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# On boundedness of unified integral operators for quasiconvex functions

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**Abstract**

This work deals with the bounds of a unified integral operator with which several fractional and conformable integral operators are directly associated. By using quasiconvex and monotone functions we establish bounds of these integral operators. We prove their boundedness and continuity. The results of this paper generalize already published results and have direct consequences for fractional and conformable integrals

**Keywords:** Quasiconvex function; Integral operators; Fractional integral operators; Conformable integral operators; Boundedness

## 1 Introduction

We start from the definition of Riemann–Liouville fractional integral operators.

**Definition 1** ([15]) Let $f \in L_1[a,b]$. Then the Riemann–Liouville fractional integrals of order $\mu$ with $\Re(\mu) > 0$ are defined by

$$^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\,dt, \quad x > a, \tag{1.1}$$

$$^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)\,dt, \quad x < b, \tag{1.2}$$

where $\Gamma$ is the gamma function.

An $k$-fractional analogues of the Riemann–Liouville integral operators are given in the next definition.

**Definition 2** ([18]) Let $f \in L_1[a,b]$. Then the $k$-fractional Riemann–Liouville integrals of order $\mu$ with $\Re(\mu) > 0$, $k > 0$, are defined by

$$^{\mu}I^k_{a^+}f(x) = \frac{1}{k\Gamma_\kappa(\mu)}\int_a^x (x-t)^{\frac{\mu}{k}-1}f(t)\,dt, \quad x > a, \tag{1.3}$$

$$^{\mu}I^k_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{\frac{\mu}{k}-1}f(t)\,dt, \quad x < b, \tag{1.4}$$

where $\Gamma_k$ is defined in [19].

We go ahead by defining the following generalized fractional integral operators:

**Definition 3** ([15]) Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Let $g$ be an increasing positive function on $(a, b]$ having a continuous derivative $g'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to a function $g$ on $[a, b]$ of order $\mu$ with $\Re(\mu) > 0$ are defined by

$$
{}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \bigl(g(x) - g(t)\bigr)^{\mu-1} g'(t)f(t)\,dt, \quad x > a, \tag{1.5}
$$

$$
{}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \bigl(g(t) - g(x)\bigr)^{\mu-1} g'(t)f(t)\,dt, \quad x < b. \tag{1.6}
$$

**Definition 4** ([16]) Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Let $g$ be an increasing positive function on $(a, b]$ having a continuous derivative $g'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to a function $g$ on $[a, b]$ of order $\mu$ with $\Re(\mu) > 0$, $k > 0$, are defined by

$$
{}_g^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \bigl(g(x) - g(t)\bigr)^{\frac{\mu}{k}-1} g'(t)f(t)\,dt, \quad x > a, \tag{1.7}
$$

$$
{}_g^{\mu}I_{b^-}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b \bigl(g(t) - g(x)\bigr)^{\frac{\mu}{k}-1} g'(t)f(t)\,dt, \quad x < b. \tag{1.8}
$$

A generalized fractional integral operator containing an extended Mittag-Leffler function is defined as follows.

**Definition 5** ([1]) Let $\omega, \mu, \alpha, l, \gamma, c \in C$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ are defined by

$$
\bigl(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\bigr)(x;p) = \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\bigl(\omega(x-t)^{\mu};p\bigr) f(t)\,dt, \tag{1.9}
$$

$$
\bigl(\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f\bigr)(x;p) = \int_x^b (t-x)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\bigl(\omega(t-x)^{\mu};p\bigr) f(t)\,dt, \tag{1.10}
$$

where

$$
E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}} \tag{1.11}
$$

is the extended generalized Mittag-Leffler function.

Recently, Farid in [7] studied the unified integral operator stated as follows (see also, [17]):

**Definition 6** Let $f, g : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$, be functions such that $f$ is positive, $f \in L_1[a, b]$, and $g$ is differentiable and strictly increasing. Also, let $\frac{\phi}{x}$ be an increasing function on $[a, \infty)$, and let $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then for $x \in [a, b]$, the left and right integral operators are defined by

$$\big({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\big)(x,\omega;p)=\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega\big(g(x)-g(t)\big)^{\mu};p\big)g'(t)f(t)\,dt, \tag{1.12}$$

$$\big({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\big)(x,\omega;p)=\int_x^b\frac{\phi(g(t)-g(x))}{g(t)-g(x)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega\big(g(t)-g(x)\big)^{\mu};p\big)g'(t)f(t)\,dt. \tag{1.13}$$

For suitable settings of functions $\phi$, $g$ and certain values of parameters included in the Mittage-Leffler function (1.11), many of the fractional integral operators defined in recent decades can be obtained simultaneously; see [17, Remarks 1 and 2].

The aim of this paper is the study of bounds of a unified integral operator by using quasiconvex functions. The results we intend to establish are directly related with fractional and conformable integral operators. All the fractional and conformable integral operators defined in [2, 3, 6, 10, 13–15, 18, 20, 21, 23–26] satisfy the results of this paper for quasiconvex functions in particular cases.

**Definition 7** ([22]) A function $f$ satisfying the inequality

$$f\big(\lambda x+(1-\lambda)y\big)\le\lambda f(x)+(1-\lambda)f(y) \tag{1.14}$$

for $\lambda\in[0,1]$ and $x,y\in C$, where $C$ is a convex set, is called a convex function on $C$.

A geometric interpretation of a convex function $f:[a,b]\to\mathbb{R}$ is visualized by the well-known Hadamard inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.15}$$

Finite convex functions defined on a finite closed interval are quasiconvex functions, whereas quasiconvex functions are defined as follows.

**Definition 8** ([12]) A function $f$ satisfying the inequality

$$f\big(\lambda a+(1-\lambda)b\big)\le\max\big\{f(a),f(b)\big\} \tag{1.16}$$

for $\lambda\in[0,1]$ and $x,y\in C$, where $C$ is a convex set, is called a quasiconvex function on $C$.

The following example distinguishes the above two definitions.

*Example* 1 ([12]) The function $f:[-2,2]\to\mathbb{R}$ given by

$$f(x)=\begin{cases}1, & x\in[-2,-1],\\ x^2, & x\in(-1,2],\end{cases}$$

is not a convex function on $[-2,2]$, but it is a quasiconvex function on $[-2,2]$.

Thus the class of quasiconvex functions contains the class of finite convex functions defined on finite closed intervals. The investigation of Hadamard inequality for quasiconvex functions is an implicit topic, and related results have been obtained independently by various authors; see, for example, [5, 11, 12] and references therein.

To get results for unified integral operators of quasiconvex functions, we follow the method from [17]. The paper is organized as: First, we obtain upper bounds of unified integral operators defined in (1.12) and (1.13), which lead to the boundedness and continuity of these operators. Then we obtain bounds in the form of a Hadamard-type inequality by imposing the symmetric property on quasiconvex functions. Finally, by defining the convolution of two functions we obtain a modulus inequality. All these results hold for almost all kinds of associated fractional and conformable integral operators. Also, some very particular cases of the proved results are already published in [4, 9, 27], and connection with them is stated in remarks.

## 2 Main results

**Theorem 1** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a positive integrable quasiconvex function. Let $g:[a,b]\longrightarrow\mathbb{R}$ be a differentiable and strictly increasing function, let $\frac{\phi}{x}$ be an increasing function on $[a,b]$, and let $g'\in L[a,b]$. If $\alpha,l,\gamma,c\in\mathbb{C}$, $p,\mu,\nu\geq 0$, $\delta\geq 0$, $0<k\leq\delta+\mu$, and $0<k\leq\delta+\nu$, then for $x\in(a,b)$, we have*

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\leq M^x_a(f)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right), \tag{2.1}$$

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(x,\omega;p)\leq M^b_x(f)\phi\left(g(b)-g(x)\right)E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega\left(g(b)-g(x)\right)^{\nu};p\right), \tag{2.2}$$

*and hence*

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(x,\omega;p)\\&\quad\leq M^x_a(f)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right)\\&\qquad+M^b_x(f)\phi\left(g(b)-g(x)\right)E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega\left(g(b)-g(x)\right)^{\nu};p\right),\end{aligned} \tag{2.3}$$

*where $M^b_a(f):=\max\{f(a),f(b)\}$.*

*Proof* Under the assumptions of the theorem, we can obtain the inequality

$$\begin{aligned}&\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right)\\&\quad\leq\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right);\quad t\in[a,x],x\in(a,b).\end{aligned} \tag{2.4}$$

By using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$ we get the following inequality:

$$\begin{aligned}&\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right)\\&\quad\leq\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right).\end{aligned} \tag{2.5}$$

Using the quasiconvexity of $f$, for $t \in [a,x]$, we have $f(t) \leq M_a^x(f)$. Therefore we get the inequality

$$\begin{aligned}&\int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)} g'(t) f(t) E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}\bigl(\omega\bigl(g(x)-g(t)\bigr)^{\mu};p\bigr)\,dt \\ &\quad\leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\bigl(\omega\bigl(g(x)-g(a)\bigr)^{\mu};p\bigr) M_a^x(f) \int_a^x g'(t)\,dt.\end{aligned}$$

By using (1.12) of Definition 6 on the left-hand side and integrating the right-hand side we obtain the inequality

$$\bigl({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\bigr)(x,\omega;p) \leq M_a^x(f)\phi\bigl(g(x)-g(a)\bigr) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\bigl(\omega\bigl(g(x)-g(a)\bigr)^{\mu};p\bigr). \tag{2.6}$$

Now, on the other hand, for $t \in (x,b]$, we have the following inequality:

$$\begin{aligned}&\frac{\phi(g(t)-g(x))}{g(t)-g(x)} g'(t) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\bigl(\omega\bigl(g(t)-g(x)\bigr)^{\mu};p\bigr) \\ &\quad\leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(t) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\bigl(\omega\bigl(g(t)-g(x)\bigr)^{\mu};p\bigr).\end{aligned} \tag{2.7}$$

By using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p) \leq E^{\gamma,\delta,k,c}_{\nu,\alpha,l}(\omega(g(b)-g(x))^{\nu};p)$ we get the inequality

$$\begin{aligned}&\frac{\phi(g(t)-g(x))}{g(t)-g(x)} g'(t) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\bigl(\omega\bigl(g(t)-g(x)\bigr)^{\mu};p\bigr) \\ &\quad\leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(t) E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\bigl(\omega\bigl(g(b)-g(x)\bigr)^{\nu};p\bigr).\end{aligned} \tag{2.8}$$

Using the quasiconvexity of $f$, for $t \in [x,b]$, we also have $f(t) \leq M_x^b(f)$. From (2.8), using (1.13) of Definition 6, we obtain

$$\bigl({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-} f\bigr)(x,\omega;p) \leq M_x^b(f)\phi\bigl(g(b)-g(x)\bigr) E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\bigl(\omega\bigl(g(b)-g(x)\bigr)^{\nu};p\bigr). \tag{2.9}$$

By adding (2.6) and (2.9) we can achieve (2.3). □

The following remark establishes connections with already known results.

*Remark* 1

(i) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ in (2.3), then the result coincides with [9, Theorem 2.1].

(ii) Under the same assumptions as we considered in (i), taking in addition $\mu = \nu$ in (2.3), the result coincides with [9, Corollary 2.2].

(iii) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ in (2.3), then the result coincides with [9, Corollary 2.3].

(iv) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.3), then the result coincides with [9, Corollary 2.4].

(v) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.3), then the result coincides with [9, Corollary 2.5].

(vi) Under the assumptions of (i), if $f$ is increasing on $[a,b]$, then the result coincides with [9, Corollary 2.6].

(vii) Under the assumptions of (i), if $f$ is decreasing on $[a,b]$, then the result coincides with [9, Corollary 2.7].

(viii) Further, if we take $\mu = \nu$ in the resulting inequality of (viii), then the result coincides with [27, Corollary 2.2].

Further consequences of Theorem 1 are studied in the following results.

**Theorem 2** *Under the assumption of Theorem 1, we have*

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\right)(b,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}\right)(a,\omega;p)\\ &\quad\leq M_a^b(f)\left(\phi\big(g(b)-g(a)\big)\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(b-a)^{\mu};p\big) + E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\big(\omega(b-a)^{\nu};p\big)\right)\right).\end{aligned} \tag{2.10}$$

*Proof* By putting $x = b$ in (2.6) we obtain

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\right)(b,\omega;p) \leq M_a^b(f)\phi\big(g(b)-g(a)\big)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(b-a)^{\mu};p\big). \tag{2.11}$$

Similarly, by putting $x = a$ in (2.9) we obtain

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}\right)(a,\omega;p) \leq M_a^b(f)\phi\big(g(b)-g(a)\big)E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\big(\omega(b-a)^{\nu};p\big). \tag{2.12}$$

By adding (2.11) and (2.12) we obtain (2.10). □

*Remark* 2

(i) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ in (2.10), then the result coincides with [9, Theorem 3.1].

(ii) If we replace $\omega$ by $\omega' = \frac{\omega}{(b-a)^{\mu}}$ and put $\phi(x) = x^{\mu}$ for the left-hand inequality, $\phi(x) = x^{\mu}$ for the right-hand one, and $g(x) = x$ in (2.10), then the result coincides with [27, Theorem 2.1].

(iii) Under the same assumptions as in (i), if in addition we take $\mu = \nu$ in (2.10), then the result coincides with [9, Corollary 3.2].

(iv) Further, if we put $\mu = k = 1$ in (ii), then the result coincides with [4, Theorem 3.3].

**Theorem 3** *Under the assumptions of Theorem 1, if $f \in L_\infty[a,b]$, then the operators $({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\cdot)(x,\omega;p), ({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\cdot)(x,\omega;p) : L_\infty[a,b] \to L_\infty[a,b]$ defined in (1.12) and (1.13) are continuous. Also, we have*

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(x,\omega;p)\right| \leq 2K\|f\|_\infty, \tag{2.13}$$

*where $K = \phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^{\mu};p)$.*

*Proof* It is clear that $({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f)(x,\omega;p)$ and $({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f)(x,\omega;p)$ are linear operators. Further, from (2.1) we have

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\right| \leq \|f\|_\infty\phi\big(g(b)-g(a)\big)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\big(g(b)-g(a)\big)^{\mu};p\right), \tag{2.14}$$

that is,

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\right| \le K\|f\|_\infty, \tag{2.15}$$

where $K = \phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$. Therefore $({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\,.)(x,\omega;p)$ is bounded and hence continuous. Similarly, (2.2) gives

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(x,\omega;p)\right| \le K\|f\|_\infty. \tag{2.16}$$

Therefore $({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f)(x,\omega;p)$ is bounded and hence continuous. From (2.15) and (2.16) we obtain (2.13). □

*Remark* 3 Theorem 2 provides the boundedness of all known operators defined in [2, 3, 6, 10, 13, 14, 18, 20, 21, 23, 25, 26]. Especially, the boundedness of the integral operator given in Definition 4, which is studied in [27].

To prove the next result, we need the following lemma.

**Lemma 1** ([8]) *Let $f:[0,\infty)\to\mathbb{R}$ be a quasiconvex function. If $f(x)=f(a+b-x)$, then for $x\in[a,b]$, we the inequality*

$$f\left(\frac{a+b}{2}\right) \le f(x). \tag{2.17}$$

The following result provides upper and lower bounds of operators (1.12) and (1.13) in the form of Hadamard inequality.

**Theorem 4** *Under the assumptions of Theorem 1, if in addition $f(x)=f(a+b-x)$, $x\in[a,b]$, then we have*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}1\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right)\\ &\quad\le \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p)\\ &\quad\le M_a^b(f)\phi\big(g(b)-g(a)\big)\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\big(g(b)-g(a)\big)^\mu;p\right)\right.\\ &\qquad\left.+E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega\big(g(b)-g(a)\big)^\nu;p\right)\right), \end{aligned} \tag{2.18}$$

*where $M_a^b(f) := \max\{f(a),f(b)\}$.*

*Proof* Under the assumptions of the theorem, we can obtain the inequality

$$\begin{aligned} &\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\big(g(x)-g(a)\big)^\mu;p\right)\\ &\quad\le\frac{\phi(g(b)-g(a))}{g(x)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\big(g(b)-g(a)\big)^\mu;p\right). \end{aligned} \tag{2.19}$$

By using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p) \le E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$ we get the inequality

$$\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\big(g(x)-g(a)\big)^\mu;p\right)$$

$$\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} g'(x) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\mu};p\big). \tag{2.20}$$

Using the quasiconvexity of $f$, for $x \in [a,b]$, we have $f(x) \leq M_a^b(f)$. Therefore we obtain the inequality

$$\begin{aligned}&\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x) f(x) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(x)-g(a)\big)^{\mu};p\big)\,dx\\ &\quad\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\mu};p\big) M_a^x(f)\int_a^b g'(x)\,dx.\end{aligned}$$

By using Definition 6 on the left-hand side and integrating the right-hand side we obtain the inequality

$$\big({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\big)(b,\omega;p) \leq M_a^b \phi\big(g(b)-g(a)\big) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\mu};p\big). \tag{2.21}$$

On the other hand, for $x \in (a,b)$, we have the inequality

$$\begin{aligned}&\frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(x)\big)^{\nu};p\big)\\ &\quad\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} g'(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(x)\big)^{\nu};p\big).\end{aligned} \tag{2.22}$$

By using $E_{\nu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(b)-g(x))^{\nu};p) \leq E_{\nu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(b)-g(a))^{\nu};p)$ we get the inequality

$$\begin{aligned}&\frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(x)\big)^{\nu};p\big)\\ &\quad\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} g'(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\nu};p\big).\end{aligned} \tag{2.23}$$

Adopting the same pattern of simplification as we did for (2.20), we can observe the following inequality for (2.23):

$$\big({}_gF_{\nu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\big)(a,\omega;p) \leq M_a^b(f) \phi\big(g(b)-g(a)\big) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\nu};p\big). \tag{2.24}$$

By adding (2.21) and (2.24) we arrive at the inequality

$$\begin{aligned}&\big({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\big)(b,\omega;p) + \big({}_gF_{\nu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\big)(a,\omega;p)\\ &\quad\leq M_a^b(f)\phi\big(g(b)-g(a)\big)\\ &\qquad\times \big(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\mu};p\big) + E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(b)-g(a)\big)^{\nu};p\big)\big).\end{aligned} \tag{2.25}$$

Multiplying both sides of (2.17) by $\frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x) E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(a))^{\mu};p)$ and integrating over $[a,b]$, we have

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(x)-g(a)\big)^{\mu};p\big)\,dx\\ &\quad\leq \int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x) f(x) E_{\nu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega\big(g(x)-g(a)\big)^{\nu};p\big)\,dx.\end{aligned}$$

From Definition 6 we obtain the inequality

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}1\right)(a,\omega;p) \le \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(a,\omega;p). \tag{2.26}$$

Similarly, multiplying both sides of (2.17) by $\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)$ and integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p) \le \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p). \tag{2.27}$$

By adding (2.26) and (2.27) we obtain the inequality

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}1\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right) \\ &\quad \le \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p). \end{aligned} \tag{2.28}$$

Using (2.25) and (2.28), we arrive at (2.18). □

The following remark establishes connections with already known results.

*Remark* 4

(i) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ in (2.18), then the result coincides with [9, Theorem 2.16].

(ii) Under the same assumptions as in (i), if in addition we take $\mu = \nu$ in (2.18), then the result coincides with [9, Corollary 2.17].

(iii) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ in (2.18), then the result coincides with [9, Corollary 2.18].

(iv) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.18), then the result coincides with [9, Corollary 2.19].

(v) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.18), then the result coincides with [9, Corollary 2.20].

(vi) Under the assumptions of (i), if $f$ is increasing on $[a,b]$, then the result coincides with [9, Corollary 2.21].

(vii) Under the assumptions of (i), if $f$ is decreasing on $[a,b]$, then the result coincides with [9, Corollary 2.22].

**Theorem 5** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function such that $|f'|$ is quasiconvex. Let $g : [a,b] \longrightarrow \mathbb{R}$ be q differentiable strictly increasing function, and let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \nu \ge 0$, $\delta \ge 0$, $0 < k \le \delta + \mu$, and $0 < k \le \delta + \nu$, then for $x \in (a,b)$, we have*

$$\begin{aligned} &\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}(f*g)\right)(x,\omega;p)\right| \\ &\quad \le \left(M^x_a\left(|f'|\right)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right)\right. \\ &\qquad \left. + M^b_x\left(|f'|\right)\phi\left(g(b)-g(x)\right)E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega\left(g(b)-g(x)\right)^{\nu};p\right)\right), \end{aligned} \tag{2.29}$$

*where*

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,\omega;p)\\&\quad:=\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right)f'(t)g'(t)\,dt,\\&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}(f*g)\right)(x,\omega;p)\\&\quad:=\int_x^a\frac{\phi(g(t)-g(x))}{g(t)-g(x)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}\left(\omega\left(g(t)-g(x)\right)^{\mu};p\right)f'(t)g'(t)\,dt,\end{aligned}$$

*and* $M_a^b(|f'|):=\max\{|f'(a)|,|f'(b)|\}$.

*Proof* Let $x\in(a,b)$ and $t\in[a,x]$. Then using the quasiconvexity of $|f'|$, we have

$$\left|f'(t)\right|\le M_a^x\left(\left|f'\right|\right). \tag{2.30}$$

Inequality (2.30) can be written as follows:

$$-M_a^x\left(\left|f'\right|\right)\le f'(t)\le M_a^x\left(\left|f'\right|\right). \tag{2.31}$$

Let us consider the left-hand side inequality of (2.31),

$$f'(t)\le M_a^x\left(\left|f'\right|\right). \tag{2.32}$$

Using (2.5) and (2.32), we obtain

$$\begin{aligned}&\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)f'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}\left(\omega\left(g(x)-g(t)\right)^{\mu};p\right)dt\\&\quad\le M_a^x\left(\left|f'\right|\right)\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right)\int_a^x g'(t)\,dt.\end{aligned}$$

By using (1.12) of Definition 6 on the left-hand side and integrating on the right-hand one, we obtain the inequality

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,\omega;p)\le M_a^x\left(\left|f'\right|\right)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right). \tag{2.33}$$

Considering the left-hand side of inequality (2.31) and adopting the same pattern as we did for the right-hand side inequality, we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,\omega;p)\ge -M_a^x\left(\left|f'\right|\right)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right). \tag{2.34}$$

From (2.33) and (2.34) we get the inequality

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x;p)\right|\le M_a^x\left(\left|f'\right|\right)\phi\left(g(x)-g(a)\right)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(g(x)-g(a)\right)^{\mu};p\right). \tag{2.35}$$

Now using the quasiconvexity of $|f'|$ on $[x,b]$, for $x\in(a,b)$, we have

$$\left|f'(t)\right|\le M_x^b\left(\left|f'\right|\right). \tag{2.36}$$

Similarly to (2.5) and (2.30), we can get following inequality from (2.8) and (2.36):

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}(f*g)\right)(x,\omega;p)\right| \leq M_x^b\left(\left|f'\right|\right)\phi\left(g(b)-g(x)\right)E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega\left(g(b)-g(x)\right)^{\nu};p\right). \quad (2.37)$$

By adding (2.35) and (2.37) we arrive at (2.29). □

The following remark establishes connections with already known results.

*Remark* 5
(i) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ in (2.29), then the result coincides with [9, Theorem 2.8].
(ii) Under the same assumptions as in (i), if in addition we take $\mu = \nu$ in (2.29), then the result coincides with [9, Corollary 2.9].
(iii) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ in (2.29), then the result coincides with [9, Corollary 2.10].
(iv) If we put $\phi(x) = x^{\frac{\mu}{k}}$ for the left-hand integral and $\phi(x) = x^{\frac{\nu}{k}}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.29), then the result coincides with [9, Corollary 2.11].
(v) If we put $\phi(x) = x^{\mu}$ for the left-hand integral and $\phi(x) = x^{\nu}$ for the right-hand one and $\omega = p = 0$ and $g(x) = x$ in (2.29), then the result coincides with [9, Corollary 2.12].
(vi) Under the assumptions of (i), if $f$ is increasing on $[a,b]$, then the result coincides with [9, Corollary 2.13].
(vii) Under the assumptions of (i), if $f$ is decreasing on $[a,b]$, then the result coincides with [9, Corollary 2.14].
(viii) Under the same assumptions as in (i), if in addition we put $x = a$ in the left-hand integral and $x = b$ in the right-hand one, then the result coincides with [9, Theorem 3.2].
(ix) Further, if we put $\mu = \nu$ in the resulting inequality obtained from (viii), then the result coincides with [9, Corollary 3.5].
(x) If we put $\mu = k = 1$ in the resulting inequality of (ix), then the result coincides with [9, Corollary 3.5].

## 3 Results for fractional integral operators containing Mittag-Leffler functions

In this section, by applying main theorems we compute results for the generalized fractional integral operators containing Mittag-Leffler functions.

**Theorem 6** *Under the assumptions of Theorem 1, we have the following inequality for the generalized integral operator containing a Mittag-Leffler function*:

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x,\omega;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\nu,\alpha,l,\omega,b^-}f\right)(x,\omega;p)\\&\quad\leq M_a^x(f)(x-a)^{\mu}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(x-a)^{\mu};p\right) + M_x^b(f)(b-x)^{\nu}E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\left(\omega(b-x)^{\nu};p\right).\end{aligned} \quad (3.1)$$

*Proof* By putting $\phi(x) = x^{\mu}$ and $g(x) = x$ in (2.1) we get the following upper bound for the left-sided generalized fractional integral operator containing a Mittag-Leffler function:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x,\omega;p) \leq M_a^x(f)(x-a)^{\mu}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(x-a)^{\mu};p\right). \quad (3.2)$$

Similarly, from (2.2) we get the following upper bound for the right-sided generalized fractional integral operator containing a Mittag-Leffler function:

$$\big(\epsilon^{\gamma,\delta,k,c}_{\nu,\alpha,l,\omega,b^-}f\big)(x,\omega;p) \le M^b_x(f)(b-x)^\nu E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\big(\omega(b-x)^\nu;p\big). \tag{3.3}$$

By adding (3.1) and (3.2) we arrive at (3.3). □

**Theorem 7** *Under the assumptions of Theorem 4, we have the following Hadamard inequality for the generalized integral operator containing a Mittag-Leffler function*:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\big(\big(\epsilon^{\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}1\big)(a,\omega;p)+\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\big)(b,\omega;p)\big) \\ &\quad\le \big(\epsilon^{\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\big)(a,\omega;p)+\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\big)(b,\omega;p) \\ &\quad\le M^b_a\big((b-a)^\mu E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(b-a)^\mu;p\big)+(b-a)^\nu E^{\gamma,\delta,k,c}_{\nu,\alpha,l}\big(\omega(b-a)^\nu;p\big). \end{aligned} \tag{3.4}$$

*Proof* By putting $\phi(x)=x^\mu$ for the left-hand integral, $\phi(x)=x^\nu$ for the right-hand one, and $g(x)=x$ in (2.18), we get the inequality for the left-sided generalized fractional integral operator containing a Mittag-Leffler function. □

**Theorem 8** *Under the assumptions of Theorem 5, we have the following modulus inequality for the generalized integral operator containing a Mittag-Leffler function*:

$$\begin{aligned} &\big|\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\big)(x,\omega;p)+\big(\epsilon^{\gamma,\delta,k,c}_{\nu,\alpha,l,\omega,b^-}f\big)(x,\omega;p)\big| \\ &\quad\le M^x_a\big(|f'|\big)(x-a)^\mu E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(x-a)^\mu;p\big)+M^b_x\big(|f'|\big)(b-x)^\nu E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(b-x)^\mu;p\big). \end{aligned} \tag{3.5}$$

*Proof* By putting $\phi(x)=x^\mu$ for the left-hand integral, $\phi(x)=x^\nu$ for the right-hand one, and $g(x)=x$ in (2.29) we get the above-mentioned inequality for the left-sided generalized fractional integral operator containing a Mittag-Leffler function. □

## 4 Concluding remarks

In this paper, we identify upper and lower bounds of various kinds of fractional and conformable integral operators of quasiconvex functions in compact and unified forms. They also ensure the boundedness and continuity of these operators. Some of the results are identified in remarks of Sect. 2 and theorems of Sect. 3 to establish the connection with already published results. The reader can deduce the results for other known fractional and conformable integral operators associated with unified integral operators (1.12) and (1.13).

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