bell polynomials and the complete Lah–Bell polynomials. Further, they are respectively multivariate versions of the r-Lah numbers and the r-extended Lah–Bell numbers.

The aim of this paper is to introduce the incomplete r-extended Lah-Bell polynomials and the complete r-extended Lah-Bell polynomials and to investigate some properties and identities for these polynomials. From these investigations we obtain some expressions for the r-Lah numbers and the r-extended Lah–Bell numbers as finite sums.

2 Complete and incomplete r-extended Lah–Bell polynomials

Let $f(t)=\frac{t}{1-t}$. Then we have

$$f^{(n)}(t)=\frac{d^n}{dt^n}f(t)=\frac{n!}{(1-t)^{n+1}}\quad (n\geq 1). \tag{19}$$

By (14) we get

$$\frac{d^n}{dt^n}e^{\frac{t}{1-t}}\bigg|_{t=0}=B_n(1!,2!,\ldots,n!). \tag{20}$$

From (2) we note that

$$\frac{d^n}{dt^n}e^{\frac{t}{1-t}}\bigg|_{t=0}=\frac{d^n}{dt^n}\sum_{k=0}^{\infty}B_k^L\frac{t^k}{k!}\bigg|_{t=0}=B_n^L. \tag{21}$$

Therefore by (20) and (21) we obtain the following theorem.

Theorem 1 *For $n\geq 1$, we have*

$$B_n^L=B_n(1!,2!,\ldots,n!)=\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{n!}{k_1!k_2!\cdots k_n!}.$$

Let us consider the incomplete Lah–Bell polynomials given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m t^m\right)^k=\sum_{n=k}^{\infty}B_{n,k}^L(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!}, \tag{22}$$

where $n,k\geq 0$ with $n\geq k$.

Note hat $B_{n,k}^L(1,1,\ldots,1)=L(n,k)$ $(n\geq k\geq 0)$.

Indeed, by (10) and (22) we get

$$B_{n,k}^{L}(x_1,x_2,\ldots,x_{n-k+1}) = B_{n,k}\bigl(1!x_1,2!x_2,\ldots,(n-k+1)!x_{n-k+1}\bigr). \tag{23}$$

From (23) we note that

$$\begin{aligned} L(n,k) &= B_{n,k}^{L}(1,1,\ldots,1) = B_{n,k}\bigl(1!,2!,\ldots,(n-k+1)!\bigr) \\ &= \sum_{\substack{j_1+j_2+\cdots+j_{n-k+1}=k \\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}. \end{aligned}$$

Therefore by (23) we obtain the following proposition.

Proposition 2 *For $n,k\geq 0$ with $n\geq k$, we have*

$$B_{n,k}^{L}(x_1,x_2,\ldots,x_{n-k+1}) = B_{n,k}\bigl(1!x_1,2!x_2,\ldots,(n-k+1)x_{n-k+1}\bigr).$$

In addition,

$$L(n,k) = \sum_{\substack{j_1+j_2+\cdots+j_{n-k+1}=k \\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}.$$

From (23) we note that

$$\begin{aligned} B_{n,k}^{L}(\alpha x_1,\alpha x_2,\ldots,\alpha x_{n-k+1}) &= B_{n,k}\bigl(\alpha\cdot 1!x_1,\alpha\cdot 2!x_2,\ldots,\alpha\cdot(n-k+1)!x_{n-k+1}\bigr) \\ &= \alpha^k B_{n,k}\bigl(1!x_1,2!x_2,\ldots,(n-k+1)!x_{n-k+1}\bigr) \\ &= \alpha^k B_{n,k}^{L}(x_1,x_2,\ldots,x_{n-k+1}). \end{aligned} \tag{24}$$

We now consider the complete Lah–Bell polynomials given by

$$\exp\left(\sum_{i=1}^{\infty} x_i t^i\right) = \sum_{n=0}^{\infty} B_n^{L}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}. \tag{25}$$

By (25) we get

$$\begin{aligned} B_n^{L}(x_1,x_2,\ldots,x_n) &= B_n(1!x_1,2!x_2,\ldots,n!x_n) \\ &= \sum_{l_1+2l_2+\cdots+nl_{n-1}=n} \frac{n!}{l_1!l_2!\cdots l_n!} x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n} \quad (n\geq 0). \end{aligned} \tag{26}$$

From (22) and (25) we note that

$$\begin{aligned} 1+\sum_{n=1}^{\infty} B_n^{L}(x_1,x_2,\ldots,x_n)\frac{t^n}{n!} &= \exp\left(\sum_{i=1}^{\infty} x_i t^i\right) \\ &= 1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{i=1}^{\infty} x_i t^i\right)^k \end{aligned} \tag{27}$$

$$= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} B_n^L(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!}$$
$$= 1 + \sum_{n=1}^{\infty}(\sum_{k=1}^{n} B_{n,k}^L(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!}.$$

Therefore by (25) and (27) we obtain the following theorem.

Theorem 3 *For $n \geq 1$, we have*

$$B_n^L(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n} B_{n,k}^L(x_1, x_2, \dots, x_{n-k+1})$$
$$= \sum_{k=1}^{n} B_{n,k}\big(1!x_1, 2!x_2, \dots, (n-k+1)!x_{n-k+1}\big).$$

In addition, for $n \geq 1$, we have

$$B_n^L(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n}\sum_{\pi(n,k)} \frac{n!}{l_1!l_2!\cdots l_{n-k+1}!} x_1^{l_1} x_2^{l_2}\cdots x_{n-k+1}^{l_{n-k+1}},$$

where $\pi(n,k)$ denotes the set of all nonnegative integers $(l_i)_{i\geq 1}$ such that $l_1 + l_2 + \cdots + l_{n-k+1} = k$ and $1 \cdot l_1 + 2 \cdot l_2 + \cdots + (n-k+1)l_{n-k+1} = n$.

By (25) we easily get

$$\sum_{n=0}^{\infty} B_n^L(1, 1, \dots, 1)\frac{t^n}{n!} = \exp\left(\sum_{i=1}^{\infty} t^i\right) = \exp\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} B_n^L \frac{t^n}{n!}. \tag{28}$$

From (28) we note that

$$B_n^L(1, 1, \dots, 1) = B_n^L \quad (n \geq 0).$$

By Proposition 2, (24), and Theorem 3 we get

$$B_n^L(x, x, \dots, x) = \sum_{k=0}^{n} B_{n,k}^L(x, x, \dots, x) = \sum_{k=0}^{n} B_{n,k}\big(1!x, 2!x, \dots, (n-k+1)!x_{n-k+1}\big) \tag{29}$$
$$= \sum_{k=0}^{n} x^k B_{n,k}\big(1!, 2!, \dots, (n-k+1)!\big) = \sum_{k=0}^{n} x^k L(n,k) = B_n^L(x).$$

Assume that $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ are sequences of positive integers. We define the *incomplete r-extended Lah–Bell polynomials* by

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} a_j t^j\right)^k \left(\sum_{i=0}^{\infty} b_{i+1} t^i\right)^{2r} = \sum_{n=k}^{\infty} B_{n+2r,k+2r}^L(a_1, a_2, \dots : b_1, b_2, \dots)\frac{t^n}{n!}, \tag{30}$$

where k, r are nonnegative integers.

From (30) we have

$$\begin{aligned} &B^{L}_{n+2r,k+2r}(a_1,a_2,\ldots:b_1,b_2,\ldots) \\ &\quad= B^{(2r)}_{n+2r,k+2r}(1!a_1,2!a_2,\ldots:0!b_1,1!b_2,\ldots) \\ &\quad= \sum_{\Lambda(n,k,2r)}\left[\frac{n!}{k_1!k_2!k_3!\cdots}a_1^{k_1}a_2^{k_2}a_3^{k_3}\cdots\right]\left[\frac{(2r)!}{r_0!r_1!r_2\cdots}b_1^{r_0}b_2^{r_1}b_3^{r_2}\cdots\right], \end{aligned}\tag{31}$$

where $\Lambda(n,k,2r)$ denotes the set of all nonnegative integers $\{k_i\}_{i\geq 1}$ and $\{r_i\}_{i\geq 0}$ such that $\sum_{i\geq 1}k_i = k$, $\sum_{i\geq 0}r_i = 2r$, and $\sum_{i\geq 1}i(k_i+r_i) = n$.

We define the *complete r-extended Lah–Bell polynomials* $B^{(L,2r)}_n(x\mid a_1,a_2,\ldots:b_1,b_2,\ldots)$ $(n\geq 0)$, which are given by

$$\exp\left(x\sum_{j=1}^{\infty}a_jt^j\right)\left(\sum_{i=0}^{\infty}b_{i+1}t^i\right)^{2r} = \sum_{n=0}^{\infty}B^{(L,2r)}_n(x\mid a_1,a_2,\ldots:b_1,b_2,\ldots)\frac{t^n}{n!}.\tag{32}$$

Thus we note that

$$\begin{aligned} &\exp\left(x\sum_{j=1}^{\infty}a_jt^j\right)\left(\sum_{i=0}^{\infty}b_{i+1}t^i\right)^{2r} \\ &\quad= \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\sum_{j=1}^{\infty}a_jt^j\right)^k\left(\sum_{i=0}^{\infty}b_{i+1}t^i\right)^{2r} \\ &\quad= \sum_{k=0}^{\infty}x^k\sum_{n=k}^{\infty}B^{L}_{n+2r,k+2r}(a_1,a_2,\ldots:b_1,b_2,\ldots)\frac{t^n}{n!} \\ &\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{n}x^kB^{L}_{n+2r,k+2r}(a_1,a_2,\ldots:b_1,b_2,\ldots)\frac{t^n}{n!}. \end{aligned}\tag{33}$$

From (32) and (33) we have

$$B^{(L,2r)}_n(x\mid a_1,a_2,\ldots:b_1,b_2,\ldots) = \sum_{k=0}^{n}x^kB^{L}_{n+2r,k+2r}(a_1,a_2,\ldots:b_1,b_2,\ldots)\quad(n\geq 0).\tag{34}$$

By (18), (31), (32), and (34) we have

$$\begin{aligned} &B^{(2r)}_n(1!a_1,2!a_2,\ldots:0!b_1,1!b_2,\ldots) \\ &\quad= \sum_{k=0}^{n}B^{(2r)}_{n+2r,k+2r}(1!a_1,2!a_2,\ldots:0!b_1,1!b_2,\ldots) \\ &\quad= \sum_{k=0}^{n}B^{L}_{n+2r,k+2r}(a_1,a_2,\ldots:b_1,b_2,\ldots) = B^{(L,2r)}_n(1\mid a_1,a_2,\ldots:b_1,b_2,\ldots). \end{aligned}\tag{35}$$

Therefore by (31) and (34) we obtain the following theorem.

Theorem 4 *For $n \geq 0$, we have*

$$\begin{aligned} B_n^{(L,2r)}(x \mid a_1, a_2, \ldots : b_1, b_2, \ldots) &= \sum_{k=0}^{n} x^k B_{n+2r,k+2r}^{L}(a_1, a_2, \ldots : b_1, b_2, \ldots) \\ &= \sum_{k=0}^{n} x^k B_{n+2r,k+2r}^{(2r)}(1!a_1, 2!a_2, \ldots : 0!b_1, 1!b_2, \ldots). \end{aligned}$$

From (30) we note that

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n+2r,k+2r}^{L}(1,1,\ldots;1,1,\ldots)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{j=1}^{\infty} t^j\right)^k \left(\sum_{i=0}^{\infty} t^i\right)^{2r} \\ &= \frac{1}{k!}\left(\frac{t}{1-t}\right)^k \left(\frac{1}{1-t}\right)^{2r} = \sum_{n=k}^{\infty} L_r(n,k)\frac{t^n}{n!}. \end{aligned} \tag{36}$$

By (32) and (36) we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(L,2r)}(x \mid 1,1,\ldots : 1,1,\ldots)\frac{t^n}{n!} &= \exp\left(x\sum_{j=1}^{\infty} t^j\right)\left(\sum_{i=0}^{\infty} t^i\right)^{2r} \\ &= e^{x(\frac{t}{1-t})} \cdot \left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k L_r(n,k)\right)\frac{t^n}{n!}. \end{aligned} \tag{37}$$

Thus by (36) and (37) we have

$$\begin{aligned} B_n^{(L,2r)}(x \mid 1,1,\ldots : 1,1,\ldots) &= \sum_{k=0}^{n} x^k B_{n+2r,k+2r}^{L}(1,1,\ldots : 1,1,\ldots) \\ &= \sum_{k=0}^{n} x^k L_r(n,k). \end{aligned} \tag{38}$$

Therefore we obtain the following theorem.

Theorem 5 *For $n \geq k \geq 0$, we have*

$$B_{n+2r,k+2r}^{L}(1,1,\ldots : 1,1,\ldots) = L_r(n,k)$$

and

$$B_n^{(L,2r)}(x \mid 1,1,\ldots : 1,1,\ldots) = \sum_{k=0}^{n} B_{n+2r,k+2r}^{L}(1,1,\ldots : 1,1,\ldots) = \sum_{k=0}^{n} x^k L_r(n,k).$$

From (36) and (31) we note that

$$\begin{aligned} L_r(n,k) &= B_{n+2r,k+2r}^{L}(1,1,\ldots : 1,1,\ldots) \\ &= B_{n+2r,k+2r}^{(2r)}(1!,2!,\ldots : 1!,2!,\ldots) \\ &= \sum_{\Lambda(n,k,2r)} \frac{n!}{k_1!k_2!\cdots}\frac{(2r)!}{r_0!r_1!\cdots}. \end{aligned}$$

Corollary 6 *For $n,k,r \geq 0$ with $n \geq k$, we have*

$$L_r(n,k) = \sum_{\Lambda(n,k,2r)} \frac{n!}{k_1!k_2!\cdots} \cdot \frac{(2r)!}{r_0!r_1!\cdots},$$

where $\Lambda(n,k,2r)$ denotes the set of all nonnegative integers $\{k_i\}_{i\geq 1}$ and $\{r_i\}_{i\geq}$ such that $\sum_{i\geq 1} k_i = k$, $\sum_{i\geq 0} r_i = 2r$, and $\sum_{i\geq 1} i(k_i + r_i) = n$.

Now we observe that

$$\begin{aligned}\exp\left(\sum_{i=1}^{\infty} x_i t^i\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{i=1}^{\infty} x_i t^i\right)^k \\ &= 1 + \frac{1}{1!}\sum_{i=1}^{\infty} x_i t^i + \frac{1}{2!}\left(\sum_{i=1}^{\infty} x_i t^i\right)^2 + \frac{1}{3!}\left(\sum_{i=1}^{\infty} x_i t^i\right)^3 + \cdots \\ &= \sum_{k=0}^{\infty} \sum_{m_1+2m_2+\cdots+km_k=k} \frac{1}{m_1!m_2!\cdots m_k!} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} t^k\end{aligned} \tag{39}$$

and

$$\left(\sum_{j=0}^{\infty} y_{j+1} t^j\right)^{2r} = \sum_{m=0}^{\infty} \sum_{l_1+\cdots+l_{2r}=m} y_{l_1+1} y_{l_2+1} \cdots y_{l_{2r}+1} t^m. \tag{40}$$

By (39) and (40) we get

$$\begin{aligned}&\exp\left(\sum_{i=1}^{\infty} x_i t^i\right)\left(\sum_{j=0}^{\infty} y_{j+1} \frac{t^j}{j!}\right)^{2r} \\ &= \sum_{n=0}^{\infty} n!\left(\sum_{k=0}^{n} \sum_{m_1+2m_2+\cdots+km_k=k} \sum_{l_1+l_2+\cdots+l_{2r}=n-k} \frac{1}{m_1!m_2!\cdots m_k!} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}\right. \\ &\quad \left. \times y_{l_1+1} y_{l_2+1} \cdots y_{l_{2r}+1}\right) \frac{t^n}{n!}.\end{aligned} \tag{41}$$

Therefore by (32) and (41) we obtain the following theorem.

Theorem 7 *For $n,r \geq 0$, we have*

$$\begin{aligned}&B_n^{(L,2r)}(1 \mid x_1, x_2, \ldots : y_1, y_2, \ldots) \\ &= n! \sum_{k=0}^{n} \sum_{m_1+2m_2+\cdots+km_k=k} \sum_{l_1+l_2+\cdots+l_{2r}=n-k} \frac{1}{m_1!m_2!\cdots m_k!} x_1^{m_1} \cdots x_k^{m_k} y_{l_1+1} y_{l_2+1} \cdots y_{l_{2r}+1}.\end{aligned}$$

Remark For $n \geq 0$, we have

$$B_{n+2r,k+2r}^{L}(x, x, \ldots : 1, 1, \ldots) = x^k B_{n+2r,k+2r}^{L}(1, 1, \ldots : 1, 1, \ldots). \tag{42}$$

Thus we note that

$$\sum_{k=0}^{n} B^{L}_{n+2r,k+2r}(x,x,\ldots:1,1,\ldots) = B^{L}_{n,r}(x) \quad (n \geq 0).$$

3 Conclusion

There are various methods of studying special numbers and polynomials, for example, generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, probability theory, orthogonal polynomials, and special functions. These ways of investigating special polynomials and numbers can be also applied to degenerate versions of such polynomials and numbers. Indeed, in recent years, many mathematicians have drawn their attention to studies of degenerate versions of many special polynomials and numbers by using the aforementioned means ([9, 10, 14] and references therein).

The incomplete and complete Bell polynomials arise in many different contexts as we stated in the Introduction. For instance, many special numbers, like Stirling numbers of both kinds, Lah numbers, and idempotent numbers, appear in many combinatorial and number-theoretic identities involving complete and incomplete Bell polynomials.

In this paper, we introduced the incomplete r-extended Lah–Bell polynomials and the complete r-extended Lah–Bell polynomials respectively as multivariate versions of r-Lah numbers and the r-extended Lah–Bell numbers and investigated some properties and identities for these polynomials. As corollaries of these results, we obtained some expressions for the r-Lah numbers and the r-extended Lah–Bell numbers as finite sums.

It would be very interesting to explore many applications of the incomplete and complete r-extended Lah–Bell polynomials as the incomplete and complete Bell polynomials have diverse applications.

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; LCJ, HL, and HYK checked the results of the paper; DSK and TK completed the revision of the paper. All authors have read and approved the final version of the manuscript.

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