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On the existence of solutions for a pointwise defined multi-singular integro-differential equation with integral boundary condition

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Abstract

It is important that we increase our ability for studying of complicate fractional integro-differential equation. In this paper, we investigates the existence of solutions for a pointwise defined multi-singular fractional differential equation under some integral boundary conditions. We provide an example to illustrate our main result.

MSC: Primary 34A08; secondary 34A60

Keywords: Multi-singularity; Pointwise defined fractional equation; The Caputo derivation

1 Preliminaries

We know that many researchers are working on fractional differential equarions from different point of view (see, for example, ([1–12] and [13]). In 2015, a new fractional derivative introduced entitled Caputo–Fabrizio and some researchers tried to obtain new techniques for studying of distinct integro-differential equations via the new derivation (see, for example, [14–19]) and new fractional models and optimal controls of different phenomena with the non-singular derivative operator (see, for example, [20–24] and [25]). Also, there has been published a lot of work about physical studies on fractional calculus and new aspects of fractional different models with Mittag-Leffler law (see, for example, [26–29] and [30]).

Most researchers like to obtain numerical solutions of fractional differential equations specially singular ones (see foe example, [24, 25] and [31]). It is natural that most softwares are not able to calculate solutions of most singular differential equations now while nowadays we can prove that most complicate problems such pointwise defined multi-singular fractional differential equations under some integral boundary conditions have solutions. But finding numerical solutions is not possible yet and this weakliness relates to the structures of the software.

In 2015, Liu and Wong investigated the fractional problem ${}^{c}D^{\alpha}x(t) = f(t,x(t),D^{\beta}x(t))$ with boundary conditions $x(0) + x'(0) = y(x)$, $\int_0^1 x(t)\,dt = m$ and $x''(0) = x^{(3)}(0) = \cdots = x^{(n-1)}(0) = 0$, where $0 < t < 1$, m is a real number, $n \geq 2$, $\alpha \in (n-1,n)$, $0 < \beta < 1$, D^{α} and

D^β are the Caputo fractional derivatives, $y \in C_{\mathbb{R}}([0,1])$ and $f : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with $f(t,x,y)$ may be singular at $t = 0$ [32]. In 2016, Shabibi et al. introduced a new type of fractional differential equations entitled pointwise defined integro-differential problems [33]. Recall that $D^\alpha x(t) + f(t) = 0$ is a pointwise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E [33]. In 2018, Baleanu et al. reviewed the existence of solutions for the pointwise defined three steps crisis integro-differential equation

$$D^\alpha x(t) + f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)\,d\xi, \phi\big(x(t)\big)\right) = 0$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that $\|\phi(x) - \phi(y)\| \leq \theta_0\|x - y\| + \theta_1\|x' - y'\|$ for some non-negative real numbers θ_0 and $\theta_1 \in [0,\infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α, $f(t, x_1(t), \ldots, x_5(t)) = f_1(t, x_1(t), \ldots, x_5(t))$ for all $t \in [0,\lambda)$, $f(t, x_1(t), \ldots, x_5(t)) = f_2(t, x_1(t), \ldots, x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), \ldots, x_5(t)) = f(t, x_1(t), \ldots, x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t, \cdot, \cdot, \cdot, \cdot)$ and $f_3(t, \cdot, \cdot, \cdot, \cdot)$ are continuous on $[0,\lambda)$ and $(\mu, 1]$ and $f_2(t, \cdot, \cdot, \cdot, \cdot)$ is multi-singular [34]. In 2019, Chergui et al. reviewed the existence and uniqueness of solution for the nonlinear fractional boundary value problem $D^q x(t) = f(t, x(t), D^r x(t))$ with non-separated type integral boundary conditions $x(0) - \lambda_1 x(T) = \mu_1 \int_0^T g(s, x(s))\,ds$ and $x'(0) - \lambda_2 x'(T) = \mu_2 \int_0^T h(s, x(s))\,ds$, where $t \in [0,T]$, $1 < q \leq 2$, $0 < r \leq 1$, D^q is the Caputo fractional derivative of order q, $f \in C_{\mathbb{R}}([0,T] \times \mathbb{R} \times \mathbb{R})$, $g, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ [35].

Motivated by the work, we investigate the existence of solutions for the nonlinear fractional differential pointwise defined problem

$$D^\alpha x(t) = f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t g(\xi)x(\xi)\,d\xi\right), \tag{1}$$

with boundary conditions $x(\mu) = \int_0^1 g(z)x(z)\,dz$ and $x(0) = x^{(j)}(0) = 0$, for $2 \leq j \leq n - 1$, where $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \beta \in (0,1)$, $g, h : [0,1] \to \mathbb{R}$ are two maps such that $g, h \in L^1[0,1]$ and $f \in L^1$ is singular at some points $[0,1]$. Here, $\|\cdot\|_1$ denotes the norm of $L^1[0,1]$. We consider the sup norm $\|\cdot\|$ for $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for $C^1[0,1]$. The Riemann–Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by $I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t - s)^{p-1} f(s)\,ds$ provided that the right-hand side is pointwise defined on (a,∞). we denote $I^p f(t)$ for $I^p_{0^+} f(t)$ [36]. The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (a,\infty) \to \mathbb{R}$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\,ds$, where $n = [\alpha] + 1$ [36]. We need the following results.

Lemma 1 ([37]) *Let $0 < n - 1 \leq \alpha < n$ and $x \in C(0,1)$. Then there exist real constants $c_0, \ldots, c_{n-1}$ such that $D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$.*

Lemma 2 ([38]) *Let X be a Banach space, C a closed and convex of X, Ω a relatively open subset of C with $0 \in \Omega$ and $F : \Omega \to C$ a continuous and compact map. Then either F has a fixed point in $\bar{\Omega}$ or there exist $y \in \partial\Omega$ and $\lambda \in (0,1)$ such that $y = \lambda F y$.*

2 Main results

Lemma 3 *Let $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \beta \in (0,1)$, $f, h : [0,1] \to \mathbb{R}$ such that $f, h \in L^1[0,1]$ and $\mu \neq \int_0^1 zh(z)\,dz$. A map x is a solution for the pointwise defined equation $D^\alpha x(t) = f(t)$ with boundary conditions $x(\mu) = \int_0^1 h(z)x(z)\,dz$ and $x(0) = x^{(j)}(0) = 0$ $(2 \leq j \leq n)$ if and only if $x(t) = \int_0^1 G(t,s)f(s)\,ds$, where*

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t[(\mu-s)^\alpha - H_\alpha(s)]}{A_\mu \Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \mu \geq s, \\ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{tH_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \leq \mu \leq s \leq t \leq 1, \\ \frac{t[(\mu-s)^\alpha - H_\alpha(s)]}{A_\mu \Gamma(\alpha)}, & 0 \leq t \leq s \leq \mu \leq 1, \\ -\frac{tH_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \mu \leq s, \end{cases}$$

$H_\alpha(t) = \int_t^1 (z-t)^{\alpha-1} h(z)\,dz$ and $A_\mu = \int_0^1 (zh(z) - \mu)\,dz$.

Proof By following the related proof in [34], we concluded that Lemma 1 is valid on $L^1[0,1]$. Now let $x(t)$ be a solution for the problem. By using Lemma 1 and $x^{(j)}(0) = 0$ for $j \geq 2$, we get $x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_0 + c_1 t$. Since $x(0) = 0$, $c_0 = 0$ and so $x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_1 t$. Hence,

$$x(\mu) = \frac{1}{\Gamma(\alpha)} \int_0^\mu (\mu - s)^{\alpha-1} f(s)\,ds + c_1 \mu \tag{2}$$

and $h(z)x(z) = \frac{1}{\Gamma(\alpha)} h(z) \int_0^z (z-s)^{\alpha-1} f(s)\,ds + c_1 zh(z)$ for all $z \in [0,1]$. Thus,

$$\int_0^1 h(z)x(z)\,dz = \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^z (z-s)^{\alpha-1} h(z)f(s)\,ds\,dz + c_1 \int_0^1 zh(z)\,dz$$

and so $\int_0^1 h(z)x(z)\,dz = \frac{1}{\Gamma(\alpha)} \int_0^1 \int_s^1 (z-s)^{\alpha-1} h(z)f(s)\,dz\,ds + c_1 \int_0^1 zh(z)\,dz$. Put

$$H_\alpha(t) = \int_t^1 (z-t)^{\alpha-1} h(z)\,dz.$$

Then we have $\int_0^1 h(z)x(z)\,dz = \frac{1}{\Gamma(\alpha)} \int_0^1 H_\alpha(s)f(s)\,ds + c_1 \int_0^1 zh(z)\,dz$. By using the assumption $x(\mu) = \int_0^1 zh(z)\,dz$ and (2), we obtain

$$\frac{1}{\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} f(s)\,ds + c_1 \mu = \frac{1}{\Gamma(\alpha)} \int_0^1 H_\alpha(s)f(s)\,ds + c_1 \int_0^1 zh(z)\,dz.$$

Hence, $c_1(\int_0^1 zh(z)\,dz - \mu) = \frac{1}{\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} f(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 H_\alpha(s)f(s)\,ds$ and so

$$c_1 = \frac{1}{A_\mu \Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} f(s)\,ds - \frac{1}{A_\mu \Gamma(\alpha)} \int_0^1 H_\alpha(s)f(s)\,ds,$$

where $A_\mu = \int_0^1 (zh(z) - \mu)\,dz$. Thus,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + \frac{t}{A_\mu \Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} f(s)\,ds$$

$$- \frac{t}{A_\mu \Gamma(\alpha)} \int_0^1 H_\alpha(s) f(s)\, ds. \tag{3}$$

If $\mu \le t$, then

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^t \right) (t-s)^{\alpha-1} f(s)\, ds + \frac{t}{A_\mu \Gamma(\alpha)} \int_0^\mu (\mu - s)^{\alpha-1} f(s)\, ds \\ &\quad - \frac{t}{A_\mu \Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^t + \int_t^1 \right) H_\alpha(s) f(s)\, ds, \end{aligned}$$

and for $t \le \mu$ we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds + \frac{t}{A_\mu \Gamma(\alpha)} \left(\int_0^t + \int_t^\mu \right) (\mu - s)^{\alpha-1} f(s)\, ds \\ &\quad - \frac{t}{A_\mu \Gamma(\alpha)} \left(\int_0^t + \int_t^\mu + \int_\mu^1 \right) H_\alpha(s) f(s)\, ds. \end{aligned}$$

Hence $x(t)$ is given by $x(t) = \int_0^1 G(t,s) f(s)\, ds$, where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t[(\mu-s)^\alpha - H_\alpha(s)]}{A_\mu \Gamma(\alpha)}, & 0 \le s \le t \le 1, \mu \ge s, \\ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le \mu \le s \le t \le 1, \\ \frac{t[(\mu-s)^\alpha - H_\alpha(s)]}{A_\mu \Gamma(\alpha)}, & 0 \le t \le s \le \mu \le 1, \\ -\frac{t H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le t \le s \le 1, \mu \le s. \end{cases}$$

The converse part could be obtained easily by some straightforward calculations. □

Note that

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(\mu-s)^\alpha - H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le s \le t \le 1, \mu \ge s, \\ \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le \mu \le s \le t \le 1, \\ \frac{(\mu-s)^\alpha - H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le t \le s \le \mu \le 1, \\ -\frac{H_\alpha(s)}{A_\mu \Gamma(\alpha)}, & 0 \le t \le s \le 1, \mu \le s \end{cases}$$

and so G and $\frac{\partial}{\partial t} G$ are continuous with respect to t. Assume that $f \in L^1([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is a map such that f is singular at some points of $[0,1]$. Define the map $F : X \to X$ by

$$\begin{aligned} F_x(t) &= \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi) x(\xi)\, d\xi \right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi) x(\xi)\, d\xi \right) ds \\ &\quad + \frac{t}{A_\mu \Gamma(\alpha)} \int_0^\mu (\mu - s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi) x(\xi)\, d\xi \right) ds \\ &\quad - \frac{t}{A_\mu \Gamma(\alpha)} \int_0^1 H_\alpha(s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi) x(\xi)\, d\xi \right) ds \end{aligned}$$

for all $t \in [0,1]$, where $g : [0,1] \to \mathbb{R}$ belongs to $L^1[0,1]$. Note that, $|D^\beta x| \leq \frac{\|x'\|}{\Gamma(2-\beta)}$ and $|\int_0^s g(\xi)x(\xi)\,d\xi| \leq m\|x\|$, where $m = \int_0^1 |g(\xi)|\,d\xi$. Let $t \in [0,1]$. Then we have

$$\begin{aligned}
F_x'(t) &= \int_0^1 \frac{\partial G}{\partial t}(t,s)f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right) ds \\
&= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right) ds \\
&\quad + \frac{1}{A_\mu \Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right) ds \\
&\quad - \frac{1}{A_\mu \Gamma(\alpha)}\int_0^1 H_\alpha(s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right) ds.
\end{aligned}$$

Note that $x_0 \in X$ is a solution for the singular pointwise defined equation (1) if and only if x_0 is a fixed point of F.

Theorem 4 *Let $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \beta \in (0,1)$, $g, h : [0,1] \to \mathbb{R}$ are mappings such that $g, h \in L^1[0,1]$ with $\mu \neq \int_0^1 zh(z)\,dz$, $\Delta := \max\{1, m, \frac{1}{\Gamma(2-\beta)}\}$ and $m := \|g\|_1$. Assume that $f : [0,1] \times X^4 \to \mathbb{R}$ is a mapping such that f is singular on some point of $[0,1]$ and for all $x_1, \dots, x_4, y_1, \dots, y_4 \in X$ and for almost all $t \in [0,1]$ we have*

$$\left|f(t, x_1, x_2, \dots, x_4) - f(t, y_1, y_2, \dots, y_4)\right| \leq \sum_{i=1}^4 a_i(t)\Lambda_i\left(|x_i - y_i|\right),$$

where $a_i : [0,1] \to \mathbb{R}^+$, $\hat{a}_i \in L^1[0,1]$, $\hat{a}_i(s) = (1-s)^{\alpha-2}a_i(s)$, $\Lambda_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing mapping with respect to all their components such that $\lim_{z\to 0^+} \frac{\Lambda_i(z)}{g_i(z)} = q_i$ for some $q_i \geq 0$. Suppose that $g_1, \dots, g_4 : \mathbb{R} \to \mathbb{R}^+$ are some functions such that $\lim_{z\to 0^+} g_i(z) = 0$ for $1 \leq i \leq 4$. Also, assume that for almost all $t \in [0,1]$ and $x_1, \dots, x_4 \in X$ we have

$$\left|f(t, x_1, x_2, \dots, x_4)\right| \leq \sum_{i=1}^{k_0} \theta_i(t) M_i(x_1, \dots, x_4) + N(x_1, \dots, x_4),$$

where $k_0 \in \mathbb{N}$, $\theta_i : [0,1] \to \mathbb{R}^+$, $\hat{\theta}_i \in L^1[0,1]$, $M_i, N : \mathbb{R}^4 \to [0,\infty)$ are nondecreasing mappings with respect to all their components, $\lim_{z\to\infty} \frac{M_i(z,\dots,z)}{z} = m_i \in (0,\infty)$ and $\lim_{z\to\infty} N(z, \dots, z) < \infty$. If $(\frac{1}{\Gamma(\alpha-1)} + \frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)})\sum_{i=1}^{k_0} m_i\|\hat{\theta}_i\|_{[0,1]} \in (0, \frac{1}{\Delta})$, then the problem (1) has a solution.

Proof We show that the map F is continuous. For $x, y \in X$ and $t \in [0,1]$, we have

$$\begin{aligned}
&\left|F_x(t) - F_y(t)\right| \\
&\quad \leq \left|\int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right) ds \right. \\
&\qquad \left. - \int_0^1 G(t,s) f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s g(\xi)y(\xi)\,d\xi\right)\right| ds \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right)\right. \\
&\qquad \left. - f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s g(\xi)y(\xi)\,d\xi\right)\right| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\bigg|f\bigg(s,x(s),x'(s),D^\beta x(s),\int_0^s g(\xi)x(\xi)\,d\xi\bigg)\\
&-f\bigg(s,y(s),y'(s),D^\beta y(s),\int_0^s g(\xi)y(\xi)\,d\xi\bigg)\bigg|\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\bigg|f\bigg(s,x(s),x'(s),D^\beta x(s),\int_0^s g(\xi)x(\xi)\,d\xi\bigg)\\
&-f\bigg(s,y(s),y'(s),D^\beta y(s),\int_0^s g(\xi)y(\xi)\,d\xi\bigg)\bigg|\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)+a_2(s)\Lambda_2\big(\big|x'(s)-y'(s)\big|\big)\\
&+a_3(s)\Lambda_3\big(\big|D^\beta x(s)-D^\beta y(s)\big|\big)+a_4(s)\Lambda_4\bigg(\bigg|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\bigg|\bigg)\bigg]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)+a_2(s)\Lambda_2\big(\big|x'(s)-y'(s)\big|\big)\\
&+a_3(s)\Lambda_3\big(\big|D^\beta x(s)-D^\beta y(s)\big|\big)+a_4(s)\Lambda_4\bigg(\bigg|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\bigg|\bigg)\bigg]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\Big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)+a_2(s)\Lambda_2\big(\big|x'(s)-y'(s)\big|\big)\\
&+a_3(s)\Lambda_3\big(\big|D^\beta x(s)-D^\beta y(s)\big|\big)+a_4(s)\Lambda_4\bigg(\bigg|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\bigg|\bigg)\bigg]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|\big)\big)\bigg]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|\big)\bigg]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|\big)\bigg]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|\big)+a_2(s)\Lambda_2\big(\Delta\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\big(\Delta\big\|x'-y'\big\|\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|\big)\big]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|\big)+a_2(s)\Lambda_2\big(\Delta\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\big(\Delta\big\|x'-y'\big\|\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|\big)\big]\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|\big)+a_2(s)\Lambda_2\big(\Delta\big\|x'-y'\big\|\big)\\
&+a_3(s)\Lambda_3\big(\Delta\big\|x'-y'\big\|\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|\big)\big]\,ds,
\end{aligned}
$$

where $\Delta := \max\{1, m, \frac{1}{\Gamma(2-\beta)}\}$. Hence,

$$\begin{aligned}
&\left|F_x(t) - F_y(t)\right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl[a_1(s)\Lambda_1\bigl(\Delta\|x-y\|_*\bigr) + a_2(s)\Lambda_2\bigl(\Delta\|x-y\|_*\bigr) \\
&\qquad + a_3(s)\Lambda_3\bigl(\Delta\|x-y\|_*\bigr) + a_4(s)\Lambda_4\bigl(\Delta\|x-y\|_*\bigr)\bigr]\,ds \\
&\qquad + \frac{t}{|A_\mu|\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} \bigl[a_1(s)\Lambda_1\bigl(\Delta\|x-y\|_*\bigr) + a_2(s)\Lambda_2\bigl(\Delta\|x-y\|_*\bigr) \\
&\qquad + a_3(s)\Lambda_3\bigl(\Delta\|x-y\|_*\bigr) + a_4(s)\Lambda_4\bigl(\Delta\|x-y\|_*\bigr)\bigr]\,ds \\
&\qquad + \frac{t}{|A_\mu|\Gamma(\alpha)} \int_0^1 \bigl|H_\alpha(s)\bigr| \bigl[a_1(s)\Lambda_1\bigl(\Delta\|x-y\|_*\bigr) + a_2(s)\Lambda_2\bigl(\Delta\|x-y\|_*\bigr) \\
&\qquad + a_3(s)\Lambda_3\bigl(\Delta\|x-y\|_*\bigr) + a_4(s)\Lambda_4\bigl(\Delta\|x-y\|_*\bigr)\bigr]\,ds. \qquad (4)
\end{aligned}$$

Since $\lim_{z\to 0^+} g_i(z) = 0$, for $\epsilon > 0$ there exists $\delta_g > 0$ such that $0 < z < \delta_g$ indicates that $|g_i(z)| < \epsilon$, for all $1 \leq i \leq 4$. On the other hand by using $\lim_{z\to 0^+} \frac{\Lambda_i(z)}{g_i(z)} = q_i$, for each $\epsilon > 0$ there exists $\delta_0 > 0$ such that $\frac{\Lambda_i(z)}{g_i(z)} < q_i + \epsilon$ for $0 < z < \delta_0$ and $1 \leq i \leq 4$. Thus,

$$\Lambda_i(z) < (q_i + \epsilon) g_i(z)$$

for $0 < z < \delta_0$. Put $\delta = \min\{\delta_0, \delta_g, \epsilon\}$. Then we have

$$\Lambda_i(z) < (q_i + \epsilon)\epsilon \qquad (5)$$

for $0 < z < \delta$ and $1 \leq i \leq 4$. Let $\|x-y\|_* < \frac{\delta}{\Delta}$. By using (5) we have

$$\Lambda_i\bigl(\Delta\|x-y\|_*\bigr) < (q_i + \epsilon)\epsilon, \qquad (6)$$

for all $1 \leq i \leq 4$. Thus by using (4), for $\|x-y\|_* < \frac{\delta}{\Delta}$ we get

$$\begin{aligned}
&\left|F_x(t) - F_y(t)\right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl[a_1(s)(q_1+\epsilon)\epsilon + \cdots + a_4(s)\Lambda_4(q_4+\epsilon)\epsilon\bigr]\,ds \\
&\qquad + \frac{t}{|A_\mu|\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} \bigl[a_1(s)(q_1+\epsilon)\epsilon + \cdots + a_4(s)\Lambda_4(q_4+\epsilon)\epsilon\bigr]\,ds \\
&\qquad + \frac{t}{|A_\mu|\Gamma(\alpha)} \int_0^1 \bigl|H_\alpha(s)\bigr| \bigl[a_1(s)(q_1+\epsilon)\epsilon + \cdots + a_4(s)\Lambda_4(q_4+\epsilon)\epsilon\bigr]\,ds \\
&\quad \leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds \\
&\qquad + \frac{\epsilon t}{|A_\mu|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds \\
&\qquad + \frac{\epsilon t}{|A_\mu|\Gamma(\alpha)} \int_0^1 \bigl|H_\alpha(s)\bigr| \sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds.
\end{aligned}$$

This implies that

$$\begin{aligned}\left|H_\alpha(s)\right| &\le \int_s^1 (z-s)^{\alpha-1}\left|h(z)\right|dz \le \int_s^1 (1-s)^{\alpha-1}\left|h(z)\right|dz\\ &\le (1-s)^{\alpha-1}\int_0^1 \left|h(z)\right|dz = (1-s)^{\alpha-1}\|h\|_1 \le (1-s)^{\alpha-2}\|h\|_1\end{aligned}$$

for all $s \in [0,1]$. Now for each $t \in [0,1]$ and $x, y \in X$ with $\|x-y\|_* < \frac{\delta}{\Delta}$, we have

$$\begin{aligned}&\left|F_x(t)-F_y(t)\right|\\ &\quad\le \frac{\epsilon}{\Gamma(\alpha)}\sum_{i=1}^4 (q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right]\\ &\qquad+\frac{\epsilon t}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^4 (q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right]\\ &\qquad+\frac{\epsilon t}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^4 (q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right]\\ &\quad=\left(\sum_{i=1}^4 (q_i+\epsilon)\|\hat{a}_i\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha)}+\frac{t}{|A_\mu|\Gamma(\alpha)}+\frac{t\|h\|_1}{|A_\mu|\Gamma(\alpha)}\right)\epsilon.\end{aligned}$$

Thus,

$$\|F_x-F_y\| \le \left(\sum_{i=1}^4 (q_i+\epsilon)\|\hat{a}_i\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha)}+\frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)}\right)\epsilon$$

for $\|x-y\|_* < \frac{\delta}{\Delta}$. Also for each $t \in [0,1]$ and $x, y \in X$, we have

$$\begin{aligned}&\left|F_x'(t)-F_y'(t)\right|\\ &\quad\le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s g(\xi)x(\xi)\,d\xi\right)\right.\\ &\qquad\left.-f\left(s,y(s),y'(s),D^\beta y(s),\int_0^s g(\xi)y(\xi)\,d\xi\right)\right|ds\\ &\qquad+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-1}\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s g(\xi)x(\xi)\,d\xi\right)\right.\\ &\qquad\left.-f\left(s,y(s),y'(s),D^\beta y(s),\int_0^s g(\xi)y(\xi)\,d\xi\right)\right|ds\\ &\qquad+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^1 \left|H_\alpha(s)\right|\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s g(\xi)x(\xi)\,d\xi\right)\right.\\ &\qquad\left.-f\left(s,y(s),y'(s),D^\beta y(s),\int_0^s g(\xi)y(\xi)\,d\xi\right)\right|ds\\ &\quad\le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\Big[a_1(s)\Lambda_1\big(\left|x(s)-y(s)\right|\big)+a_2(s)\Lambda_2\big(\left|x'(s)-y'(s)\right|\big)\\ &\qquad+a_3(s)\Lambda_3\big(\left|D^\beta x(s)-D^\beta y(s)\right|\big)+a_4(s)\Lambda_4\left(\left|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\right|\right)\Big]ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)+a_2(s)\Lambda_2\big(\big|x'(s)-y'(s)\big|\big)\\
&+a_3(s)\Lambda_3\big(\big|D^\beta x(s)-D^\beta y(s)\big|\big)+a_4(s)\Lambda_4\bigg(\bigg|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\bigg|\bigg)\bigg]\,ds\\
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\Big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)+a_2(s)\Lambda_2\big(\big|x'(s)-y'(s)\big|\big)\\
&+a_3(s)\Lambda_3\big(\big|D^\beta x(s)-D^\beta y(s)\big|\big)+a_4(s)\Lambda_4\bigg(\bigg|\int_0^s g(\xi)\big(x(\xi)-y(\xi)\big)\,d\xi\bigg|\bigg)\bigg]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\|x'-y'\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|)\big)\Big]\,ds\\
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\|x'-y'\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|\big)\Big]\,ds\\
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\Big[a_1(s)\Lambda_1\big(\|x-y\|\big)+a_2(s)\Lambda_2\big(\|x'-y'\|\big)\\
&+a_3(s)\Lambda_3\bigg(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\bigg)+a_4(s)\Lambda_4\big(m\|x-y\|\big)\Big]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|_*\big)+a_2(s)\Lambda_2\big(\Delta\|x-y\|_*\big)\\
&+a_3(s)\Lambda_3\big(\Delta\|x-y\|_*\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|_*\big)\big]\,ds\\
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|_*\big)+a_2(s)\Lambda_2\big(\Delta\|x-y\|_*\big)\\
&+a_3(s)\Lambda_3\big(\Delta\|x-y\|_*\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|_*\big)\big]\,ds\\
&+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\big[a_1(s)\Lambda_1\big(\Delta\|x-y\|_*\big)+a_2(s)\Lambda_2\big(\Delta\|x-y\|_*\big)\\
&+a_3(s)\Lambda_3\big(\Delta\|x-y\|_*\big)+a_4(s)\Lambda_4\big(\Delta\|x-y\|_*\big)\big]\,ds.
\end{aligned}
$$

Thus, by using (5), we get

$$
\begin{aligned}
&\big|F'_x(t)-F'_y(t)\big|\\
&\quad\le\frac{\epsilon}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds\\
&\qquad+\frac{\epsilon}{|A_\mu|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds\\
&\qquad+\frac{\epsilon}{|A_\mu|\Gamma(\alpha)}\int_0^1\big|H_\alpha(s)\big|\sum_{i=1}^4 a_i(s)(q_i+\epsilon)\,ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\epsilon}{\Gamma(\alpha-1)}\sum_{i=1}^{4}(q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right] \\
&\quad+\frac{\epsilon}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{4}(q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right] \\
&\quad+\frac{\epsilon}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{4}(q_i+\epsilon)\left[\int_0^1 (1-s)^{\alpha-2}a_i(s)\,ds\right] \\
&=\left(\sum_{i=1}^{4}(q_i+\epsilon)\|\hat{a}_i\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|A_\mu|\Gamma(\alpha)}+\frac{\|h\|_1}{|A_\mu|\Gamma(\alpha)}\right)\epsilon
\end{aligned}$$

whenever $\|x-y\|_* < \frac{\delta}{\Delta}$. Hence, $\|F'_x - F'_y\| \leq (\sum_{i=1}^4 (q_i+\epsilon)\|\hat{a}_i\|_{[0,1]})(\frac{1}{\Gamma(\alpha-1)} + \frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)})\epsilon$ whenever $\|x-y\|_* < \frac{\delta}{\Delta}$ and so

$$\begin{aligned}
\|F_x - F_y\|_* &= \max\left\{\|F_x - F_y\|, \left\|F'_x - F'_y\right\|\right\} \\
&\leq \left(\sum_{i=1}^{4}(q_i+\epsilon)\|\hat{a}_i\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha-1)}+\frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)}\right)\epsilon
\end{aligned}$$

whenever $\|x-y\|_* < \frac{\delta}{\Delta}$. Since $\epsilon > 0$ was arbitrary, $\|F_x - F_y\|_* \to 0$ whenever $\|x-y\|_* \to 0$. This implies that F is continuous. Since $\lim_{z\to\infty} \frac{M_i(\Delta z,\ldots,\Delta z)}{\Delta z} = m_i$ for all $1 \leq i \leq k_0$, for each $\epsilon > 0$ there exists $r(\epsilon) > 0$ such that $|\frac{M_i(\Delta z,\ldots,\Delta z)}{\Delta z} - m_i| < \epsilon$ whenever $z \in [r(\epsilon),\infty)$. Thus,

$$M_i(\Delta z,\ldots,\Delta z) < (m_i+\epsilon)\Delta z \tag{7}$$

for all $z \in [r(\epsilon),\infty)$. Since $\lim_{z\to\infty} N(\Delta z,\ldots,\Delta z) < \infty$, $\lim_{z\to\infty} \frac{N(\Delta z,\ldots,\Delta z)}{\Delta z} = 0$. Hence, there exists $r'(\epsilon) > 0$ such that $\frac{N(\Delta z,\ldots,\Delta z)}{\Delta z} < \epsilon$ and

$$N(\Delta z,\ldots,\Delta z) < \Delta z\epsilon, \tag{8}$$

for all $z \in [r'(\epsilon),\infty)$. Since $(\frac{1}{\Gamma(\alpha-1)} + \frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)})\sum_{i=1}^{k_0} m_i\|\hat{\theta}_i\|_{[0,1]} \in (0,\frac{1}{\Delta})$, there exists $\epsilon_0 > 0$ such that $(\frac{1}{\Gamma(\alpha-1)} + \frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)})\sum_{i=1}^{k_0}(m_i+\epsilon)\|\hat{\theta}_i\|_{[0,1]} + (\frac{1}{\Gamma(\alpha+1)} + \frac{\mu^\alpha+\|h\|_1}{|A_\mu|\Gamma(\alpha+1)})\epsilon_0 \in (0,\frac{1}{\Delta})$. Now, put $r_0 := \max\{r(\epsilon_0), r'(\epsilon_0)\}$. By using (7) and (8) for $z = r_0$, we get $M_i(\Delta r_0,\ldots,\Delta r_0) < (m_i+\epsilon_0)\Delta r_0$ and $N(\Delta r_0,\ldots,\Delta r_0) < \Delta r_0\epsilon_0$. Define $\Omega = \{x \in X : \|x\|_* < r_0\}$. Let $x_0 \in \partial\Omega$ and $\lambda \in (0,1)$ be such that $x_0 = \lambda F_{x_0}$. Then $\|x_0\|_* = r$. Now for each $t \in [0,1]$, we have

$$x_0(t) = \lambda\int_0^1 G(t,s)f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s g(\xi)x(\xi)\,d\xi\right)ds$$

and so

$$\begin{aligned}
&\left|x_0(t)\right| \\
&\quad= \left|\lambda\int_0^1 G(t,s)f\left(s, x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi)x_0(\xi)\,d\xi\right)ds\right| \\
&\quad\leq \lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sum_{i=1}^{k_0}\theta_i(s)M_i\left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi)x_0(\xi)\,d\xi\right)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} N\biggl(x_0(s),x_0'(s),D^\beta x_0(s),\int_0^s g(\xi)x_0(\xi)\,d\xi\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-1}\sum_{i=1}^{k_0}\theta_i(s)M_i\biggl(x_0(s),x_0'(s),D^\beta x_0(s),\int_0^s g(\xi)x_0(\xi)\,d\xi\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-1} N\biggl(x_0(s),x_0'(s),D^\beta x_0(s),\int_0^s g(\xi)x_0(\xi)\,d\xi\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1 H_\alpha(s)\sum_{i=1}^{k_0}\theta_i(s)M_i\biggl(x_0(s),x_0'(s),D^\beta x_0(s),\int_0^s g(\xi)x_0(\xi)\,d\xi\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1 H_\alpha(s)\sum_{i=1}^{k_0}\theta_i(s)N\biggl(x_0(s),x_0'(s),D^\beta x_0(s),\int_0^s g(\xi)x_0(\xi)\,d\xi\biggr)\,ds\biggr]\\
&\le \lambda\biggl[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^t (t-s)^{\alpha-1}\theta_i(s)M_i\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^\mu (\mu-s)^{\alpha-1}\theta_i(s)M_i\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-1}N\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^1 H_\alpha(s)\theta_i(s)M_i\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\int_0^1 H_\alpha(s)\sum_{i=1}^{k_0}\theta_i(s)N\biggl(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\biggr)\,ds\biggr]\\
&\le \lambda\biggl[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1 (1-s)^{\alpha-2}\theta_i(s)\,ds\\
&+\frac{1}{\Gamma(\alpha)}N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^t (t-s)^{\alpha-1}\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1 (1-s)^{\alpha-2}\theta_i(s)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^\mu (\mu-s)^{\alpha-1}\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\|h\|_1\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1 (1-s)^{\alpha-2}\theta_i(s)\,ds\\
&+\frac{t}{|A_\mu|\Gamma(\alpha)}\|h\|_1 N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1 (1-s)^{\alpha-1}\,ds\biggr]\\
&\le \lambda\biggl[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\Delta(m_i+\epsilon_0)r_0\|\hat{\theta}_i\|_{[0,1]}+\frac{t^\alpha}{\Gamma(\alpha+1)}\Delta r_0\epsilon_0
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{t}{|A_\mu|\Gamma(\alpha)} \sum_{i=1}^{k_0} \Delta(m_i + \epsilon_0) r_0 \|\hat{\theta}_i\|_{[0,1]} + \frac{t\mu^\alpha}{|A_\mu|\Gamma(\alpha+1)} \Delta r_0 \epsilon_0 \\
&+ \frac{t\|h\|_1}{|A_\mu|\Gamma(\alpha)} \sum_{i=1}^{k_0} \Delta(m_i + \epsilon_0) r_0 \|\hat{\theta}_i\|_{[0,1]} + \frac{t\|h\|_1}{|A_\mu|\Gamma(\alpha+1)} \Delta r_0 \epsilon_0 \Bigg]
\end{aligned}
$$

for all $t \in [0,1]$. Hence,

$$
\begin{aligned}
\|x_0\| &\le \lambda \Bigg[\left(\frac{1}{\Gamma(\alpha)} + \frac{1 + \|h\|_1}{|A_\mu|\Gamma(\alpha)} \right) \sum_{i=1}^{k_0} (m_i + \epsilon_0) \|\hat{\theta}_i\|_{[0,1]} \\
&\quad + \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\mu^\alpha + \|h\|_1}{|A_\mu|\Gamma(\alpha+1)} \right) \epsilon_0 \Bigg] \Delta r_0 \\
&\le \lambda \Bigg[\left(\frac{1}{\Gamma(\alpha-1)} + \frac{1 + \|h\|_1}{|A_\mu|\Gamma(\alpha)} \right) \sum_{i=1}^{k_0} (m_i + \epsilon_0) \|\hat{\theta}_i\|_{[0,1]} \\
&\quad + \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\mu^\alpha + \|h\|_1}{|A_\mu|\Gamma(\alpha+1)} \right) \epsilon_0 \Bigg] \Delta r_0 \\
&< r_0.
\end{aligned}
$$

Also for each $t \in [0,1]$, we have

$$
\begin{aligned}
&\left| x_0'(t) \right| \\
&= \left| \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s) f\left(s, x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \right| \\
&\le \lambda \Bigg[\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \sum_{i=1}^{k_0} \theta_i(s) M_i \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} N \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \\
&\quad + \frac{1}{|A_\mu|\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} \sum_{i=1}^{k_0} \theta_i(s) M_i \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \\
&\quad + \frac{1}{|A_\mu|\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} N \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \\
&\quad + \frac{1}{|A_\mu|\Gamma(\alpha)} \int_0^1 H_\alpha(s) \sum_{i=1}^{k_0} \theta_i(s) M_i \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \\
&\quad + \frac{1}{|A_\mu|\Gamma(\alpha)} \int_0^1 H_\alpha(s) \sum_{i=1}^{k_0} \theta_i(s) N \left(x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s g(\xi) x_0(\xi)\, d\xi \right) ds \Bigg] \\
&\le \lambda \Bigg[\frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_0^t (t-s)^{\alpha-2} \theta_i(s) M_i \left(\|x_0\|_*, \|x_0\|_*, \frac{\|x_0\|_*}{\Gamma(2-\beta)}, m\|x_0\|_* \right) ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} N \left(\|x_0\|_*, \|x_0\|_*, \frac{\|x_0\|_*}{\Gamma(2-\beta)}, m\|x_0\|_* \right) ds
\end{aligned}
$$

$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^\mu(\mu-s)^{\alpha-1}\theta_i(s)M_i\left(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\right)ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^\mu(\mu-s)^{\alpha-1}N\left(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\right)ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^1 H_\alpha(s)\theta_i(s)M_i\left(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\right)ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\int_0^1 H_\alpha(s)\sum_{i=1}^{k_0}\theta_i(s)N\left(\|x_0\|_*,\|x_0\|_*,\frac{\|x_0\|_*}{\Gamma(2-\beta)},m\|x_0\|_*\right)ds\Bigg]$$
$$\le\lambda\Bigg[\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1(1-s)^{\alpha-2}\theta_i(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha-1)}N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^t(t-s)^{\alpha-2}\,ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1(1-s)^{\alpha-2}\theta_i(s)\,ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^\mu(\mu-s)^{\alpha-1}\,ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\|h\|_1\sum_{i=1}^{k_0}M_i(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1(1-s)^{\alpha-2}\theta_i(s)\,ds$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\|h\|_1N(\Delta r_0,\Delta r_0,\Delta r_0,\Delta r_0)\int_0^1(1-s)^{\alpha-1}\,ds\Bigg]$$
$$\le\lambda\Bigg[\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Delta(m_i+\epsilon_0)r_0\|\hat{\theta}_i\|_{[0,1]}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\Delta r_0\epsilon_0$$
$$+\frac{1}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\Delta(m_i+\epsilon_0)r_0\|\hat{\theta}_i\|_{[0,1]}+\frac{\mu^\alpha}{|A_\mu|\Gamma(\alpha+1)}\Delta r_0\epsilon_0$$
$$+\frac{\|h\|_1}{|A_\mu|\Gamma(\alpha)}\sum_{i=1}^{k_0}\Delta(m_i+\epsilon_0)r_0\|\hat{\theta}_i\|_{[0,1]}+\frac{\|h\|_1}{|A_\mu|\Gamma(\alpha+1)}\Delta r_0\epsilon_0\Bigg].$$

Thus, $\|x_0'\|\le\lambda[(\frac{1}{\Gamma(\alpha-1)}+\frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)})\sum_{i=1}^{k_0}(m_i+\epsilon_0)\|\hat{\theta}_i\|_{[0,1]}+(\frac{1}{\Gamma(\alpha+1)}+\frac{\mu^\alpha+\|h\|_1}{|A_\mu|\Gamma(\alpha+1)})\epsilon_0]\Delta r_0<r_0$ and so $\|x_0\|_*=\max\{\|x_0\|,\|x_0'\|\}<r_0$. Now by using Lemma 2, F has a fixed point in $\bar{\Omega}$ and so the problem (1) has a solution. □

Example 1 Consider the problem

$$D^{\frac{5}{2}}x(t)+\theta(t)M\left(x(t),x'(t),D^{\frac{1}{2}}x(t),\int_0^t\xi x(\xi)\,d\xi\right)+N\left(x(t),x'(t),D^{\frac{1}{2}}x(t),\int_0^t\xi x(\xi)\,d\xi\right)$$
$$=0$$

with boundary conditions $x(\frac{2}{3})=\int_0^1 zx(z)\,dz$ and $x(0)=x''(0)=0$, where $M(x_1,\dots,x_4)=\Sigma_{i=1}^4|x_i|$, $N(x_1,\dots,x_4)=\sum_{i=1}^4\frac{|x_i|}{1+|x_i|}$, $\theta(t)=\frac{1}{50\sqrt{1-tp(t)}}$ and $p(t)=0$ whenever $t\in[0,1]\cap\mathcal{Q}$

and $p(t) = 1$ as $t \in [0,1] \cap \mathcal{Q}^c$. Put $f(t,(x_1,\dots,x_4) = \theta(t)M(x_1,\dots,x_4) + N(x_1,\dots,x_4)$, $k_0 = 1$, $M_1(x_1,\dots,x_4) := M(x_1,\dots,x_4)$, $\theta_1(t) := \theta(t)$, $a_1(t) = \cdots = a_4(t) := 1 + \theta(t)$, $\Lambda_1(z) = \cdots = \Lambda_4(z) = z$, $g_1(z) = \cdots = g_4(z) = g(t) = h(t) = t$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\lambda = \frac{1}{2}$ and $\mu = \frac{2}{3}$. Then we have

$$\begin{aligned}
&\bigl|f(t,x_1,x_2,\dots,x_4) - f(t,y_1,y_2,\dots,y_4)\bigr| \\
&\quad = \theta(t)\left|\sum_{i=1}^{4} |x_i| - |y_i|\right| + \left|\sum_{i=1}^{4} \frac{(1+|y_i|)|x_i| - (1+|x_i|)|y_i|}{(1+|x_i|)(1+|y_i|)}\right| \\
&\quad \le \theta(t)\sum_{i=1}^{4} |x_i - y_i| + \sum_{i=1}^{4} \frac{|x_i - y_i|}{(1+|x_i|)(1+|y_i|)} \\
&\quad \le \theta(t)\sum_{i=1}^{4} |x_i - y_i| + \sum_{i=1}^{4} |x_i - y_i| \\
&\quad = \bigl(\theta(t)+1\bigr)\sum_{i=1}^{4} |x_i - y_i| = \sum_{i=1}^{4} a_i(t)\Lambda_i\bigl(|x_i - y_i|\bigr).
\end{aligned}$$

One can see that $\hat{a}_i(t) = (1-s)^{\alpha-2}a_i(t) \in L^1[0,1]$, $\Lambda_i : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, $\lim_{z\to 0^+} \frac{\Lambda_i(z)}{g_i(z)} = 1 := q_i \in [0,\infty)$ and $\lim_{z\to 0^+} g_i(z) = 0$ for $1 \le i \le 4$. Also, M and N are nondecreasing with respect to all their components, $\lim_{z\to\infty} \frac{M_1(z,\dots,z)}{z} = 4 := m_1 \in (0,\infty)$ and $\lim_{z\to\infty} N(z,\dots,z) = 4 < \infty$. Note that, $g, h \in L^1[0,1]$, $\int_0^1 zh(z)\,dz = \frac{1}{3} \neq \mu$, $m := \|g\|_1 = \frac{1}{2}$, $A_\mu = \int_0^1 (zh(z) - \mu)\,dz = \int_0^1 (z^2 - \frac{2}{3})\,dz = \frac{1}{3}$, $\Delta := \max\{1, m, \frac{1}{\Gamma(2-\beta)}\} = \frac{2}{\sqrt{\pi}}$ and $\|\hat{\theta}_1\|_{[0,1]} = \frac{1}{50}$. Finally, we have

$$\begin{aligned}
&\left(\frac{1}{\Gamma(\alpha-1)} + \frac{1+\|h\|_1}{|A_\mu|\Gamma(\alpha)}\right)\sum_{i=1}^{k_0} m_i\|\hat{\theta}_i\|_{[0,1]} \\
&\quad = \left(\frac{1}{\Gamma(\frac{3}{2})} + \frac{1+\frac{1}{2}}{\frac{1}{3}\Gamma(\frac{5}{2})}\right) \times 4 \times \frac{1}{50} \\
&\quad = \left(\frac{2}{\sqrt{\pi}} + 6\sqrt{\pi}\right) \times 4 \times \frac{1}{50} \in \left(0, \frac{2}{\sqrt{\pi}}\right) = \left(0, \frac{1}{\Delta}\right).
\end{aligned}$$

Thus by using Theorem 4, this problem has a solution.

3 Conclusion

It is important that we increase our ability for studying of complicate fractional integro-differential equation. One of such equations are pointwise defined multi-singular fractional differential equations. Solving of such equations prepares us for modeling of most phenomena without removing most parameters which play a role in the phenomena. It is natural that most software is not able to calculate solutions of most singular differential equations now, while this weakliness relates to the structures of the softwares. In this work, we study the existence of solutions for a pointwise defined multi-singular fractional differential equation under some integral boundary conditions.

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