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On fractional boundary value problems involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions

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Abstract
In this paper, we consider two classes of boundary value problems for nonlinear implicit differential equations with nonlinear integral conditions involving Atangana–Baleanu–Caputo fractional derivatives of orders $0 < \vartheta \leq 1$ and $1 < \vartheta \leq 2$. We structure the equivalent fractional integral equations of the proposed problems. Further, the existence and uniqueness theorems are proved with the aid of fixed point theorems of Krasnoselskii and Banach. Lastly, the paper includes pertinent examples to justify the validity of the results.

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Keywords: Implicit fractional differential equations; Atangana–Baleanu–Caputo fractional derivative; Integral conditions; Fixed point theorems

1 Introduction

Fractional calculus [1–3] has continued to attract the attention of many authors in the past three decades. Recently, new fractional derivatives (FDs) which interpolate the Riemann–Liouville, Caputo, Hilfer, Hadamard, and generalized FDs have appeared, see [4–9]. Some investigators have recognized that innovation for novel FDs with various nonsingular or singular kernels is necessary to address the need to model more realistic problems in various areas of engineering and science. Caputo and Fabrizio [10] introduced a new kind of FDs where the kernel is based on the exponential function. Losada and Nieto [11] studied some properties of this new operator. In [12, 13], the authors presented new interesting FDs where the kernel relies on Mittag-Leffler function, the so-called Atangana–Baleanu–Caputo (AB–Caputo) which is basically a generalization of the Caputo FD. Then in [14, 15], the authors deliberated the discrete versions of those new operators. For modeling in the framework of nonsingular kernels and fractal-fractional derivatives, we refer to [16–18]. There are many works pertinent to ABC problem in medical science and engineering. Hence we highlight medical, as well as engineering, applications by referring to [19–21].

On the other hand, the fixed point theory is a collection of results saying that a mapping T will have at least one fixed point (i.e., $T(x) = x$), under some conditions on T. Results of this kind are of paramount importance in many areas of mathematics, other sciences, and engineering. So, some recent articles which are pertinent to the fixed point theory can found in [22–29]. The existence and uniqueness of solutions for different classes of fractional differential equations (FDEs) with initial or boundary conditions have been studied by several researchers; see [30–38] and the references therein. Some recent contributions on FDEs involving ABC-FDs can be found in the following articles series: [39–49]. For instance, AB–Caputo fractional IVP is one of the studied problems by Jarad et al. [39], and has the form

$$\begin{cases} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) = f(\mathfrak{r},\varsigma(\mathfrak{r})), & \mathfrak{r}\in[a,T], 0<\vartheta\le 1,\\ \varsigma(a)=\varsigma_a. \end{cases}$$

The BVP of AB-Caputo FD, presented by Abdeljawad in [40], is also one of the recent problems through which the higher fractional orders are addressed:

$$\begin{cases} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) + q(\mathfrak{r})\varsigma(\mathfrak{r}) = 0, & \mathfrak{r}\in[a,T], 1<\vartheta\le 2,\\ \varsigma(a)=\varsigma(T)=0. \end{cases}$$

Motivated by the above arguments, the intent of this work is to investigate two AB–Caputo-type implicit FDEs with nonlinear integral conditions described by

$$\begin{cases} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) = f(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})), & \mathfrak{r}\in[a,T], 0<\vartheta\le 1,\\ \varsigma(a)-\varsigma'(a)=\int_a^T g(\mathfrak{s},\varsigma(\mathfrak{s}))\,d\mathfrak{s} \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) = f(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})), & \mathfrak{r}\in[a,T], 1<\vartheta\le 2,\\ \varsigma(a)=0, \quad \varsigma(T)=\int_a^T g(\mathfrak{s},\varsigma(\mathfrak{s}))\,d\mathfrak{s} \end{cases} \tag{1.2}$$

where ${}^{ABC}\mathbb{D}^{\vartheta}_{a^+}$ is the AB–Caputo FD of order ϑ, while $f:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:[a,T]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Some fixed point theorems (FPTs) are applied to establish the existence and uniqueness theorems for the problems (1.1) and (1.2). The proposed problems are more general, and the results generalize those obtained in recent studies; we also provide an extension of the development of FDEs involving this new operator. Moreover, the analysis of the results was limited to the minimum assumptions.

Many other recent works have investigated similar topics using the same concepts; one can see [50–55].

The rest of the paper is structured as follows. In Sect. 2, we give some useful preliminaries related to main consequences. Section 3 is devoted to obtaining formulas of solution to the proposed problems. Moreover, the existence and uniqueness theorems for the problems at hand are proved by means of various techniques for FPTs. Ultimately, illustrative examples are offered in Sect. 4.

2 Background materials and preliminaries

Here we recollect some requisite definitions and preliminary concepts related to our work.

Let $\mathfrak{Z} = [a, T] \subset \mathbb{R}$, $C(\mathfrak{Z}, \mathbb{R})$ be the space of continuous functions $\varsigma : \mathfrak{Z} \to \mathbb{R}$ with the norm

$$\|\varsigma\| = \max\{|\varsigma(\mathfrak{r})| : \mathfrak{r} \in \mathfrak{Z}\},$$

Clearly, $C(\mathfrak{Z}, \mathbb{R})$ is a Banach space with the norm $\|\varsigma\|$.

Definition 2.1 ([12, 13]) Let $\vartheta \in (0, 1]$ and $\mathfrak{p} \in H^1(\mathfrak{Z})$. Then the AB–Caputo and AB–Riemann–Liouville FDs of order ϑ for a function $\mathfrak{p}$ are described by

$${}^{ABC}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \frac{\mathfrak{N}(\vartheta)}{1-\vartheta}\int_a^{\mathfrak{r}} \mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{\vartheta-1}(\mathfrak{r}-\mathfrak{s})^{\vartheta}\right)\mathfrak{p}'(\mathfrak{s})\,d\mathfrak{s}, \quad \mathfrak{r} > a,$$

and

$${}^{ABR}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \frac{\mathfrak{N}(\vartheta)}{1-\vartheta}\frac{d}{d\mathfrak{r}}\int_a^{\mathfrak{r}} \mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{\vartheta-1}(\mathfrak{r}-\mathfrak{s})^{\vartheta}\right)\mathfrak{p}(\mathfrak{s})\,d\mathfrak{s}, \quad \mathfrak{r} > a,$$

respectively, where $\mathbb{E}_{\vartheta}$ is called the Mittag-Leffler function and described by

$$\mathbb{E}_{\vartheta}(\mathfrak{p}) = \sum_{k=0}^{\infty}\frac{\mathfrak{p}^k}{\Gamma(\mathfrak{Z}\vartheta+1)}, \qquad \mathrm{Re}(\vartheta) > 0, \quad \mathfrak{p} \in \mathbb{C}.$$

The associated AB fractional integral is specified by

$${}^{AB}I^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\mathfrak{p}(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\mathfrak{p}(\mathfrak{s})\,d\mathfrak{s}, \quad \mathfrak{r} > a,$$

where $\mathfrak{N}(\vartheta) > 0$ is a normalization function satisfying $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$.

Definition 2.2 ([13]) In particular, if $a = 0$, the Laplace transform of AB–Caputo FD of $\mathfrak{p}(\mathfrak{r})$ is specified by

$$\mathcal{L}\left[{}^{ABC}D^{\vartheta}_{0^+}\mathfrak{p}(\mathfrak{r})\right] = \frac{\mathfrak{N}(\vartheta)}{\mathfrak{s}^{\vartheta}(1-\vartheta)+\vartheta}\left[\mathfrak{s}^{\vartheta}\mathcal{L}\left[\mathfrak{p}(\mathfrak{r})\right] - \mathfrak{s}^{\vartheta-1}\mathfrak{p}(0)\right].$$

Lemma 2.1 ([14]) *Let $\vartheta \in (0, 1]$ and $\mathfrak{p} \in H^1(\mathfrak{Z})$, if AB–Caputo FD exists, then we have*

$${}^{ABR}D^{\vartheta}_{a^+}{}^{AB}I^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r})$$

and

$${}^{AB}I^{\vartheta}_{a^+}{}^{ABC}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r}) - \mathfrak{p}(a).$$

Definition 2.3 ([40]) Let $\vartheta \in (n, n+1]$ and $\mathfrak{p}$ be such that $\mathfrak{p}^n \in H^1(\mathfrak{Z})$. Set $\mathfrak{v} = \vartheta - n$ where $\mathfrak{v} \in (0, 1]$. Then the AB–Caputo and AB–Riemann–Liouville FDs of order ϑ for a function $\mathfrak{p}$ are described by

$${}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{ABC}D^{\mathfrak{v}}_{a^+}\mathfrak{p}^{(n)}(\mathfrak{r})$$

and

$$^{ABR}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{ABR}D^{\mathfrak{v}}_{a^+}\mathfrak{p}^{(n)}(\mathfrak{r}),$$

respectively. The associated AB fractional integral is specified by

$$^{AB}\mathbb{I}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = I^{n}_{a^+}{}^{AB}I^{\mathfrak{v}}_{a^+}\mathfrak{p}(\mathfrak{r}).$$

Remark 2.1 If $\vartheta \in (0,1]$, we have $\vartheta = \mathfrak{v}$. Hence

$$\begin{aligned}
&^{AB}\mathbb{I}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{AB}I^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}),\\
&^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{ABC}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}),\\
&^{ABR}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{ABR}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}).
\end{aligned}$$

Definition 2.4 ([40]) The relation between the AB–Riemann–Liouville and AB–Caputo FDs is

$$^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = {}^{ABR}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) - \frac{\mathfrak{N}(\vartheta)}{1-\vartheta}\mathfrak{p}(a)\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{\vartheta-1}(\mathfrak{r}-a)^{\vartheta}\right). \tag{2.1}$$

Lemma 2.2 ([40]) *For $n-1<\vartheta\le n$, $n\in\mathbb{N}_0$, and $\mathfrak{p}(\mathfrak{r})$ defined on $\mathfrak{Z}$, we have*:

(i) $^{ABR}\mathbb{D}^{\vartheta}_{a^+}{}^{AB}\mathbb{I}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r})$;
(ii) $^{AB}\mathbb{I}^{\vartheta}_{a^+}{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r}) - \sum_{k=0}^{n}\frac{\mathfrak{p}^{(k)}(a)}{k!}(\mathfrak{r}-a)^k$;
(iii) $^{AB}\mathbb{I}^{\vartheta}_{a^+}{}^{ABR}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r}) - \sum_{k=0}^{n-1}\frac{\mathfrak{p}^{(k)}(a)}{k!}(\mathfrak{r}-a)^k$.

Remark 2.2 With the help of (2.1), for any ϑ, it can be shown that

$$^{ABC}\mathbb{D}^{\vartheta}_{a^+}{}^{AB}\mathbb{I}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r}) - \mathfrak{p}(a). \tag{2.2}$$

Hence, under the condition that $\mathfrak{p}(a)=0$, we get the identity

$$^{ABC}\mathbb{D}^{\vartheta}_{a^+}{}^{AB}\mathbb{I}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \mathfrak{p}(\mathfrak{r}). \tag{2.3}$$

Lemma 2.3 ([40]) *Let $n<\vartheta\le n+1$. Then $^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r})=0$, if $\mathfrak{p}(\mathfrak{r})$ is constant function.*

Lemma 2.4 ([13]) *Let $\vartheta>0$. Then $^{AB}\mathbb{I}^{\vartheta}_{a^+}$ is bounded from $C(\mathfrak{Z},\mathbb{R})$ into $C(\mathfrak{Z},\mathbb{R})$.*

Lemma 2.5 *Let $n<\vartheta\le n+1$. Then $^{ABC}\mathbb{D}^{\vartheta}_{a^+}(\mathfrak{r}-a)^k=0$, for $k=0,1,\dots,n$.*

Proof Let $\mathfrak{p}(\mathfrak{r})=(\mathfrak{r}-a)^k$. By Definition 2.3, we have

$$\begin{aligned}
^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) &= {}^{ABC}D^{\mathfrak{v}}_{a^+}\mathfrak{p}^{(n)}(\mathfrak{r})\\
&= {}^{ABC}D^{\mathfrak{v}}_{a^+}\left[(\mathfrak{r}-a)^k\right]^{(n)}\\
&= {}^{ABC}D^{\mathfrak{v}}_{a^+}\left(\frac{d}{d\mathfrak{r}}\right)^n(\mathfrak{r}-a)^k.
\end{aligned}$$

Since $k < n \in \mathbb{N}$, we have $(\frac{d}{d\mathfrak{r}})^n(\mathfrak{r}-a)^k = 0$. It follows from Lemma 2.3 that

$$ {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = 0. \qquad \square $$

Lemma 2.6 ([39]) *Let $\vartheta \in (0,1]$ and $\varpi \in C(\mathfrak{Z},\mathbb{R})$ with $\varpi(a) = 0$. Then the solution of the following problem*

$$ {}^{ABC}D^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \varpi(\mathfrak{r}), \quad \mathfrak{r} \in \mathfrak{Z}, $$
$$ \mathfrak{p}(a) = c $$

is given by

$$ \mathfrak{p}(\mathfrak{r}) = c + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}. $$

Lemma 2.7 ([40]) *Let $\vartheta \in (1,2]$ and $\varpi \in C(\mathfrak{Z},\mathbb{R})$ with $\varpi(a) = 0$. Then the solution of the following problem*

$$ \begin{cases} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\mathfrak{p}(\mathfrak{r}) = \varpi(\mathfrak{r}), & \mathfrak{r} \in \mathfrak{Z}, \\ \mathfrak{p}(a) = c_1, & \mathfrak{p}'(a) = c_2 \end{cases} $$

is given by

$$ \mathfrak{p}(\mathfrak{r}) = c_1 + c_2(\mathfrak{r}-a) + \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}\varpi(\mathfrak{s})\,d\mathfrak{s} + \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}. $$

Definition 2.5 ([56]) Let $\mathcal{J}$ be a Banach space. The operator $\mathfrak{B}: \mathcal{J} \to \mathcal{J}$ is a contraction if

$$ \|\mathfrak{B}x_1 - \mathfrak{B}x_2\| \le p\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in \mathfrak{D}, 0 < p < 1. $$

Theorem 2.1 (Banach FPT, [56]) *Let $\mathcal{J}$ be a Banach space, and $\mathfrak{K}$ be a nonempty closed subset of $\mathcal{J}$. If $\mathfrak{B}: \mathfrak{K} \longrightarrow \mathfrak{K}$ is a contraction, then there exists a unique fixed point of $\mathfrak{B}$.*

Theorem 2.2 (Krasnoselskii FPT, [56]) *Let $\mathfrak{K}$ be a nonempty, closed, convex subset of a Banach space $\mathcal{J}$. Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be two operators such that (i) $\mathfrak{B}_1u + \mathfrak{B}_2v \in \mathfrak{K}$, $\forall u, v \in \mathfrak{K}$; (ii) $\mathfrak{B}_1$ is compact and continuous; (iii) $\mathfrak{B}_2$ is a contraction mapping. Then, there exists $w \in \mathfrak{K}$ such that $\mathfrak{B}_1w + \mathfrak{B}_2w = w$.*

3 Main results

This section is devoted to obtaining formulas of solutions to linear problems corresponding to (1.1) and (1.2). Moreover, we prove the existence and uniqueness theorems to suggested problems by applying Theorems 2.1 and 2.2.

3.1 Solution formulas

Theorem 3.1 *Let $0 < \vartheta \le 1$, and let $\varpi, g \in C(\mathfrak{Z},\mathbb{R})$ with $\varpi(a) = \varpi'(\mathfrak{a}) = 0$. A function $\varsigma \in C(\mathfrak{Z},\mathbb{R})$ is a solution of the fractional integral equation (FIE)*

$$ \varsigma(\mathfrak{r}) = \int_a^T g(\mathfrak{s})\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}, \quad \mathfrak{r} \in \mathfrak{Z}, \tag{3.1} $$

if and only if ς is a solution of the ABC-problem

$$\begin{aligned} &{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) = \varpi(\mathfrak{r}), \quad \mathfrak{r} \in \mathfrak{Z}, \\ &\varsigma(a) - \varsigma'(a) = \int_a^T g(\mathfrak{s})\,d\mathfrak{s}. \end{aligned} \tag{3.2}$$

Proof Assume ς satisfies the first equation of (3.2). From Lemma 2.6, we have

$$\varsigma(\mathfrak{r}) = \varsigma(a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}, \tag{3.3}$$

Also,

$$\varsigma'(\mathfrak{r}) = \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi'(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta-1)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-2}\varpi(\mathfrak{s})\,d\mathfrak{s}. \tag{3.4}$$

Taking $\mathfrak{r} \to a$ on both sides of (3.4), we have

$$\varsigma'(a) = \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi'(\mathfrak{a}).$$

Using the integral condition, we obtain

$$\varsigma(a) = \varsigma'(a) + \int_a^T g(\mathfrak{s})\,d\mathfrak{s} = \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi'(\mathfrak{a}) + \int_a^T g(\mathfrak{s})\,d\mathfrak{s}. \tag{3.5}$$

From (3.3) and (3.5), and from fact that $\varpi'(\mathfrak{a}) = 0$, we get

$$\varsigma(\mathfrak{r}) = \int_a^T g(\mathfrak{s})\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\varpi(\mathfrak{r}) + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}, \quad \mathfrak{r} \in \mathfrak{Z}.$$

Thus (3.1) is satisfied.

Conversely, suppose that ς satisfies equation (3.1). Applying ${}^{ABC}\mathbb{D}^{\vartheta}_{a^+}$ on both sides of (3.1), then using Remark 2.2 and Lemma 2.3, we find that

$$\begin{aligned} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) &= {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\int_a^T g(\mathfrak{s})\,d\mathfrak{s} + {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}{}^{AB}\mathbb{I}^{\vartheta}_{a^+}\varpi(\mathfrak{r}) \\ &= \varpi(\mathfrak{r}). \end{aligned}$$

Thus, we can simply infer that

$$\varsigma(a) - \varsigma'(a) = \int_a^T g(\mathfrak{s})\,d\mathfrak{s}.$$

□

Theorem 3.2 *Let $1 < \vartheta \leq 2$, and let $\varpi, g \in C(\mathfrak{Z}, \mathbb{R})$ with $\varpi(a) = 0$. A function $\varsigma \in C(\mathfrak{Z}, \mathbb{R})$ is a solution of the FIE*

$$\begin{aligned} \varsigma(\mathfrak{r}) = {}& \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T g(\mathfrak{s})\,d\mathfrak{s} + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T \varpi(\mathfrak{s})\,d\mathfrak{s} \\ &+ \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T (T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s} \end{aligned}$$

$$+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}\varpi(\mathfrak{s})\,d\mathfrak{s}+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s} \tag{3.6}$$

if and only if ς *is a solution of the ABC-problem*

$$\begin{aligned}&{}^{ABC}D_{a^+}^{\vartheta}\varsigma(\mathfrak{r})=\varpi(\mathfrak{r}),\quad \mathfrak{r}\in\mathfrak{Z},\\&\varsigma(a)=0,\qquad \varsigma(T)=\int_a^T g(\mathfrak{s})\,d\mathfrak{s}.\end{aligned} \tag{3.7}$$

Proof Assume ς satisfies the first equation of (3.7). From Lemma 2.7, we have

$$\varsigma(\mathfrak{r})=c_1+c_2(\mathfrak{r}-a)+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}\varpi(\mathfrak{s})\,d\mathfrak{s}+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}, \tag{3.8}$$

for some $c_1,c_2\in\mathbb{R}$. Since $\varsigma(a)=0$, we get $c_1=0$. Hence

$$\varsigma(T)=c_2(T-a)+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{T}\varpi(\mathfrak{s})\,d\mathfrak{s}+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{T}(T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}. \tag{3.9}$$

Using the integral condition $\varsigma(T)=\int_a^T g(\mathfrak{s})\,d\mathfrak{s}$, we get

$$\begin{aligned}c_2={}&\frac{1}{(T-a)}\int_a^T g(\mathfrak{s})\,d\mathfrak{s}-\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\frac{1}{(T-a)}\int_a^T\varpi(\mathfrak{s})\,d\mathfrak{s}\\&-\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{1}{(T-a)}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}.\end{aligned} \tag{3.10}$$

Substituting the values of c_1 and c_2 into (3.8), we obtain

$$\begin{aligned}\varsigma(\mathfrak{r})={}&\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T g(\mathfrak{s})\,d\mathfrak{s}+\frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T\varpi(\mathfrak{s})\,d\mathfrak{s}\\&+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}\\&+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}\varpi(\mathfrak{s})\,d\mathfrak{s}+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}.\end{aligned}$$

Thus (3.6) is satisfied.

Conversely, assume that ς satisfies (3.6). Applying ${}^{ABC}\mathbb{D}_{a^+}^{\vartheta}$ on both sides of (3.6), then using Lemmas 2.2, 2.3, and 2.5, we find that

$$\begin{aligned}{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{r})={}&\frac{1}{(T-a)}\int_a^T g(\mathfrak{s})\,d\mathfrak{s}\,{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}(\mathfrak{r}-a)\\&+\frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{1}{(T-a)}\int_a^T\varpi(\mathfrak{s})\,d\mathfrak{s}\,{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}(\mathfrak{r}-a)\\&+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{1}{(T-a)}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s}\,{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}(\mathfrak{r}-a)\\&+{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\,{}^{AB}\mathbb{I}_{a^+}^{\vartheta}\varpi(\mathfrak{r})\\={}&\varpi(\mathfrak{r}).\end{aligned}$$

Clearly, $\varsigma(a) = 0$. Thus, we can simply infer that

$$\begin{aligned}\varsigma(T) &= \int_a^T g(\mathfrak{s})\,d\mathfrak{s} + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\int_a^T \varpi(\mathfrak{s})\,d\mathfrak{s} \\ &\quad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^T (T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s} \\ &\quad + \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^T \varpi(\mathfrak{s})\,d\mathfrak{s} + \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^T (T-\mathfrak{s})^{\vartheta-1}\varpi(\mathfrak{s})\,d\mathfrak{s} \\ &= \int_a^T g(\mathfrak{s})\,d\mathfrak{s}. \end{aligned}$$ □

Before proceeding with the main findings, we are obligated to provide the following assumptions:

(A_1) $f : \mathfrak{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $L_f > 0$ and $0 < K_f < 1$ such that

$$\left|f(\mathfrak{r},u,\overline{u}) - f(\mathfrak{r},v,\overline{v})\right| \le L_f|u-v| + K_f|\overline{u}-\overline{v}|, \quad \mathfrak{r} \in \mathfrak{Z} \text{ and } u,v,\overline{u},\overline{v} \in \mathbb{R};$$

(A_2) $g : \mathfrak{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constant $L_f > 0$ such that

$$\left|g(\mathfrak{r},u) - g(\mathfrak{r},v)\right| \le L_g|u-v|, \quad \mathfrak{r} \in \mathfrak{Z} \text{ and } u,v \in \mathbb{R};$$

(A_3)
$$\left[L_g(T-a) + \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)}\right)\frac{L_f}{1-K_f}\right] < 1.$$

3.2 Existence and uniqueness theorems for (1.1)

As a result of Theorem 3.1, we have the following theorem:

Theorem 3.3 *Let $0 < \vartheta \le 1$, and let $f : \mathfrak{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathfrak{Z} \times \mathbb{R} \to \mathbb{R}$ be continuous with $f(a,\varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = f'(a,\varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$. If $\varsigma \in C(\mathfrak{Z},\mathbb{R})$ then ς satisfies (1.1) if and only if ς fulfills*

$$\begin{aligned}\varsigma(\mathfrak{r}) &= \int_a^T g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} f\big(\mathfrak{r},\varsigma(\mathfrak{r}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f\big(\mathfrak{s},\varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\big)\,d\mathfrak{s}.\end{aligned}$$

Theorem 3.4 *Let (A_1) and (A_3) be fulfilled. Then the ABC-problem (1.1) has a unique solution.*

Proof Set

$$\mathfrak{D} = \left\{\varsigma \in C(\mathfrak{Z},\mathbb{R}) : {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma \in C(\mathfrak{Z},\mathbb{R})\right\}.$$

By Theorem 3.3, we define the operator $\mathbb{T} : \mathfrak{D} \to \mathfrak{D}$ by

$$\begin{aligned}(\mathbb{T}\varsigma)(\mathfrak{r}) &= \int_a^T g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} f\big(\mathfrak{r},\varsigma(\mathfrak{r}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f\big(\mathfrak{s},\varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\big)\,d\mathfrak{s}.\end{aligned}$$

This $\mathbb{T}$ is well defined, that is, $\mathbb{T}(\mathfrak{D}) \subseteq \mathfrak{D}$. Indeed, for any $\varsigma \in C(\mathfrak{Z},\mathbb{R}), f(\cdot,\varsigma(\cdot), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\cdot))$ is continuous. Besides, by Lemma 2.4, $\mathbb{T}\varsigma \in C(\mathfrak{Z},\mathbb{R})$. Also, by Lemma 2.1 and Remark 2.1, we end up with

$$\begin{aligned} {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}(\mathbb{T}\varsigma)(\mathfrak{r}) &= \int_a^T g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\,d\mathfrak{s}\,{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}(1) + {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\,{}^{ABC}\mathbb{I}^{\vartheta}_{a^+}f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) \\ &= {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\,{}^{ABC}\mathbb{I}^{\vartheta}_{a^+}f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) \\ &= f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big). \end{aligned}$$

Since $f(\mathfrak{r},\cdot,\cdot)$ is continuous on $[a,T]$, one has ${}^{ABC}\mathbb{D}^{\vartheta}_{a^+}(\mathbb{T}\varsigma)(\mathfrak{r}) \in C(\mathfrak{Z},\mathbb{R})$.

Now, we need to prove that $\mathbb{T}$ is a contraction. Let $\varsigma,\overline{\varsigma} \in \mathfrak{D}$ and $\mathfrak{r} \in \mathfrak{Z}$. Then

$$\begin{aligned} &\big|(\mathbb{T}\varsigma)(\mathfrak{r}) - (\mathbb{T}\overline{\varsigma})(\mathfrak{r})\big| \\ &\quad\leq \int_a^T \big|g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big) - g\big(\mathfrak{s},\overline{\varsigma}(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ &\qquad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) - f\big(\mathfrak{r},\overline{\varsigma}(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{r})\big)\big| \\ &\qquad + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\big|f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\big) - f\big(\mathfrak{s},\overline{\varsigma}(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s})\big)\big|\,d\mathfrak{s}. \end{aligned}$$

Using (A_1) and the fact that ${}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) = f(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}))$, we obtain

$$\begin{aligned} &\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) - f\big(\mathfrak{r},\overline{\varsigma}(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{r})\big)\big| \\ &\quad\leq L_f\big|\varsigma(\mathfrak{r}) - \overline{\varsigma}(\mathfrak{r})\big| + K_f\big|{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r}) - {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{r})\big| \\ &\quad= L_f\big|\varsigma(\mathfrak{r}) - \overline{\varsigma}(\mathfrak{r})\big| + K_f\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) - f\big(\mathfrak{r},\overline{\varsigma}(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{r})\big)\big|, \end{aligned}$$

which implies

$$\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) - f\big(\mathfrak{r},\overline{\varsigma}(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{r})\big)\big| \leq \frac{L_f}{1-K_f}\big|\varsigma(\mathfrak{r}) - \overline{\varsigma}(\mathfrak{r})\big|. \tag{3.11}$$

By (A_2) and (3.11), for $\mathfrak{r} \in \mathfrak{Z}$,

$$\begin{aligned} \big|(\mathbb{T}\varsigma)(\mathfrak{r}) - (\mathbb{T}\overline{\varsigma})(\mathfrak{r})\big| &\leq L_g\int_a^T \big|\varsigma(\mathfrak{s}) - \overline{\varsigma}(\mathfrak{s})\big|\,d\mathfrak{s} \\ &\quad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\frac{L_f}{1-K_f}\big|\varsigma(\mathfrak{r}) - \overline{\varsigma}(\mathfrak{r})\big| \\ &\quad + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{L_f}{1-K_f}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\big|\varsigma(\mathfrak{s}) - \overline{\varsigma}(\mathfrak{s})\big|\,d\mathfrak{s} \\ &\leq \left[L_g(T-a) + \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)}\right)\frac{L_f}{1-K_f}\right]\|\varsigma - \overline{\varsigma}\|. \end{aligned}$$

Condition (A_3) shows that $\mathbb{T}$ is a contraction. Hence, by Theorem 2.1, $\mathbb{T}$ has a unique fixed point. □

Theorem 3.5 *Suppose* (A_1) *and* (A_3) *are fulfilled. Then there exists at least one solution of the problem* (1.1).

Proof Choose $(\mathbb{T}\varsigma)(\mathfrak{r}) = (\mathbb{T}_1\varsigma)(\mathfrak{r}) + (\mathbb{T}_2\varsigma)(\mathfrak{r})$, where

$$(\mathbb{T}_1\varsigma)(\mathfrak{r}) = \int_a^T g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) \tag{3.12}$$

and

$$(\mathbb{T}_2\varsigma)(\mathfrak{r}) = \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\big)\,d\mathfrak{s}. \tag{3.13}$$

Set $\mu_f := \max\{|f(\mathfrak{r},0,0)|;\mathfrak{r}\in\mathfrak{Z}\} < \infty$ and $\mu_g := \max\{|g(\mathfrak{r},0)|;\mathfrak{r}\in\mathfrak{Z}\} < \infty$. Let

$$B_\xi = \big\{\varsigma\in\mathfrak{D} : \|\varsigma\| \le \xi\big\} \tag{3.14}$$

with the radius

$$\xi \ge \frac{\mu_g(T-a) + (\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)})\frac{\mu_f}{1-K_f}}{1-(L_g(T-a) + (\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)})\frac{L_f}{1-K_f})}. \tag{3.15}$$

We will complete the proof in several steps.
Step 1. We show that $\mathbb{T}_1\varsigma + \mathbb{T}_2\upsilon \in B_\xi$, for all $\varsigma,\upsilon\in B_\xi$.
By (3.12),

$$\big|(\mathbb{T}_1\varsigma)(\mathfrak{r})\big| \le \int_a^T \big|g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\big|\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big)\big|. \tag{3.16}$$

From (A_1) and (A_2), we have

$$\begin{aligned}\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big)\big| &\le \big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big) - f(\mathfrak{r},0,0)\big| + \big|f(\mathfrak{r},0,0)\big| \\ &\le L_f\big|\varsigma(\mathfrak{r})\big| + K_f\big|{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big| + \mu_f \\ &= L_f\big|\varsigma(\mathfrak{r})\big| + K_f\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big)\big| + \mu_f,\end{aligned}$$

which gives

$$\big|f\big(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\big)\big| \le \frac{L_f|\varsigma(\mathfrak{r})| + \mu_f}{1-K_f} \tag{3.17}$$

and

$$\begin{aligned}\big|g\big(\mathfrak{r},\varsigma(\mathfrak{r})\big)\big| &= \big|g\big(\mathfrak{r},\varsigma(\mathfrak{r})\big) - g(\mathfrak{r},0)\big| + \big|g(\mathfrak{r},0)\big| \\ &\le L_g\big|\varsigma(\mathfrak{r})\big| + \mu_g.\end{aligned} \tag{3.18}$$

By substituting (3.17) and (3.18) into (3.16), we have for $\varsigma\in B_\xi$,

$$\big|(\mathbb{T}_1\varsigma)(\mathfrak{r})\big| \le \int_a^T \big(L_g\|\varsigma\| + \mu_g\big)\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\frac{L_f\|\varsigma\| + \mu_f}{1-K_f}$$

$$\begin{aligned} &\leq (L_g\xi + \mu_g)(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f\xi + \mu_f}{1-K_f} \\ &= \left(L_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f}\right)\xi + \mu_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{\mu_f}{1-K_f}. \end{aligned} \tag{3.19}$$

Also, by (3.13),

$$\left|(\mathbb{T}_2\upsilon)(\mathfrak{r})\right| = \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} \left|f\left(\mathfrak{s}, \upsilon(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\upsilon(\mathfrak{s})\right)\right| d\mathfrak{s}.$$

From (3.17), then for $\upsilon \in B_\xi$,

$$\begin{aligned} \left|(\mathbb{T}_2\upsilon)(\mathfrak{r})\right| &\leq \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} \frac{L_f\|\upsilon\| + \mu_f}{1-K_f}\, d\mathfrak{s} \\ &\leq \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{L_f\xi + \mu_f}{1-K_f} \\ &= \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{\mu_f}{1-K_f} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{L_f}{1-K_f}\xi. \end{aligned} \tag{3.20}$$

Inequalities (3.19) and (3.20) give

$$\begin{aligned} \left|(\mathbb{T}_1\varsigma)(\mathfrak{r}) + (\mathbb{T}_2\upsilon)(\mathfrak{r})\right| &\leq \left|(\mathbb{T}_1\varsigma)(\mathfrak{r})\right| + \left|(\mathbb{T}_2\upsilon)(\mathfrak{r})\right| \\ &\leq \left(L_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f}\right)\xi + \mu_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{\mu_f}{1-K_f}. \\ &\quad + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{\mu_f}{1-K_f} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{L_f}{1-K_f}\xi \\ &= \left(L_g(T-a) + \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)}\right) \frac{L_f}{1-K_f}\right)\xi \\ &\quad + \mu_g(T-a) + \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)}\right) \frac{\mu_f}{1-K_f}. \end{aligned}$$

Using (A_3) and (3.15), for $\mathfrak{r} \in \mathfrak{Z}$ and $\varsigma, \upsilon \in B_\xi$,

$$\|\mathbb{T}_1\varsigma + \mathbb{T}_2\upsilon\| \leq \xi.$$

Thus, $\mathbb{T}_1\varsigma + \mathbb{T}_2\upsilon \in B_\xi$, for all $\varsigma, \upsilon \in B_\xi$.

Step 2. We prove that $\mathbb{T}_1$ is a contraction.

From (A_1), we have

$$\left|f\left(\mathfrak{r}, \varsigma(\mathfrak{r}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{r})\right) - f\left(\mathfrak{r}, \varsigma^*(\mathfrak{r}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma^*(\mathfrak{r})\right)\right| \leq \frac{L_f}{1-K_f}\left|\varsigma(\mathfrak{r}) - \varsigma^*(\mathfrak{r})\right|. \tag{3.21}$$

From (A_2) and (3.21), for $\varsigma, \varsigma^* \in B_\xi$,

$$\begin{aligned} \left|(\mathbb{T}_1\varsigma)(\mathfrak{r}) - (\mathbb{T}_1\varsigma^*)(\mathfrak{r})\right| &\leq \int_a^T \left|g\left(\mathfrak{s}, \varsigma(\mathfrak{s})\right) - g\left(\mathfrak{s}, \varsigma^*(\mathfrak{s})\right)\right| d\mathfrak{s} \\ &\quad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \left|f\left(\mathfrak{r}, \varsigma(\mathfrak{r}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{r})\right) - f\left(\mathfrak{r}, \varsigma^*(\mathfrak{r}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma^*(\mathfrak{r})\right)\right| \end{aligned}$$

$$\leq L_g \int_a^T \left|\varsigma(\mathfrak{s}) - \varsigma^*(\mathfrak{s}))\right| d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f} \left|\varsigma(\mathfrak{r}) - \varsigma^*(\mathfrak{r})\right|$$
$$\leq L_g(T-a)\left\|\varsigma - \varsigma^*\right\| + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f} \left\|\varsigma - \varsigma^*\right\|$$
$$\leq \left(L_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f}\right) \left\|\varsigma - \varsigma^*\right\|.$$

Since (A_3) holds, $(L_g(T-a) + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)} \frac{L_f}{1-K_f}) < 1$. Hence, $\mathbb{T}_1$ is a contraction.

Step 3. $\mathbb{T}_2$ is compact and continuous.

The map $\mathbb{T}_2 : B_\xi \to B_\xi$ is continuous due to the continuity of f. Next, $\mathbb{T}_2$ is uniformly bounded on B_ξ by (3.20), because for any $\varsigma \in B_\xi$ and $\mathfrak{r} \in \mathfrak{Z}$, we have

$$\left|(\mathbb{T}_2\varsigma)(\mathfrak{r})\right| \leq \frac{(T-a)^\vartheta}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)(1-K_f)}(\mu_f + L_f\xi).$$

This leads to a conclusion that $\mathbb{T}_2$ is uniformly bounded on B_ξ.

Now, we show that $\mathbb{T}_2(B_\xi)$ is equicontinuous. In order to establish that, let $\varsigma \in B_\xi$ and $a \leq \mathfrak{r}_1 < \mathfrak{r}_2 \leq T$. Then

$$\begin{aligned}
&\left|(\mathbb{T}_2\varsigma)(\mathfrak{r}_2) - (\mathbb{T}_2\varsigma)(\mathfrak{r}_1)\right| \\
&\quad = \left|\frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}_2} (\mathfrak{r}_2 - \mathfrak{s})^{\vartheta-1} f\left(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^\vartheta \varsigma(\mathfrak{s})\right) d\mathfrak{s}\right. \\
&\qquad \left. - \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}_1} (\mathfrak{r}_1 - \mathfrak{s})^{\vartheta-1} f\left(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^\vartheta \varsigma(\mathfrak{s})\right) d\mathfrak{s}\right| \\
&\quad \leq \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} (\mathfrak{r}_2 - \mathfrak{s})^{\vartheta-1} \left|f\left(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^\vartheta \varsigma(\mathfrak{s})\right)\right| d\mathfrak{s} \\
&\qquad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}_1} \left|(\mathfrak{r}_2 - \mathfrak{s})^{\vartheta-1} - (\mathfrak{r}_1 - \mathfrak{s})^{\vartheta-1}\right| \left|f\left(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^\vartheta \varsigma(\mathfrak{s})\right)\right| d\mathfrak{s}.
\end{aligned}$$

Using (3.17), for $\varsigma \in B_\xi$,

$$\begin{aligned}
&\left|(\mathbb{T}_2\varsigma)(\mathfrak{r}_2) - (\mathbb{T}_2\varsigma)(\mathfrak{r}_1)\right| \\
&\quad \leq \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} (\mathfrak{r}_2 - \mathfrak{s})^{\vartheta-1} \left(\frac{L_f|\varsigma(\mathfrak{r})| + \mu_f}{1-K_f}\right) d\mathfrak{s} \\
&\qquad + \frac{\vartheta}{\mathfrak{N}(\vartheta)} \frac{1}{\Gamma(\vartheta)} \int_a^{\mathfrak{r}_1} \left|(\mathfrak{r}_2 - \mathfrak{s})^{\vartheta-1} - (\mathfrak{r}_1 - \mathfrak{s})^{\vartheta-1}\right| \left(\frac{L_f\xi + \mu_f}{1-K_f}\right) d\mathfrak{s} \\
&\quad = \frac{1}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)} \frac{L_f\xi + \mu_f}{1-K_f} \left[2(\mathfrak{r}_2 - \mathfrak{r}_1)^\vartheta + (\mathfrak{r}_1 - a)^\vartheta - (\mathfrak{r}_2 - a)^\vartheta)\right] \\
&\quad \leq \frac{2(L_f\xi + \mu_f)}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)(1-K_f)} (\mathfrak{r}_2 - \mathfrak{r}_1)^\vartheta.
\end{aligned}$$

Observe that $|(\mathbb{T}_2\varsigma)(\mathfrak{r}_2) - (\mathbb{T}_2\varsigma)(\mathfrak{r}_1)| \to 0$ as $t_2 \to t_1$. In light of the former steps, together with Arzela–Ascoli theorem, we derive that $(\mathbb{T}_2 B_\xi)$ is relatively compact, and hence $\mathbb{T}_2$ is completely continuous. So, Theorem 2.2 shows that (1.1) has at least one solution. □

3.3 Existence and uniqueness theorems for (1.2)

As a result of Theorem 3.2, we have the following theorem:

Theorem 3.6 *Let* $1 < \vartheta \leq 2$, *and let* $f : \mathfrak{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathfrak{Z} \times \mathbb{R} \to \mathbb{R}$ *be continuous with* $f(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$. *If* $\varsigma \in C(\mathfrak{Z}, \mathbb{R})$, *then* ς *satisfies* (1.2) *if and only if* ς *fulfills*

$$\begin{aligned}\varsigma(\mathfrak{r}) = {}& \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T g(\mathfrak{s}, \varsigma(\mathfrak{s}))\,d\mathfrak{s} + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T (T-\mathfrak{s})^{\vartheta-1} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s}.\end{aligned}$$

Theorem 3.7 *Suppose* (A_1) *and* (A_2) *are satisfied. If*

$$L_g(T-a) < 1, \tag{3.22}$$

then the problem (1.2) *has a unique solution.*

Proof In view of Theorem 3.6, we consider $\mathbb{T}^* : \mathfrak{D} \to \mathfrak{D}$ defined by

$$\begin{aligned}(\mathbb{T}^*\varsigma)(\mathfrak{r}) = {}& \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T g(\mathfrak{s}, \varsigma(\mathfrak{s}))\,d\mathfrak{s} + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T (T-\mathfrak{s})^{\vartheta-1} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s} \\ &+ \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s}))\,d\mathfrak{s},\end{aligned}$$

From the continuity of g and f, $\mathbb{T}^*$ is well defined, that is, $\mathbb{T}^*(\mathfrak{D}) \subseteq \mathfrak{D}$.
Now, let $\varsigma, \overline{\varsigma} \in \mathfrak{D}$ and $\mathfrak{r} \in \mathfrak{Z}$. Then

$$\begin{aligned}&\left|(\mathbb{T}^*\varsigma)(\mathfrak{r}) - (\mathbb{T}^*\overline{\varsigma})(\mathfrak{r})\right| \\ &\quad\leq \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T \left|g(\mathfrak{s}, \varsigma(\mathfrak{s})) - g(\mathfrak{s}, \overline{\varsigma}(\mathfrak{s}))\right| d\mathfrak{s} \\ &\qquad+ \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T \left|f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})) - f(\mathfrak{s}, \overline{\varsigma}(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s}))\right| d\mathfrak{s} \\ &\qquad+ \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)} \\ &\qquad\times \int_a^T (T-\mathfrak{s})^{\vartheta-1}\left|f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})) - f(\mathfrak{s}, \overline{\varsigma}(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s}))\right| d\mathfrak{s} \\ &\qquad+ \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}} \left|f(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})) - f(\mathfrak{s}, \overline{\varsigma}(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s}))\right| d\mathfrak{s}\end{aligned}$$

$$+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}$$
$$\times\int_a^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\vartheta-1}\left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\right)-f\left(\mathfrak{s},\overline{\varsigma}(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s})\right)\right|d\mathfrak{s}.$$

Using (A_1) and same arguments used to get (3.11), we obtain

$$\left|f\left(\mathfrak{t},\varsigma(\mathfrak{t}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{t})\right)-f\left(\mathfrak{s},\overline{\varsigma}(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\overline{\varsigma}(\mathfrak{s})\right)\right|\leq\frac{L_f}{1-K_f}\left|\varsigma(\mathfrak{t})-\overline{\varsigma}(\mathfrak{t})\right|. \tag{3.23}$$

By (A_2) and (3.23), for $\mathfrak{t}\in\mathfrak{Z}$,

$$\begin{aligned}
&\left|\left(\mathbb{T}^*\varsigma\right)(\mathfrak{t})-\left(\mathbb{T}^*\overline{\varsigma}\right)(\mathfrak{t})\right|\\
&\quad\leq\frac{(\mathfrak{t}-a)}{(T-a)}L_g\int_a^T\left|\varsigma(\mathfrak{s})-\overline{\varsigma}(\mathfrak{s})\right|d\mathfrak{s}\\
&\qquad+\frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{t}-a)}{(T-a)}\frac{L_f}{1-K_f}\int_a^T\left|\varsigma(\mathfrak{s})-\overline{\varsigma}(\mathfrak{s})\right|d\mathfrak{s}\\
&\qquad+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{t}-a)}{(T-a)}\frac{L_f}{1-K_f}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\left|\varsigma(\mathfrak{s})-\overline{\varsigma}(\mathfrak{s})\right|d\mathfrak{s}\\
&\qquad+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\frac{L_f}{1-K_f}\int_a^{\mathfrak{t}}\left|\varsigma(\mathfrak{s})-\overline{\varsigma}(\mathfrak{s})\right|d\mathfrak{s}\\
&\qquad+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{L_f}{1-K_f}\int_a^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\vartheta-1}\left|\varsigma(\mathfrak{s})-\overline{\varsigma}(\mathfrak{s})\right|d\mathfrak{s}\\
&\quad\leq L_g(T-a)\|\varsigma-\overline{\varsigma}\|+\frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{L_f(T-a)}{1-K_f}\|\varsigma-\overline{\varsigma}\|\\
&\qquad+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\frac{L_f(T-a)^{\vartheta}}{1-K_f}\|\varsigma-\overline{\varsigma}\|+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\frac{L_f(T-a)}{1-K_f}\|\varsigma-\overline{\varsigma}\|\\
&\qquad+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\frac{L_f(T-a)^{\vartheta}}{1-K_f}\|\varsigma-\overline{\varsigma}\|\\
&\quad=L_g(T-a)\|\varsigma-\overline{\varsigma}\|.
\end{aligned}$$

Condition (3.22) shows that $\mathbb{T}^*$ is a contraction. Hence, by Theorem 2.1, $\mathbb{T}^*$ has a unique fixed point. □

Theorem 3.8 *Suppose that* (A_1) *and* (A_2) *are satisfied. If*

$$\left(L_g(T-a)+\left(\frac{(\vartheta-2)(T-a)}{\mathfrak{N}(\vartheta-1)}+\frac{(1-\vartheta)(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{L_f}{1-K_f}\right)<1, \tag{3.24}$$

then there exists at least one solution of the problem (1.2).

Proof Choose $(\mathbb{T}^*\varsigma)(\mathfrak{t})=(\mathbb{T}^*_1\varsigma)(\mathfrak{t})+(\mathbb{T}^*_2\varsigma)(\mathfrak{t})$ where

$$\begin{aligned}
\left(\mathbb{T}^*_1\varsigma\right)(\mathfrak{t})={}&\frac{(\mathfrak{t}-a)}{(T-a)}\int_a^T g\left(\mathfrak{s},\varsigma(\mathfrak{s})\right)d\mathfrak{s}+\frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{t}-a)}{(T-a)}\int_a^T f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\right)d\mathfrak{s}\\
&+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{t}-a)}{(T-a)}\int_a^T(T-\mathfrak{s})^{\vartheta-1}f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\right)d\mathfrak{s}
\end{aligned} \tag{3.25}$$

and

$$\big(\mathbb{T}_2^*\varsigma\big)(\mathfrak{r}) = \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\big)\,d\mathfrak{s} + \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\big)\,d\mathfrak{s}. \tag{3.26}$$

Let B_ξ be defined by (3.14) with the radius

$$\xi \geq \frac{\mu_g(T-a)}{1-L_g(T-a)}, \tag{3.27}$$

where μ_g is as in Theorem 3.5. The proof will be complete in several steps:

Claim 1. $\mathbb{T}_1^*\varsigma + \mathbb{T}_2^*\upsilon \in B_\xi$, for all $\varsigma, \upsilon \in B_\xi$.

By (3.25),

$$\begin{aligned}\big|\big(\mathbb{T}_1^*\varsigma\big)(\mathfrak{r})\big| &\leq \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T\big|g\big(\mathfrak{s},\varsigma(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ &\quad+ \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T\big|f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ &\quad+ \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\big|f\big(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\big)\big|\,d\mathfrak{s}.\end{aligned}$$

From (A_1), (A_2), and for $\varsigma \in B_\xi$, we get $|f(\mathfrak{r},\varsigma(\mathfrak{r}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{r}))| \leq \frac{L_f\xi+\mu_f}{1-K_f}$ (where μ_f is as in Theorem 3.5) and $|g(\mathfrak{s},\varsigma(\mathfrak{s}))| \leq (L_g\xi+\mu_g)$. Hence,

$$\begin{aligned}\big|\big(\mathbb{T}_1^*\varsigma\big)(\mathfrak{r})\big| &\leq (\mathfrak{r}-a)(L_g\xi+\mu_g) + \frac{(\vartheta-2)(\mathfrak{r}-a)}{\mathfrak{N}(\vartheta-1)}\frac{(L_f\xi+\mu_f)}{1-K_f} \\ &\quad+ \frac{(1-\vartheta)(\mathfrak{r}-a)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\frac{(T-a)^{\vartheta-1}(L_f\xi+\mu_f)}{1-K_f} \\ &\leq \left(L_g(\mathfrak{r}-a) + \left(\frac{(\vartheta-2)}{\mathfrak{N}(\vartheta-1)} + \frac{(1-\vartheta)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{L_f(\mathfrak{r}-a)}{1-K_f}\right)\xi \\ &\quad+ \mu_g(\mathfrak{r}-a) + \left(\frac{(\vartheta-2)}{\mathfrak{N}(\vartheta-1)} + \frac{(1-\vartheta)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{\mu_f(\mathfrak{r}-a)}{1-K_f}.\end{aligned} \tag{3.28}$$

Also, by (3.26),

$$\begin{aligned}\big|\big(\mathbb{T}_2^*\upsilon\big)(\mathfrak{r})\big| &\leq \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\big|f\big(\mathfrak{r},\upsilon(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\upsilon(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ &\quad+ \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_a^{\mathfrak{r}}\big|f\big(\mathfrak{r},\upsilon(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\upsilon(\mathfrak{s})\big)\big|\,d\mathfrak{s}.\end{aligned}$$

For $\upsilon \in B_\xi$,

$$\begin{aligned}\big|\big(\mathbb{T}_2^*\upsilon\big)(\mathfrak{r})\big| &\leq \frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{L_f\xi+\mu_f}{1-K_f}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\vartheta-1}\,d\mathfrak{s} \\ &\quad+ \frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\frac{L_f\xi+\mu_f}{1-K_f}\int_a^{\mathfrak{r}}d\mathfrak{s}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\vartheta-1)(\mathfrak{r}-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)} \frac{L_f\xi+\mu_f}{1-K_f} + \frac{(2-\vartheta)(\mathfrak{r}-a)}{\mathfrak{N}(\vartheta-1)} \frac{L_f\xi+\mu_f}{1-K_f} \\
&= \left(\frac{(\vartheta-1)(\mathfrak{r}-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)} + \frac{(2-\vartheta)(\mathfrak{r}-a)}{\mathfrak{N}(\vartheta-1)}\right) \frac{\mu_f}{1-K_f} \\
&\quad + \left(\frac{(\vartheta-1)(\mathfrak{r}-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)} + \frac{(2-\vartheta)(\mathfrak{r}-a)}{\mathfrak{N}(\vartheta-1)}\right) \frac{L_f}{1-K_f}\xi.
\end{aligned} \tag{3.29}
$$

From (3.28), (3.29), and for $\mathfrak{r} \in \mathfrak{Z}$, we get

$$
\begin{aligned}
\left\|\mathbb{T}_1^*\varsigma + \mathbb{T}_2^*\upsilon\right\| &\le \left\|\mathbb{T}_1^*\varsigma\right\| + \left\|\mathbb{T}_2^*\upsilon\right\| \\
&= \max_{\mathfrak{r}\in\mathfrak{Z}}\left|\left(\mathbb{T}_1^*\varsigma\right)(\mathfrak{r})\right| + \max_{\mathfrak{r}\in\mathfrak{Z}}\left|\left(\mathbb{T}_2^*\upsilon\right)(\mathfrak{r})\right| \\
&\le \left(L_g(T-a) + \left(\frac{(\vartheta-2)(T-a)}{\mathfrak{N}(\vartheta-1)} + \frac{(1-\vartheta)(T-a)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{L_f}{1-K_f}\right)\xi \\
&\quad + \mu_g(T-a) + \left(\frac{(\vartheta-2)(T-a)}{\mathfrak{N}(\vartheta-1)} + \frac{(1-\vartheta)(T-a)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{\mu_f}{1-K_f} \\
&\quad + \left(\frac{(\vartheta-1)(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)} + \frac{(2-\vartheta)(T-a)}{\mathfrak{N}(\vartheta-1)}\right)\frac{\mu_f}{1-K_f} \\
&\quad + \left(\frac{(\vartheta-1)(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)} + \frac{(2-\vartheta)(T-a)}{\mathfrak{N}(\vartheta-1)}\right)\frac{L_f}{1-K_f}\xi \\
&< L_g(T-a)\xi + \mu_g(T-a).
\end{aligned}
$$

Here we used fact that $(T-a) < (T-a)^{\vartheta}$ for all $1 < \vartheta \le 2$. By (3.24), we conclude that $L_g(T-a) < 1$, it follows from (3.27) that

$$
\left\|\mathbb{T}_1^*\varsigma + \mathbb{T}_2^*\upsilon\right\| \le \xi.
$$

Thus, $\mathbb{T}_1^*\varsigma + \mathbb{T}_2^*\upsilon \in B_\xi$, for all $\varsigma, \upsilon \in B_\xi$.

Claim 2. $\mathbb{T}_1^*$ is a contraction.

From (A_1) and (A_2). Then for each $\varsigma, \varsigma^* \in B_\xi$ and $\mathfrak{r} \in \mathfrak{Z}$,

$$
\begin{aligned}
&\left|\left(\mathbb{T}_1^*\varsigma\right)(\mathfrak{r}) - \left(\mathbb{T}_1^*\varsigma^*\right)(\mathfrak{r})\right| \\
&\quad \le \frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T \left|g\left(\mathfrak{s},\varsigma(\mathfrak{s})\right) - g\left(\mathfrak{s},\varsigma^*(\mathfrak{s})\right)\right| d\mathfrak{s} \\
&\qquad + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\int_a^T \left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\right) - f\left(\mathfrak{s},\varsigma^*(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma^*(\mathfrak{s})\right)\right| d\mathfrak{s} \\
&\qquad + \frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)} \\
&\qquad \times \int_a^T (T-\mathfrak{s})^{\vartheta-1}\left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma(\mathfrak{s})\right) - f\left(\mathfrak{s},\varsigma^*(\mathfrak{s}), {}^{ABC}\mathbb{D}_{a^+}^{\vartheta}\varsigma^*(\mathfrak{s})\right)\right| d\mathfrak{s} \\
&\quad \le \frac{L_g(\mathfrak{r}-a)}{(T-a)}\int_a^T \left|\varsigma(\mathfrak{s}) - \varsigma^*(\mathfrak{s})\right| d\mathfrak{s} + \frac{\vartheta-2}{\mathfrak{N}(\vartheta-1)}\frac{(\mathfrak{r}-a)}{(T-a)}\frac{L_f}{1-K_f}\int_a^T \left|\varsigma(\mathfrak{s}) - \varsigma^*(\mathfrak{s})\right| d\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1-\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\frac{(\mathfrak{r}-a)}{(T-a)}\frac{L_f}{1-K_f}\int_a^T(T-\mathfrak{s})^{\vartheta-1}\left|\varsigma(\mathfrak{s})-\varsigma^*(\mathfrak{s})\right|d\mathfrak{s}\\
&\leq\left(L_g(T-a)+\left(\frac{(\vartheta-2)(T-a)}{\mathfrak{N}(\vartheta-1)}+\frac{(1-\vartheta)(T-a)^\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\right)\frac{L_f}{1-K_f}\right)\|\varsigma-\varsigma^*\|.
\end{aligned}
$$

Condition (3.24) shows that $\mathbb{T}_1^*$ is a contraction.

Claim 3. $\mathbb{T}_2^*$ is compact and continuous.

The map $\mathbb{T}_2^*:B_\xi\to B_\xi$ is continuous due to the continuity of f. Next, let $\varsigma\in B_\xi$ and $\mathfrak{r}\in\mathfrak{Z}$. Then by using (3.29), we have

$$
\|\mathbb{T}_2^*\varsigma\|\leq\frac{(\mu_f+L_f\xi)}{1-K_f}\left(\frac{(\vartheta-1)(T-a)^\vartheta}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}+\frac{(2-\vartheta)(T-a)}{\mathfrak{N}(\vartheta-1)}\right).
$$

This leads to the conclusion that $\mathbb{T}_2^*$ is uniformly bounded on B_ξ.

Now, we show that $\mathbb{T}_2^*(B_\xi)$ is equicontinuous. Let $\varsigma\in B_\xi$ and $a\leq\mathfrak{r}_1<\mathfrak{r}_2\leq T$. Then

$$
\begin{aligned}
&\left|\left(\mathbb{T}_2^*\varsigma\right)(\mathfrak{r}_2)-\left(\mathbb{T}_2^*\varsigma\right)(\mathfrak{r}_1)\right|\\
&\leq\left|\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\left[\int_a^{\mathfrak{r}_2}(\mathfrak{r}_2-\mathfrak{s})^{\vartheta-1}f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)d\mathfrak{s}\right.\right.\\
&\quad\left.\left.-\int_a^{\mathfrak{r}_1}(\mathfrak{r}_1-\mathfrak{s})^{\vartheta-1}f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)d\mathfrak{s}\right]\right|\\
&\quad+\left|\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\left[\int_a^{\mathfrak{r}_2}f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)d\mathfrak{s}-\int_a^{\mathfrak{r}_1}f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)d\mathfrak{s}\right]\right|\\
&\leq\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}(\mathfrak{r}_2-\mathfrak{s})^{\vartheta-1}\left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)\right|d\mathfrak{s}\\
&\quad+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}_1}\left|(\mathfrak{r}_2-\mathfrak{s})^{\vartheta-1}-(\mathfrak{r}_1-\mathfrak{s})^{\vartheta-1}\right|\left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)\right|d\mathfrak{s}\\
&\quad+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left|f\left(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s})\right)\right|d\mathfrak{s}.
\end{aligned}
$$

Since $|f(\mathfrak{s},\varsigma(\mathfrak{s}),{}^{ABC}\mathbb{D}_{a^+}^\vartheta\varsigma(\mathfrak{s}))|\leq\frac{L_f\xi+\mu_f}{1-K_f}$, for $\varsigma\in B_\xi$, we have

$$
\begin{aligned}
&\left|\left(\mathbb{T}_2^*\varsigma\right)(\mathfrak{r}_2)-\left(\mathbb{T}_2^*\varsigma\right)(\mathfrak{r}_1)\right|\\
&\leq\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}(\mathfrak{r}_2-\mathfrak{s})^{\vartheta-1}\left(\frac{L_f\xi+\mu_f}{1-K_f}\right)d\mathfrak{s}\\
&\quad+\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta)}\int_a^{\mathfrak{r}_1}\left|(\mathfrak{r}_2-\mathfrak{s})^{\vartheta-1}-(\mathfrak{r}_1-\mathfrak{s})^{\vartheta-1}\right|\left(\frac{L_f\xi+\mu_f}{1-K_f}\right)d\mathfrak{s}\\
&\quad+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\frac{L_f\xi+\mu_f}{1-K_f}\right)d\mathfrak{s}\\
&=\frac{\vartheta-1}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}\frac{L_f\xi+\mu_f}{1-K_f}\left[2(\mathfrak{r}_2-\mathfrak{r}_1)^\vartheta+(\mathfrak{r}_1-a)^\vartheta-(\mathfrak{r}_2-a)^\vartheta)\right]\\
&\quad+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}\left(\frac{L_f\xi+\mu_f}{1-K_f}\right)(\mathfrak{r}_2-\mathfrak{r}_1)\\
&\leq\left(\frac{2(\vartheta-1)}{\mathfrak{N}(\vartheta-1)\Gamma(\vartheta+1)}(\mathfrak{r}_2-\mathfrak{r}_1)^\vartheta+\frac{2-\vartheta}{\mathfrak{N}(\vartheta-1)}(\mathfrak{r}_2-\mathfrak{r}_1)\right)\frac{L_f\xi+\mu_f}{1-K_f}.
\end{aligned}
$$

Observe that $|(\mathbb{T}_2^*\varsigma)(\mathfrak{r}_2) - (\mathbb{T}_2^*\varsigma)(\mathfrak{r}_1)| \to 0$ as $t_2 \to t_1$. In view of the preceding claims, together with Arzela–Ascoli theorem, we infer that $(\mathbb{T}_2^* B_\xi)$ is relatively compact. Hence, Claim 3 holds. So, Theorem 2.2 shows that (1.2) has at least one solution. □

4 Examples

Example 4.1 In this example, we justify the validity of Theorem 3.4. For $\vartheta \in (0,1]$, we consider the following ABC fractional problem:

$$\begin{cases} {}^{ABC}\mathbb{D}_{0^+}^{\vartheta}\varsigma(\mathfrak{r}) = \frac{\mathfrak{r}^2}{9e^{\mathfrak{r}-1}}\left(\frac{|\varsigma(\mathfrak{r})|}{1+|\varsigma(\mathfrak{r})|} + \frac{|{}^{ABC}\mathbb{D}_{0^+}^{\vartheta}\varsigma(\mathfrak{r})|}{1+|{}^{ABC}\mathbb{D}_{0^+}^{\vartheta}\varsigma(\mathfrak{r})|}\right), & \mathfrak{r} \in [0,1], \\ \varsigma(0) - \varsigma'(0) = \int_0^T \frac{|\varsigma(\mathfrak{s})|}{10+|\varsigma(\mathfrak{s})|}\,d\mathfrak{s}. \end{cases} \tag{4.1}$$

Set $f(\mathfrak{r},\varsigma(\mathfrak{r}),\overline{\varsigma}(\mathfrak{r})) = \frac{\mathfrak{r}^2}{9e^{\mathfrak{r}-1}}(\frac{\varsigma(\mathfrak{r})}{1+\varsigma(\mathfrak{r})} + \frac{\overline{\varsigma}(\mathfrak{r})}{1+\overline{\varsigma}(\mathfrak{r})})$ and $g(\mathfrak{r},\varsigma(\mathfrak{r})) = \frac{\varsigma(\mathfrak{r})}{10+\varsigma(\mathfrak{r})}$, for $\mathfrak{r} \in [0,1]$, $\varsigma, \overline{\varsigma} \in \mathbb{R}$.

Clearly, $f(0,\varsigma(0),\overline{\varsigma}(0)) = f'(0,\varsigma(0),\overline{\varsigma}(0)) = 0$. Let $\mathfrak{r} \in [0,1]$ and $\varsigma, \overline{\varsigma}, \upsilon, \overline{\upsilon} \in \mathbb{R}$. Then

$$\begin{aligned} \left|f(\mathfrak{r},\varsigma,\overline{\varsigma}) - f(\mathfrak{r},\upsilon,\overline{\upsilon})\right| &\le \frac{\mathfrak{r}^2}{9e^{\mathfrak{r}-1}}\left(\left|\frac{\varsigma}{1+\varsigma} - \frac{\overline{\varsigma}}{1+\overline{\varsigma}}\right| + \left|\frac{\upsilon}{1+\upsilon} - \frac{\overline{\upsilon}}{1+\overline{\upsilon}}\right|\right) \\ &\le \frac{1}{9}|\varsigma - \overline{\varsigma}| + \frac{1}{9}|\upsilon - \overline{\upsilon}| \end{aligned}$$

and

$$\left|g(\mathfrak{r},\varsigma) - g(\mathfrak{r},\upsilon)\right| = \left|\frac{\varsigma}{10+\varsigma} - \frac{\upsilon}{10+\upsilon}\right| \le \frac{10|\varsigma-\upsilon|}{(10+\varsigma)(10+\upsilon)} \le \frac{1}{10}|\varsigma - \upsilon|.$$

Therefore, the hypotheses (A_1) and (A_2) hold with $L_f = K_f = \frac{1}{9}$ and $L_g = \frac{1}{10}$. We shall examine that the condition (A_3) holds too, with $\vartheta = \frac{1}{2}$ and $\mathfrak{N}(\vartheta) = 1$. Indeed,

$$\left[L_g(T-a) + \left(\frac{1-\vartheta}{\mathfrak{N}(\vartheta)} + \frac{(T-a)^{\vartheta}}{\mathfrak{N}(\vartheta)\Gamma(\vartheta)}\right)\frac{L_f}{1-K_f}\right] \approx 0.23 < 1.$$

Thus by Theorem 3.4, the problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 The following example validates Theorem 3.7. For $\vartheta \in (1,2]$, we consider the following ABC fractional problem:

$$\begin{cases} {}^{ABC}\mathbb{D}_{1^+}^{\vartheta}\varsigma(\mathfrak{r}) = \frac{\mathfrak{r}-1}{9+e^{\mathfrak{r}-1}}\left(\frac{|\varsigma(\mathfrak{r})|}{1+|\varsigma(\mathfrak{r})|} + \frac{|{}^{ABC}\mathbb{D}_{0^+}^{\vartheta}\varsigma(\mathfrak{r})|}{1+|{}^{ABC}\mathbb{D}_{0^+}^{\vartheta}\varsigma(\mathfrak{r})|}\right), & \mathfrak{r} \in [1,2], \\ \varsigma(1) - \varsigma(2) = \int_1^2 \frac{|\varsigma(\mathfrak{s})|}{100+|\varsigma(\mathfrak{s})|}\,d\mathfrak{s}. \end{cases} \tag{4.2}$$

Set $f(\mathfrak{r},\varsigma(\mathfrak{r}),\overline{\varsigma}(\mathfrak{r})) = \frac{\mathfrak{r}-1}{9+e^{\mathfrak{r}-1}}(\frac{\varsigma(\mathfrak{r})}{1+\varsigma(\mathfrak{r})} + \frac{\overline{\varsigma}(\mathfrak{r})}{1+\overline{\varsigma}(\mathfrak{r})})$ and $g(\mathfrak{r},\varsigma(\mathfrak{r})) = \frac{\varsigma(\mathfrak{r})}{100+\varsigma(\mathfrak{r})}$, for $\mathfrak{r} \in [1,2]$, $\varsigma, \overline{\varsigma} \in \mathbb{R}$.

Clearly, $f(1,\varsigma(1),\overline{\varsigma}(1)) = 0$. Let $\mathfrak{r} \in [1,2]$ and $\varsigma, \overline{\varsigma}, \upsilon, \overline{\upsilon} \in \mathbb{R}$. Then

$$\left|f(\mathfrak{r},\varsigma,\overline{\varsigma}) - f(\mathfrak{r},\upsilon,\overline{\upsilon})\right| \le \frac{1}{9}|\varsigma - \overline{\varsigma}| + \frac{1}{9}|\upsilon - \overline{\upsilon}|$$

and

$$\left|g(\mathfrak{r},\varsigma) - g(\mathfrak{r},\upsilon)\right| \le \frac{1}{100}|\varsigma - \upsilon|.$$

Therefore, the hypotheses (A_1) and (A_2) hold with $L_f = K_f = \frac{1}{9}$ and $L_g = \frac{1}{100}$. Also, for $1 < \vartheta \leq 2$, $a = 1$, $T = 2$, and $\mathfrak{N}(\vartheta) = 1$, the condition (A_3) holds, that is, $L_g(T - a) = \frac{1}{100} < 1$. Thus by Theorem 3.7, the problem (4.2) has a unique solution on $[1, 2]$.

Remark 4.1 In Theorems 3.3, 3.4, and 3.5, if $f'(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) \neq 0$, then the formula of solution of the problem (1.1) becomes

$$\begin{aligned} \varsigma(\mathfrak{r}) = {}& \int_a^T g\bigl(\mathfrak{s}, \varsigma(\mathfrak{s})\bigr)\,d\mathfrak{s} + \frac{1-\vartheta}{\mathfrak{N}(\vartheta)}\bigl[f'\bigl(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)\bigr) + f\bigl(\mathfrak{r}, \varsigma(\mathfrak{r}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{r})\bigr)\bigr] \\ & + \frac{\vartheta}{\mathfrak{N}(\vartheta)}\frac{1}{\Gamma(\vartheta)}\int_a^{\mathfrak{r}} (\mathfrak{r}-\mathfrak{s})^{\vartheta-1} f\bigl(\mathfrak{s}, \varsigma(\mathfrak{s}), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(\mathfrak{s})\bigr)\,d\mathfrak{s}, \end{aligned}$$

for $\mathfrak{r} \in \mathfrak{Z}$. Accordingly, the analysis of the results remains valid with the addition of the Lipschitz-type condition on f', that is,

(A_4) $f' : \mathfrak{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $L^* > 0$ and $0 < K^* < 1$ such that

$$\bigl|f'(\mathfrak{r}, u, \overline{u}) - f'(\mathfrak{r}, v, \overline{v})\bigr| \leq L^*|u - v| + K^*|\overline{u} - \overline{v}|, \quad \mathfrak{r} \in \mathfrak{Z} \text{ and } u, v, \overline{u}, \overline{v} \in \mathbb{R}.$$

5 Conclusions

The theory of fractional operators containing nonsingular kernels is novel and of considerable recent interest, thus there is a need to study the qualitative properties of differential equations involving such operators. In this work, we have considered two classes of boundary value problems for fractional implicit differential equations with nonlinear integral conditions in the framework of Atangana–Baleanu–Caputo derivatives. Krasnoselskii and Banach fixed point theorems were applied to establish the existence and uniqueness theorems for the problems (1.1) and (1.2). For problem (1.1), we realized that one must always have the necessary conditions $f(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$ and $f'(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$ to guarantee a unique solution, whereas for problem (1.2) we needed $f(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$ to confirm the initial data for the solution. To avoid the condition $f'(a, \varsigma(a), {}^{ABC}\mathbb{D}^{\vartheta}_{a^+}\varsigma(a)) = 0$ in Theorem 3.3, one can use condition (A_4) mentioned in Remark 4.1 to obtain the same results. The proposed problems are more general, also the results obtained generalize the recent studies and offer an extension of the development of FDEs that involve this new operator. Moreover, the analysis of the results was limited to the minimum assumptions. The problems scrutinized include some special cases, in other words, they could be reduced to the corresponding problems that contain Caputo–Fabrizio operator. We are certain that the communicated results here are rather interesting, and will have a positive effect on the development of more applications in engineering and sciences.

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