Li and Pang *Advances in Difference Equations* (2020) 2020:67
https://doi.org/10.1186/s13662-020-2529-y

Advances in Difference Equations
a SpringerOpen Journal

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Application of Adomian decomposition method to nonlinear systems

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Abstract

In this paper, we study the Adomian decomposition method (ADM for short) including its iterative scheme and convergence analysis, which is a simple and effective technique in dealing with some nonlinear problems. We take algebraic equations and fractional differential equations as applications to illustrate ADM's efficiency.

MSC: 35C

Keywords: Nonlinear systems; Approximate solutions; Adomian decomposition method

1 Introduction

As Alan Turing said: "Science is a differential equation"; many real-world physical phenomena are described by differential equations including linear differential equations and nonlinear differential equations. In order to see the nature of the background of these phenomena, we have to solve differential equations. Generally speaking, we are good at dealing with linear problems rather than nonlinear problems. However, in practical applications, we are faced with more and more nonlinear problems. Hence we want to approximate the exact solutions to nonlinear equations by all kinds of techniques, such as linearization method, decomposition method, homotopy method, perturbation method.

With the development of science and technique, more and more phenomena cannot be well described by the classical differential equations. For example, various physical process have memory and hereditary properties and cannot be well depicted by the classical local differential operators. We have to face these new problems and obtain a new excellent tool (fractional differential equations, described by nonlocal operators) to describe these nonlocal process. Similarly, we are faced with solving fractional differential equations as well, and many methods have been proposed to tackle these problems: the residual power series method [1–13], the homotopy perturbation method [14–16], the homotopy analysis method [17–19], the tanh method [20], the extended tanh-function method [21, 22], the sine–cosine method [22, 23], the exp-function method [22, 24, 25], implicit hybrid methods [26], trigonometric basic functions [27], the polynomial least square method [28], the reproducing kernel algorithm [29], and so on [30, 31].

In this paper, we are concerned with the Adomian decomposition method (ADM for short). ADM is a kind of algorithm, based on a technique of decomposition, to construct the approximate solutions and even exact solutions with suitable initial data for nonlinear systems. This method has many advantages, such as

- It is very easy to apply and can solve wide classes of nonlinear systems including algebraic equation, ordinary differential equations, partial differential equations, integral equations, integro-differential equations, and so on and so forth [32].
- It avoids the cumbersome integrations of the Picard method.
- It can solve some nonlinear problems which cannot be solved by other numerical methods (iterative, etc.).
- The solutions series do not depend explicitly on this number, hence the number of variables is not an inconvenience in applications.

In recent years, more and more researchers have applied this method to solving nonlinear systems [33–35]. We firstly study the algorithm and convergence analysis of ADM, and then apply ADM to constructing approximate solutions for nonlinear equations with initial data, including algebraic equations, fractional ordinary differential equations and fractional partial differential equations.

This paper is organized as follows: In Sect. 2, we introduce the iterative scheme and convergence analysis of ADM. In Sect. 3, we apply ADM to construct approximate solutions for algebraic equations, time-fractional Riccati equations, time-fractional Kawahara equations and modified time-fractional Kawahara equations. In Sect. 4, we make a concluding remark about this paper.

2 Model: $u - Nu = f$

In this section, we introduce the ADM for the following functional equations:

$$u - Nu = f, \tag{2.1}$$

where N is a nonlinear operator from a Hilbert space H into H, f is a given function (system input) in H, and u is an unknown function (system output) in H [36], such a system is called a nonlinear system if it contains the nonlinear term $N(u)$. We are interested in finding the exact solutions or approximate solutions to (2.1).

We will introduce the iterative scheme of ADM, and its convergence analysis obtained by Cherruault [36], and then apply the ADM to obtain the approximate solutions to (2.1).

2.1 Adomian decomposition method

We are interested in constructing approximate solutions for (2.1). Assume that the solution to (2.1) is unique with the form

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.2}$$

Hence the following problem is to determine every term u_n. However, notice that there is a nonlinear term N in this equation, which brings about a great difficulty to complete our goal. In the present paper, we introduce the ADM [36] to overcome this difficulty. The key step is to decompose the nonlinear term N as

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where A_n are Adomian polynomials of $u_0, u_1, \ldots, u_n$, that is,

$$A_n = A_n(u_0, u_1, \ldots, u_n). \tag{2.4}$$

Unluckily, we do not know the specific form of A_n. We shall complete this task by introducing an external parameter. For the sake of simplicity, set

$$v = \sum_{n=0}^{\infty} \lambda^n u_n \quad \text{and} \quad Nv = \sum_{n=0}^{\infty} \lambda^n A_n, \tag{2.5}$$

where λ is a parameter. It should be emphasized that if $\lambda = 1$, then $v = u$ and $Nv = Nu$; if $\lambda \neq 1$, the desired A_n is the same as the case of $\lambda = 1$. Hence it is easy to see that

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \bigg|_{\lambda=0}. \tag{2.6}$$

Once A_n are obtained by (2.6), plugging (2.2) and (2.3) into (2.1), then one can see that

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} A_n + f. \tag{2.7}$$

In order to obtain u_n, define the following recurrent equations (we call these recurrent equations by Adomian relation):

$$u_0 = f, \qquad u_{n+1} = A_n(u_0, u_1, \ldots, u_n) \quad (n \in \mathbb{N}), \tag{2.8}$$

from which u_n are solvable formally. If we can solve u_n up to $n \leq N$, then $u = \sum_{n \leq N} u_n$ is called the Nth approximate solution to (2.1).

Remark 2.1 If the nonlinearity has the form $Nu = g(u)$, where g is a smooth function of u, then one can obtain the first several Adomian polynomials as [32]:

$$A_0 = g'(u_0); \tag{2.9}$$

$$A_1 = g'(u_0)u_1; \tag{2.10}$$

$$A_2 = g'(u_0)u_2 + g''(u_0)\frac{u_1^2}{2!}; \tag{2.11}$$

$$A_3 = g'(u_0)u_3 + g''(u_0)u_1u_2 + g^{(3)}(u_0)\frac{u_1^3}{3!}; \tag{2.12}$$

$$A_4 = g'(u_0)u_4 + g''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3 \right) + g^{(3)}(u_0)\frac{u_1^2 u_2}{2!} + g^{(4)}(u_0)\frac{u_1^4}{4!}. \tag{2.13}$$

2.2 Convergence analysis of ADM

In the above discussions, we obtain the approximate solutions to (2.1), but we do not know whether the following two basic but essential questions:

- Is $\sum u_n$ convergent?
- Is $\sum u_n$ a solution to (2.1)?

are answered 'yes' or 'no'. In [36], Cherruault answered the above two questions. He proved that if N is contractive, then $\sum u_n$ is convergent to the solution of (2.1). This result is stated as the following theorem.

Theorem 2.2 ([36, Theorem 3.1]) *Set*

$$S_0 = 0; \qquad S_n = u_1 + u_2 + \cdots + u_n, \quad n = 1, 2, \ldots. \tag{2.14}$$

For every sequence $y_0 + S_n$, approximating $N(y_0 + S_n)$ by

$$N_n(u_0 + S_n) = A_0 + A_1 + \cdots + A_n. \tag{2.15}$$

Then

(1) *the Adomian relation* (2.8) *is equivalent the following recurrent equations*:

$$S_0 = 0; \qquad S_{n+1} = N_n(u_0 + S_n), \quad n = 1, 2, \ldots. \tag{2.16}$$

(2) *If N is a contraction* (*i.e.*, $\|N\| = \delta < 1$), *and* $\|N_n - N\| = \epsilon_n \to 0$ $(n \to \infty)$ (*satisfied in our case*), *then the sequence S_n given by* (2.16) *converges towards the solution S of* $S = N(u_0 + S)$.

Remark 2.3 We modify the original proof given by Cherruault [36] to obtain the above iterative inequality, based on which we know that S_n converges to the solution S of $S = N(y_0 + S)$.

Remark 2.4 The recurrent relation (2.16) may be associated with the functional equation

$$N(y_0 + S) = S. \tag{2.17}$$

For Eq. (2.17), if N is contractive, then the sequence S_n defined by (2.16) converges to the only solution of (2.16). Also, for Eq. (2.1), if S^* is a solution to (2.17), then $u^* \triangleq f + S^*$ is a solution to (2.1).

Remark 2.5 In [36], Cherruault discussed the convergence of ADM in more general situations and obtained the following important theorem.

Theorem 2.6 ([36, Theorem 4.1]) *Set $R = N - I$, where I denotes the identity element. Assume R satisfies the following hypotheses*:

(1) *R is hemicontinuous.*
(2) *For any $u, v \in H$, there exists a constant $k > 0$ such that*

$$\langle Ru - Rv, u - v \rangle \geq k\|u - v\|. \tag{2.18}$$

(3) *For any* $N > 0$, *there exists a constant* $C(N) > 0$, *such that, for any* $u, v \in H$ *with* $\|u\| \leq N$, $\|v\| \leq N$, *we have*

$$\langle Ru - Rv, w\rangle \leq C(N)\|u - v\|\|w\|, \quad \forall w \in H. \tag{2.19}$$

Then we have the following results:

(i) *For every* $f \in H'$ (*dual of Hilbert space* H), (2.1) *is solvable in* H.

(ii) *The sequence* S_n *defined by*

$$S_{n+1} = S_n - \rho N(u_0 + S_n), \quad \rho > 0 \tag{2.20}$$

is strongly convergent in H *and its limit* S *the solution of*

$$S = N(u_0 + S). \tag{2.21}$$

Corollary 2.7 $S + u_0$ *with* $u_0 = f$ *is a solution to* (2.1).

Remark 2.8 In [35], Turkyilmazoglu speeded up the convergence of ADM, that is to say, the convergence region of the series approximation is found to be enlarged to a bigger physical domain.

3 Applications

3.1 Algebraic equations

In this subsection, we consider the approximate solution to algebraic equation with the following general form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0, \tag{3.1}$$

where the degree $\mathbb{N} \ni n \geq 2$ and the leading coefficient $a_n \neq 0$. For the algebraic equation (3.1), mathematicians have obtained the following celebrated theorem.

Theorem 3.1 ([37, Fundamental theorem of algebra]) *Every polynomial with real or complex coefficients has at least one complex root.*

From the fundamental theorem of algebra 3.1, the algebraic equation (3.1) is solvable in the complex field. However, it does not tell us the specific form, or "distribution" of these roots, which is also important in practical applications. As a result on the specific form of solutions to (3.1), Abel proved the following celebrated theorem.

Theorem 3.2 ([38, Abel–Ruffini theorem or Abel's impossibility theorem]) *A general algebraic equation of degree* $\mathbb{N} \ni n \geq 5$ *cannot be solved in radicals. This means that there does not exist any formula which would express the roots of such equation as functions of the coefficients by means of the algebraic operations* $(+, -, \times, \div)$ *and roots of natural degrees.*

From the Abel–Ruffini theorem, Theorem 3.2, it is impossible to obtain a specific formula for algebraic equation (3.1) as $n \geq 5$. However, it is important to have some information about (3.1). Hence, algebraic equation (3.1) is hoped to be solved approximately.

Next, we shall apply the ADM to solve algebraic equation (3.1) approximately, and take the quadratic equation

$$au^2 + bu + c = 0, \tag{3.2}$$

where a, b, c are constants and $a \neq 0$, as an example to illustrate the validity of ADM.

In [39], Adomian and Rach applied the ADM to obtain an approximate solution of (3.2). Here we will introduce the specific process of the ADM to solve (3.2).

In order to apply the ADM, rewrite (3.2) as

$$u = f + Nu, \tag{3.3}$$

where

$$f = -\frac{c}{a} \quad \text{and} \quad Nu = g(u) = -\frac{a}{b}u^2. \tag{3.4}$$

By the ADM, we assume that Eq. (3.3) has a series solution

$$u_* = u_0 + u_1 + u_2 + \cdots + u_n + \cdots \tag{3.5}$$

and the nonlinear term can be decomposed as

$$g(u) = \sum_{n=0}^{\infty} A_n, \tag{3.6}$$

where

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} g\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\bigg|_{\lambda=0} \tag{3.7}$$

$$= -\frac{a}{b}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^2\bigg|_{\lambda=0} \tag{3.8}$$

$$= -\frac{a}{b}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^{2i} u_i^2 + 2\sum_{\ell=0}^{\infty}\sum_{0\le j<k\le\ell}\lambda^{j+k}u_j u_k\right)\bigg|_{\lambda=0}. \tag{3.9}$$

Explicitly

$$A_0 = -\frac{a}{b}u_0^2, \tag{3.10}$$

$$A_1 = -\frac{a}{b}\cdot 2u_0u_1, \tag{3.11}$$

$$A_2 = -\frac{a}{b}\cdot\left(2u_0u_2 + u_1^2\right), \tag{3.12}$$

$$A_3 = -\frac{b}{2}\cdot(2u_0u_3 + 2u_1u_2), \tag{3.13}$$

$$\vdots \tag{3.14}$$

$$A_{2m-1} = -\frac{a}{b} \cdot (2u_0u_{2m-1} + 2u_1u_{2m-2} + \cdots + 2u_iu_{2m-1-i} + 2u_{m-1}u_m) \tag{3.15}$$

$$= -\frac{a}{b} \cdot \left(2\sum_{i=0}^{2m-1} u_iu_{2m-1-i}\right), \tag{3.16}$$

$$A_{2m} = -\frac{a}{b} \cdot \left(2u_0u_{2m} + 2u_1u_{2m-1} + \cdots + 2u_iu_{2m-i} + 2u_{m-1}u_{m+1} + u_m^2\right) \tag{3.17}$$

$$= -\frac{a}{b} \cdot \left(2\sum_{i=0}^{2m} u_iu_{2m-i} + u_m^2\right). \tag{3.18}$$

Hence

$$u_0 = -\frac{a}{b} \cdot \frac{bc}{a^2}, \tag{3.19}$$

$$u_1 = -\frac{a}{b}u_0^2, \tag{3.20}$$

$$u_2 = -\frac{a}{b} \cdot 2u_0u_1, \tag{3.21}$$

$$u_3 = -\frac{a}{b} \cdot \left(2u_0u_2 + u_1^2\right), \tag{3.22}$$

$$u_4 = -\frac{a}{b} \cdot (2u_0u_3 + 2u_1u_2), \tag{3.23}$$

$$\vdots \tag{3.24}$$

$$u_{2m-2} = -\frac{a}{b} \cdot (2u_0u_{2m-1} + 2u_1u_{2m-2} + \cdots + 2u_iu_{2m-1-i} + 2u_{m-1}u_m) \tag{3.25}$$

$$= -\frac{a}{b} \cdot \left(2\sum_{i=0}^{2m-1} u_iu_{2m-1-i}\right), \tag{3.26}$$

$$u_{2m-1} = -\frac{a}{b} \cdot \left(2u_0u_{2m} + 2u_1u_{2m-1} + \cdots + 2u_iu_{2m-i} + 2u_{m-1}u_{m+1} + u_m^2\right) \tag{3.27}$$

$$= -\frac{a}{b} \cdot \left(2\sum_{i=0}^{2m} u_iu_{2m-i} + u_m^2\right). \tag{3.28}$$

We claim that

$$u_* = \sum_{n=0}^{\infty} u_n \tag{3.29}$$

is a solution to (3.2). For the sake of convenience, permutating the coefficients of u_n ($n = 1, 2, \ldots$) as $(-\frac{b}{a})$ times every element of the following table:

$$\begin{array}{cccccc} u_0u_0 & u_0u_1 & u_0u_2 & u_0u_3 & u_0u_4 & \cdots \\ u_1u_0 & u_1u_1 & u_1u_2 & u_1u_3 & \cdots & \cdots \\ u_2u_0 & u_2u_1 & u_2u_2 & \cdots & \cdots & \cdots \\ u_3u_0 & u_3u_1 & \cdots & \cdots & \cdots & \cdots \\ u_4u_0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{3.30}$$

and one can see that

$$u_* = u_0 + u_1 + u_2 + \cdots + u_n + \cdots \tag{3.31}$$

$$= -\frac{c}{a} - \frac{a}{b}\left(u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + \cdots\right) \tag{3.32}$$

$$= -\frac{c}{a} - \frac{a}{b} \cdot u_*^2 \quad \left(u_*^2 \text{ can be computed by the table above}\right). \tag{3.33}$$

Hence u satisfies

$$au_*^2 + bu_* + c = 0, \tag{3.34}$$

and this indicates that u_* is a solution to (3.2).

3.2 Fractional differential equations

In this subsection, we will apply the ADM to constructing approximate solutions to the Cauchy problems of fractional differential equations, including fractional ordinary differential equations and fractional partial differential equations.

At first, we introduce the definitions of fractional derivative and fractional integral, together with some basic formulas, which will be used frequently in the specific examples.

3.2.1 Fractional calculus

The concept of fractional calculus could be traced back to 1695 [40, 41]. With the development of operator theory, fractional derivative has taken a huge leap, and there are many kinds of fractional derivative, such as the Caputo derivative, the Riemann–Liouville integral. Unluckily, there is not an uniform definition for a fractional derivative and fractional integral, which is an open problem. It should be emphasized that fractional calculus is not only a simple generalization of the classical calculus, but also it is an excellent instrument for the description of memory and hereditary properties of various physical process [42, 43], because it is defined by an integral with a singular integral kernel.

Next we introduce some definitions and lemmas of fractional calculus used frequently below.

Definition 3.3 ([44, Gamma Fuction]) The Gamma function is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad \operatorname{Re}(x) > 0. \tag{3.35}$$

Remark 3.4 For the Gamma function, we have the following important and useful properties:

- $\Gamma(1) = 1$ (this is simply the Euler integral).
- $\Gamma(1/2) = \sqrt{\pi}$ (replace t by τ^2, and then apply multiple integral to calculate it).
- (Iterative formula) For any x such that $\operatorname{Re}(x) > 0$, $\Gamma(x+1) = x\Gamma(x)$. Especially, for any $n \in \mathbb{N}$, $\Gamma(n+1) = n\Gamma(n) = \cdots = n!$, from which it can be seen that the Gamma function is a generalization of factorial operation (using integral by parts with the "LIATE rule").

Some more details of the Gamma function could found in the book [45] by Artin.

Definition 3.5 ([46, Mittag-Leffler function]) The Mittag-Leffler function is defined by

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \operatorname{Re}(\alpha) > 0. \tag{3.36}$$

Definition 3.6 ([44, Riemann–Liouville integral]) The (left sided) Riemann–Liouville fractional integral of order β ($\beta > 0$) of a function $u(x,t) \in C_p$ ($p \geq -1$) is denoted by $\mathrm{I}^\beta u(x,t)$ (with respect to t) and defined as

$$\mathrm{I}^\beta u(x,t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(x,\tau)\,\mathrm{d}\tau. \tag{3.37}$$

Definition 3.7 ([44, Caputo derivative]) The (left sided) Caputo fractional derivative of order β ($\beta > 0$) of a function $u(x,t) \in C_l^m$ is denoted by $\mathrm{D}^\beta u(x,t)$ (with respect to t) and defined as

$$\mathrm{D}^\beta u(x,t) := \begin{cases} \partial_t^m u(x,t), & \beta = m \in \mathbb{N}^*, \\ \mathrm{I}^{m-\beta} \partial_t^m u(x,t), & m-1 < \beta < m. \end{cases} \tag{3.38}$$

Lemma 3.8 ([44, Integral formula]) *For any* $m-1 < \beta \leq m \in \mathbb{N}^*$,

$$\mathrm{I}^\beta \mathrm{D}^\beta u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \partial^k u(x,0) \frac{t^k}{k!}. \tag{3.39}$$

Lemma 3.9 ([44, polynomial]) *For any* β, γ,

$$\mathrm{I}^\beta t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)} t^{\beta+\gamma}. \tag{3.40}$$

3.2.2 Time-fractional Riccati equations

In this subsection, we take the following fractional Riccati equation:

$$\mathrm{D}^\alpha u = u^2 + t \tag{3.41}$$

as an example to illustrate the ADM, where D^α is the fractional derivative in the sense of Caputo and $0 < \alpha < 1$ is a fraction. For the study of the fractional Riccati equation, Syam et al. [26] modified the implicit hybrid method to solve the Cauchy problem of the fractional Riccati equation in the sense of a conformable fractional derivative . In [27], Agheli proposed a new method based on trigonometric basic functions to obtain an approximate solution to a time-fractional Riccati equation. In [28], Bota and Caruntu applied the polynomial least square method to construct the analytical approximate solutions for quadratic Riccati differential equations. In addition, there are also many other methods to study time-fractional Riccati equations. In this paper, we shall apply the ADM to construct approximate solutions to time-fractional Riccati equations in the Caputo sense.

Clearly, by (3.39), one can see that the solution to (3.41) is equivalent to the following integral equation:

$$u(t) = u_0 + \mathrm{I}^\alpha u^2 + \mathrm{I}^\alpha t. \tag{3.42}$$

Furthermore, it follows (3.40) that

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha+2)} t^{1+\alpha} + \mathrm{I}^\alpha u^2. \tag{3.43}$$

Let

$$f \triangleq u_0 + \frac{1}{\Gamma(\alpha+2)} t^{1+\alpha}, \qquad Nu \triangleq \mathrm{I}^\alpha u^2. \tag{3.44}$$

Then (3.43) has the following simple form:

$$u = f + Nu. \tag{3.45}$$

By the idea of ADM, suppose (3.45) has a solution of the form

$$u_* = \sum_{n=0}^{\infty} u_n \tag{3.46}$$

and the nonlinear term can be decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{3.47}$$

where

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\Bigg|_{\lambda=0} \tag{3.48}$$

$$= \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \mathrm{I}^\alpha \left(\sum_{i=0}^{\infty} \lambda^i u_i\right)^2\Bigg|_{\lambda=0} \tag{3.49}$$

$$= \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \mathrm{I}^\alpha \left(\sum_{i=0}^{\infty} \lambda^{2i} u_i^2 + 2\sum_{\ell=1}^{\infty} \sum_{0\le i<j\le \ell} \lambda^{i+j} u_i u_j\right)\Bigg|_{\lambda=0} \tag{3.50}$$

$$= \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\sum_{i=0}^{\infty} \lambda^{2i} \mathrm{I}^\alpha u_i^2 + 2\sum_{\ell=1}^{\infty} \sum_{0\le i<j\le \ell} \lambda^{i+j} \mathrm{I}^\alpha u_i u_j\right)\Bigg|_{\lambda=0}. \tag{3.51}$$

Hence we obtain the iterative equations $u_0 = f$ and

$$u_{n+1} = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\sum_{i=0}^{\infty} \lambda^{2i} \mathrm{I}^\alpha u_i^2 + 2\sum_{\ell=1}^{\infty} \sum_{0\le i<j\le \ell} \lambda^{i+j} \mathrm{I}^\alpha u_i u_j\right)\Bigg|_{\lambda=0}. \tag{3.52}$$

After a "simple" computation, one can obtain the specific form of u_n as follows:

$$u_1 = \mathrm{I}^\alpha u_0^2 \tag{3.53}$$

$$u_2 = \mathrm{I}^\alpha 2u_0 u_1, \tag{3.54}$$

$$u_3 = \mathrm{I}^\alpha \left(u_1^2 + 2u_0 u_2\right), \tag{3.55}$$

$$u_4 = \mathrm{I}^\alpha(2u_0u_3 + 2u_1u_2), \tag{3.56}$$

$$u_5 = \mathrm{I}^\alpha\left(u_2^2 + 2u_0u_4 + 2u_1u_3\right), \tag{3.57}$$

$$u_6 = \mathrm{I}^\alpha(2u_0u_5 + 2u_1u_4 + 2u_2u_3), \tag{3.58}$$

$$u_7 = \mathrm{I}^\alpha\left(u_3^2 + 2u_0u_6 + 2u_1u_5 + 2u_2u_4\right), \tag{3.59}$$

$$\vdots$$

$$u_{2k-1} = \mathrm{I}^\alpha\left(u_{k-1}^2 + 2u_0u_{2k-2} + 2u_1u_{2k-3} + \cdots + 2u_{k-2}u_k\right), \tag{3.60}$$

$$u_{2k} = \mathrm{I}^\alpha(2u_0u_{2k-1} + 2u_1u_{2k-2} + \cdots + 2u_{k-1}u_k), \tag{3.61}$$

for $k = 2, 3, \ldots$. By a similar discussion to the table in (3.30), one can see that

$$u_* := f + \sum_{n=1}^{\infty} u_n \tag{3.62}$$

satisfies the identity

$$u_* - f = \mathrm{I}^\alpha u_*^2. \tag{3.63}$$

By the operation of a fractional derivative and integral, we have

$$\mathrm{D}^\alpha u_* - t = u_*^2, \tag{3.64}$$

that is,

$$\mathrm{D}^\alpha u_* = u_*^2 + t, \tag{3.65}$$

which indicates that u_* is exactly a solution to (3.41).

Remark 3.10 From the discussions above, it is easy to see that if we know the initial condition of u, i.e. u_0, then we can obtain an approximate solution for (3.41) using successive iterations.

3.2.3 Time-fractional Kawahara equation

Consider the following time-fractional Kawahara equation:

$$\mathrm{D}^\alpha u + u\partial_x u + p\partial_{xxx}u - q\partial_{xxxxx}u = 0, \tag{3.66}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R} \times \mathbb{R}$. This equation was derived by Hasimoto as a model of capillary-gravity waves in an infinitely long canal over a flat bottom in a long wave regime when the Bond number is nearly one third. This type of equation was first found by Kakutani and Ono in an analysis of magnet-acoustic waves in a cold collision free plasma. Then Hasimoto derived the above equation from capillary-gravity waves. Kawahara studied this type of equation numerically and observed that the equation has both oscillatory and monotone solitary-wave solutions. Some more details on Kawahara equations can be found in [47–50]. In fact, the Kawahara equation can be viewed as a type of the fifth KdV equation,

$$u_t + \frac{105}{16}\beta^2 u\partial_x u + \frac{13}{4}\delta\partial_{xxx}u + \gamma\partial_{xxxxx}u = 0, \tag{3.67}$$

for some special coefficients.

Next we shall apply the ADM to construct approximate solution to (3.66) with initial condition.

Obviously, the time-fractional Kawahara equation (3.66) is equivalent to the following integral form:

$$u = f + Nu, \tag{3.68}$$

where

$$f = u(x,0), \qquad Nu = -\mathrm{I}^\alpha\{u\partial_x u + p\partial_{xxx}u - q\partial_{xxxxx}u\}. \tag{3.69}$$

By the ADM, suppose that (3.68) has a solution of the following form:

$$u = \sum_{n=0}^{\infty} u_n \tag{3.70}$$

and the nonlinear term could be decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{3.71}$$

where

$$A_n \tag{3.72}$$

$$= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\bigg|_{\lambda=0} \tag{3.73}$$

$$= -\mathrm{I}^\alpha \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \tag{3.74}$$

$$\times \left\{\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\cdot\left(\sum_{i=0}^{\infty}\lambda^i\partial_x u_i\right) + p\left(\sum_{i=0}^{\infty}\lambda^i\partial_{xxx}u_i\right) - q\left(\sum_{i=0}^{\infty}\lambda^i\partial_{xxxxx}u_i\right)\right\}\bigg|_{\lambda=0} \tag{3.75}$$

$$= -\mathrm{I}^\alpha\left\{\sum_{j=0}^{n} u_i\partial_x u_{n-i} + p\partial_{xxx}u_n - q\partial_{xxxxx}u_n\right\}. \tag{3.76}$$

Furthermore

$$u_1 = A_0(u_0) \tag{3.77}$$

$$= N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\bigg|_{\lambda=0} \tag{3.78}$$

$$= -\mathrm{I}^{\alpha} \tag{3.79}$$

$$\times \left\{ \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n \right) + p \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n \right) - q \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n \right) \right\} \Bigg|_{\lambda=0} \tag{3.80}$$

$$= -\mathrm{I}^{\alpha} \{ u_0 \partial_x u_0 + p \partial_{xxx} u_0 - q \partial_{xxxxx} u_0 \}, \tag{3.81}$$

$$u_2 = A_1(u_0, u_1) \tag{3.82}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda} N \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \Bigg|_{\lambda=0} \tag{3.83}$$

$$= -\mathrm{I}^{\alpha} \frac{\mathrm{d}}{\mathrm{d}\lambda} \tag{3.84}$$

$$\times \left\{ \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n \right) + p \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n \right) - q \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n \right) \right\} \Bigg|_{\lambda=0} \tag{3.85}$$

$$= -\mathrm{I}^{\alpha} \{ u_0 \partial_x u_1 + u_1 \partial_x u_0 + p \partial_{xxx} u_1 - q \partial_{xxxxx} u_1 \}, \tag{3.86}$$

$$u_3 = A_2(u_0, u_1, u_2) \tag{3.87}$$

$$= \frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} N \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \Bigg|_{\lambda=0} \tag{3.88}$$

$$= -\mathrm{I}^{\alpha} \frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \tag{3.89}$$

$$\times \left\{ \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n \right) + p \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n \right) - q \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n \right) \right\} \Bigg|_{\lambda=0} \tag{3.90}$$

$$= -\mathrm{I}^{\alpha} \{ u_0 \partial_x u_2 + u_1 \partial_x u_1 + u_2 \partial_x u_0 + p \partial_{xxx} u_2 - q \partial_{xxxxx} u_2 \}, \tag{3.91}$$

$$u_4 = A_2(u_0, u_1, u_2, u_3) \tag{3.92}$$

$$= \frac{1}{3!} \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3} N \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \Bigg|_{\lambda=0} \tag{3.93}$$

$$= -\mathrm{I}^{\alpha} \frac{1}{3!} \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3} \tag{3.94}$$

$$\times \left\{ \left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n \right) + p \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n \right) - q \left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n \right) \right\} \Bigg|_{\lambda=0} \tag{3.95}$$

$$= -\mathrm{I}^{\alpha} \{ u_0 \partial_x u_3 + u_1 \partial_x u_2 + u_2 \partial_x u_1 + u_3 \partial_x u_0 + p \partial_{xxx} u_3 - q \partial_{xxxxx} u_3 \}, \tag{3.96}$$

and so on.

Remark 3.11 We claim that with $u_* = \sum_{n=0}^{\infty} u_n$ we have Eq. (3.66), and we do not verify it, which is the same as the examples above.

3.2.4 Modified time-fractional Kawahara equations

Consider the following modified time-fractional Kawahara equation:

$$\mathrm{D}^{\alpha}u + u^2\partial_x u + p\partial_{xxx}u - q\partial_{xxxxx}u = 0, \tag{3.97}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R} \times \mathbb{R}$. The modified Kawahara equation is known as the critical surface-tension model. This equation arises in the modeling of weakly nonlinear waves in a wide variety of media. A variety of physical phenomena, like magneto acoustic waves in a plasma, shallow-water waves with surface tension and capillary-gravity water waves, are described by the modified Kawahara equation [47].

Now we start to construct an approximate solution to (3.97) with initial value via the ADM.

Clearly, the modified time-fractional Kawahara equation (3.97) is equivalent to the following integral form:

$$u = f + Nu, \tag{3.98}$$

where

$$f = u(x,0), \qquad Nu = -\mathrm{I}^{\alpha}\left\{u^2\partial_x u + p\partial_{xxx}u - q\partial_{xxxxx}u\right\}. \tag{3.99}$$

By the ADM, suppose that (3.98) has a solution of the following form:

$$u = \sum_{n=0}^{\infty} u_n \tag{3.100}$$

and the nonlinear term could be decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{3.101}$$

where

$$A_n \tag{3.102}$$

$$= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\bigg|_{\lambda=0} \tag{3.103}$$

$$= -\mathrm{I}^{\alpha}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \tag{3.104}$$

$$\times \left\{\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^2 \cdot \left(\sum_{i=0}^{\infty}\lambda^i \partial_x u_i\right) + p\left(\sum_{i=0}^{\infty}\lambda^i \partial_{xxx}u_i\right) - q\left(\sum_{i=0}^{\infty}\lambda^i \partial_{xxxxx}u_i\right)\right\}\bigg|_{\lambda=0} \tag{3.105}$$

$$= \begin{cases} -\mathrm{I}^{\alpha}\{\sum_{j=0}^{k} u_j\partial_x u_{2k-2j} + p\partial_{xxx}u_n - q\partial_{xxxxx}u_n\}, & n = 2k; \\ -\mathrm{I}^{\alpha}\{\sum_{j=0}^{k} u_j\partial_x u_{2k+1-2j} + p\partial_{xxx}u_n - q\partial_{xxxxx}u_n\}, & n = 2k+1. \end{cases} \tag{3.106}$$

Furthermore

$$u_1 = A_0(u_0) \tag{3.107}$$

$$= N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\bigg|_{\lambda=0} \tag{3.108}$$

$$= -\mathrm{I}^{\alpha} \tag{3.109}$$

$$\times \left\{\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)^2 \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n\right) + p\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n\right) - q\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n\right)\right\}\bigg|_{\lambda=0} \tag{3.110}$$

$$= -\mathrm{I}^{\alpha}\left\{u_0^2 \partial_x u_0 + p\partial_{xxx} u_0 - q\partial_{xxxxx} u_0\right\}, \tag{3.111}$$

$$u_2 = A_1(u_0, u_1) \tag{3.112}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda} N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\bigg|_{\lambda=0} \tag{3.113}$$

$$= -\mathrm{I}^{\alpha} \frac{\mathrm{d}}{\mathrm{d}\lambda} \tag{3.114}$$

$$\times \left\{\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)^2 \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n\right) + p\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n\right) - q\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n\right)\right\}\bigg|_{\lambda=0} \tag{3.115}$$

$$= -\mathrm{I}^{\alpha}\left\{u_0^2 \partial_x u_1 + p\partial_{xxx} u_1 - q\partial_{xxxxx} u_1\right\}, \tag{3.116}$$

$$u_3 = A_2(u_0, u_1, u_2) \tag{3.117}$$

$$= \frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\bigg|_{\lambda=0} \tag{3.118}$$

$$= -\mathrm{I}^{\alpha} \frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \tag{3.119}$$

$$\times \left\{\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)^2 \cdot \left(\sum_{n=0}^{\infty} \lambda^n \partial_x u_n\right) + p\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxx} u_n\right) - q\left(\sum_{n=0}^{\infty} \lambda^n \partial_{xxxxx} u_n\right)\right\}\bigg|_{\lambda=0} \tag{3.120}$$

$$= -\mathrm{I}^{\alpha}\left\{u_0^2 \partial_x u_2 + u_2 \partial_x u_0 + p\partial_{xxx} u_2 - q\partial_{xxxxx} u_2\right\}, \tag{3.121}$$

and so on.

4 Concluding remark

In this paper, we study the Adomian decomposition method (ADM) including its convergence analysis obtained by Cherrault, and it is really an effective technique in dealing with nonlinear problems with initial data. By applying the ADM, one can construct approximate solutions to algebraic equations, fractional ordinary differential equations (time-fractional Riccati equations etc.), fractional partial differential equations (time-fractional Kawahara equations, modified time-fractional Kawahara equations etc.), and even integro-differential equations, differential algebraic equations and so on. In practical applications, we can take a finite sum according to the accuracy we need.

Acknowledgements
The authors sincerely thank the editor and anonymous reviewer for their careful reviews and useful suggestions on improving the presentation of the paper.

Funding
The research of YP was partially supported by Application of Virtual Reality Technology in Experimental Teaching in Colleges (451170404309).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 27 August 2019 Accepted: 23 January 2020 Published online: 11 February 2020

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