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Fuzzy stochastic differential equations driven by fractional Brownian motion

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Abstract

In this paper, we consider fuzzy stochastic differential equations (FSDEs) driven by fractional Brownian motion (fBm). These equations can be applied in hybrid real-world systems, including randomness, fuzziness and long-range dependence. Under some assumptions on the coefficients, we follow an approximation method to the fractional stochastic integral to study the existence and uniqueness of the solutions. As an example, in financial models, we obtain the solution for an equation with linear coefficients.

Keywords: Fuzzy set theory; Fuzzy stochastic processes; Fuzzy stochastic differential equation; Fractional Brownian motion

1 Introduction

The FSDEs are used in real-world systems that the phenomena are related to randomness and fuzziness as two kinds of uncertainties, such as in economics and finance. There are papers on FSDEs that each one is different from the others in the approach. The author in [9] presented a definition of the fuzzy stochastic Itô integral. In [12–15] the author used the fuzzy non-anticipating stochastic processes and Wiener process to drive the fuzzy Itô stochastic integral. The method deals with the embedding of a crisp Itô stochastic integral into fuzzy space to build a fuzzy random variable.

On the other hand, in the modeling of many stochastic systems, the fBm which shows a long-range dependence, is suggested to replace the Brownian motion as the driving process. The fBm with $H \in (0, 1)$ as Hurst parameter is a Gaussian process with beneficial properties, long-range dependence, self-similarity and stationary of increments. This process is appropriate for the analysis of phenomena which present long-range and scale-invariant correlations. Nevertheless, when $H \neq \frac{1}{2}$, the fBm is not a semimartingale.

In this paper, we introduce FSDEs with respect to the fBm. These equations can be helpful in the modeling of hybrid dynamic systems, including randomness, fuzziness and long-range dependence. We apply an approximation procedure to fractional stochastic integral to find the explicit solutions. We consider the Liouville form of the fBm with parameter $H \in (\frac{1}{2}, 1)$ to study the existence and uniqueness of strong solutions. Moreover, we consider an application of the equations in financial models.

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The organization of the paper is as follows. In Sect. 2, the definition of the fBm and the Liouville form of this process are reviewed. Then some preliminaries on fuzzy stochastic processes and fuzzy stochastic integrals are recalled. In Sect. 3, a class of FSDEs driven by fBm is introduced. Besides, an approximation approach is used to prove the existence and uniqueness of the solutions. Finally, some conclusions are given in Sect. 4.

2 Preliminaries

2.1 Fractional Brownian motion

The fBm $B^H = \{B^H(t), t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process with the following covariance function:

$$R_{B^H}(t, s) = \mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}). \tag{2.1}$$

This process was introduced in [10] and studied in [16], where a stochastic integral representation was established in terms of the Brownian motion. The long-range dependence and self-similarity properties of this process, for $H > 1/2$, yield a suitable driving noise in stochastic models, such as networks, finance, and physics. The classical Itô theory cannot be used to construct a stochastic integral in terms of the fBm, because B^H is not a semimartingale if $H \neq 1/2$. Two approaches have been used to define stochastic integrals with respect to fBm. In the first one, the Riemann–Stieltjes stochastic integral can be defined using Young’s integral [19] in the case of $H > 1/2$. The second approach to defining a stochastic integral with respect to the fBm is based on the Malliavin calculus (see [2–4, 6]). In [16] a representation of $B_H(t)$ was given as follows:

$$B^H(t) = \frac{1}{\Gamma(1 + \alpha)} \left(\int_{-\infty}^0 [(t - s)^\alpha - (-s)^\alpha] dW(s) + B_H(t) \right),$$

where W is a Brownian motion, $\alpha = H - \frac{1}{2}$ and $B_H(t) = \int_0^t (t - s)^\alpha dW(s)$. The process $B_H(t)$, with $H \in (0, 1)$ is called the Liouville form of a fractional Brownian motion (LfBm) which holds many properties of the fBm except that it has non-stationary of increments. In [2] the Malliavin calculus technique was used to approximate $B_H(t)$ by a semimartingale process as follows:

$$B_{H,\epsilon}(t) = \int_0^t (t - s + \epsilon)^\alpha dW(s), \quad \epsilon > 0. \tag{2.2}$$

Furthermore,

$$B_{H,\epsilon}(t) = \alpha \int_0^t \varphi^\epsilon(s) ds + \epsilon^\alpha W(t), \tag{2.3}$$

where

$$\varphi^\epsilon(t) = \int_0^t (t - s + \epsilon)^{\alpha-1} dW(s). \tag{2.4}$$

The process $B_{H,\epsilon}(t)$ converges to $B_H(t)$ in $L^2(\Omega)$ when ϵ tends to zero [18].

2.2 Fuzzy background

In this section, we provide some preliminaries on fuzzy random variable, fuzzy stochastic process and fuzzy stochastic integral (see [7, 11, 13]). Let us denote by $\mathcal{K}(\mathbb{R})$ the family of all nonempty, compact and convex subsets of \mathbb{R} . The Hausdorff metric, denoted by d_H , is defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b| \right\}.$$

The space $\mathcal{K}(\mathbb{R})$ is a complete and separable metric space with respect to d_H . If $A, B, C \in \mathcal{K}(\mathbb{R})$, then

$$d_H(A + C, B + C) = d_H(A, B).$$

Let (Ω, \mathcal{A}, P) be a probability space. The mapping $F : \Omega \rightarrow \mathcal{K}(\mathbb{R})$ is called \mathcal{A} -measurable if it satisfies

$$\{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \mathcal{A},$$

for every closed set $C \subset \mathbb{R}$. Let $\mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}))$ denote a family of \mathcal{A} -measurable multifunctions with values in $\mathcal{K}(\mathbb{R})$. A multifunction $F \in \mathcal{M}$ is said to be L^p -integrably bounded, for $p \geq 1$, if there exists $h \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}_+)$ such that $\|F\| \leq h$ P-a.e, $\mathbb{R}_+ = [0, \infty)$, and

$$\|F\| = d_H(F, \{0\}) = \sup_{f \in F} |f|.$$

It is well known that (see [8]) $F \in \mathcal{M}$ is L^p -integrably bounded if and only if $\|F\| \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}_+)$. Let us denote

$$\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbb{R})) = \{F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R})) : \|F\| \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}_+)\}.$$

The membership function $u : \mathbb{R} \rightarrow [0, 1]$ is defined for a fuzzy set $u \in \mathbb{R}$, where $u(x)$ is the degree of membership of x in the fuzzy set u . Let us denote by $\mathcal{F}(\mathbb{R})$ the fuzzy sets $u : \mathbb{R} \rightarrow [0, 1]$ such that $[u]^\alpha \in \mathcal{K}(\mathbb{R})$ for every $\alpha \in [0, 1]$, where $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$.

Define $d_\infty : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \rightarrow [0, \infty)$ by

$$d_\infty(u, v) = \sup_{\alpha \in [0, 1]} d_H([u]^\alpha, [v]^\alpha),$$

then d_∞ is a metric in $\mathcal{F}(\mathbb{R})$ and $(\mathcal{F}(\mathbb{R}), d_\infty)$ is a complete metric space. For every $u, v, w, z \in \mathcal{F}(\mathbb{R})$, $\lambda \in \mathbb{R}$, we have the following properties:

- $d_\infty(u + w, v + w) = d_\infty(u, v)$,
- $d_\infty(u + v, w + z) = d_\infty(u, w) + d_\infty(v, z)$
- $d_\infty(u, v) \leq d_\infty(u, w) + d_\infty(w, v)$
- $d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v)$.

We use $\langle 0 \rangle \in \mathcal{F}(\mathbb{R})$ as $\langle 0 \rangle := \mathbb{1}_{\{0\}}$, where for $y \in \mathbb{R}$, $\mathbb{1}_{\{y\}}(x) = 1$ if $x = y$ and $\mathbb{1}_{\{y\}}(x) = 0$ if $x \neq y$.

Definition 2.1 ([17]) Let (Ω, \mathcal{A}, P) be a probability space. A fuzzy random variable is a function $X : \Omega \rightarrow \mathcal{F}(\mathbb{R})$, if the mapping $[X]^\alpha : \Omega \rightarrow \mathcal{K}(\mathbb{R})$ is an \mathcal{A} -measurable multifunction for all $\alpha \in [0, 1]$.

Let us consider a metric ρ in the set $\mathcal{F}(\mathbb{R})$, and σ -algebra \mathcal{B}_ρ generated by the topology induced by ρ . A fuzzy random variable can be viewed as a measurable mapping between two measurable spaces, namely (Ω, \mathcal{A}) and $(\mathcal{F}(\mathbb{R}), \mathcal{B}_\rho)$, we call X is $\mathcal{A}|\mathcal{B}_\rho$ -measurable. Consider the following metric:

$$d_s(u, v) := \inf_{\lambda \in \Lambda} \max \left\{ \sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} d_H(\mathcal{X}_u(t), \mathcal{X}_u(\lambda(t))) \right\},$$

where Λ denotes the set of strictly increasing continuous functions $\lambda : [0, 1] \rightarrow [0, 1]$ such that $\lambda(0) = 0, \lambda(1) = 1$, and $\mathcal{X}_u, \mathcal{X}_v : [0, 1] \rightarrow \mathcal{F}(\mathbb{R})$ are càdlàg representations for the fuzzy sets $u, v \in \mathcal{F}(\mathbb{R})$ (see [5]). The space $(\mathcal{F}(\mathbb{R}), d_\infty)$ is complete and non-separable, and the space $(\mathcal{F}(\mathbb{R}), d_s)$ is a Polish metric space.

For a mapping $X : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ on the probability space (Ω, \mathcal{A}, P) , we have:

- X is a fuzzy random variable if and only if X is $\mathcal{A}|\mathcal{B}_{d_s}$ -measurable.
- If X is $\mathcal{A}|\mathcal{B}_{d_\infty}$ -measurable, then it is a fuzzy random variable; the opposite is not true.

Definition 2.2 A fuzzy random variable $X : \Omega \rightarrow \mathcal{F}(\mathbb{R})$, is L^p -integrably bounded, for $p \geq 1$, if $[X]^\alpha \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbb{R}))$, for every $\alpha \in [0, 1]$.

Let us denote by $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}))$ the set of all L^p -integrably bounded fuzzy random variables. The random variables $X, Y \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}))$ are identical if $P(d_\infty(X, Y) = 0) = 1$. For a fuzzy random variable $X : \Omega \rightarrow \mathcal{F}(\mathbb{R})$, and $p \geq 1$, the following conditions are equivalent:

- a) $X \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}))$,
- b) $[X]^0 \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbb{R}))$,
- c) $\| [X]^0 \| \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}_+)$.

Let $I := [0, T]$, and (Ω, \mathcal{A}, P) be a complete probability space with a filtration $\{\mathcal{A}_t\}_{t \in I}$ satisfying the hypotheses, an increasing and right continuous family of sub σ -algebras of \mathcal{A} , and containing all P -null sets.

Definition 2.3 If the mapping $X(t) : \Omega \rightarrow \mathcal{F}(\mathbb{R})$, for every $t \in I$, is a fuzzy random variable, then $X : I \times \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy stochastic process.

Definition 2.4 A fuzzy stochastic process X is d_∞ -continuous, if almost all its trajectories, i.e. the mappings $X(\cdot, \omega) : I \times \Omega \rightarrow \mathcal{F}(\mathbb{R})$ are d_∞ -continuous functions.

A fuzzy stochastic process X is a measurable, if $[X]^\alpha : I \times \Omega \rightarrow \mathcal{K}(\mathbb{R})$ is $\mathcal{B}(I) \otimes \mathcal{A}$ -measurable multifunction for all $\alpha \in [0, 1]$, where $\mathcal{B}(I)$ denotes the Borel σ -algebra of subsets of I .

A process X is nonanticipating if and only if for every $\alpha \in [0, 1]$, the multifunction $[X]^\alpha$ is measurable with respect to the σ -algebra \mathcal{N} , which is defined as follows

$$\mathcal{N} := \{A \in \mathcal{B}(I) \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\},$$

where $A^t = \{\omega : (t, \omega) \in A\}$.

Definition 2.5 A fuzzy stochastic process X is called L^p -integrably bounded ($p \geq 1$), if there exists a real-valued stochastic process $h \in L^p(I \times \Omega, \mathcal{N}; \mathbb{R}_+)$ such that

$$\| [X(t, \omega)]^0 \| \leq h(t, \omega),$$

for almost all $(t, \omega) \in I \times \Omega$.

Let us denote by $\mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$ the set of nonanticipating and L^p -integrably bounded fuzzy stochastic processes. Let $X \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$, by the Fubini theorem, the fuzzy integral is defined by

$$\int_0^T X(s, \omega) ds,$$

for $\omega \in \Omega \setminus N_x$, where $N_x \in \mathcal{A}$ and $P(N_x) = 0$. The fuzzy integral $\int_0^T X(s, \omega) ds$ can be defined level-wise. For every $\alpha \in [0, 1]$, and every $\omega \in \Omega \setminus N_x$, the Aumann integral $\int_0^T [X(s, \omega)]^\alpha ds$ belongs to $\mathcal{K}(\mathbb{R})$, so a fuzzy random variable $\int_0^T X(s, \omega) ds$ belongs to $\mathcal{F}(\mathbb{R})$ for every $\omega \in \Omega \setminus N_x$.

Definition 2.6 The fuzzy stochastic Lebesgue–Aumann integral of $X \in \mathcal{L}^1(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$ is defined as

$$L_x(t, \omega) = \begin{cases} \int_0^t \mathbb{1}_{[0,t]}(s) X(s, \omega) ds & \text{for every } \omega \in \Omega \setminus N_x, \\ \langle 0 \rangle & \text{for every } \omega \in N_x. \end{cases} \tag{2.5}$$

Proposition 2.1 ([13]) *For the integral L_x , one can show the following properties:*

- 1) Let $p \geq 1$. If $X \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$, then $L_x(\cdot, \cdot) \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$.
- 2) Let $X \in \mathcal{L}^1(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$, then $\{L_x(t)\}_{t \in I}$ is d_∞ -continuous.
- 3) Let $X, Y \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$, for $p \geq 1$, then

$$\sup_{u \in [0,t]} d_\infty^p(L_{t,x}(u), L_{t,y}(u)) \leq t^{p-1} \int_0^t d_\infty^p(X(s), Y(s)) ds, \quad a.e.$$

Let us denote by $\langle \cdot \rangle : \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R})$ an embedding of \mathbb{R} into $\mathcal{F}(\mathbb{R})$ i.e. for $r \in \mathbb{R}$,

$$\langle r \rangle(a) = \begin{cases} 1 & \text{for } a = r, \\ 0 & \text{for } a \in \mathbb{R} \setminus \{r\}. \end{cases}$$

If $X : \Omega \rightarrow \mathbb{R}$ is a random variable on the probability space (Ω, \mathcal{A}, P) , then $\langle X \rangle : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy random variable. For stochastic processes we have a similar property.

We define the fuzzy stochastic Itô integral by using the fuzzy random variable as $\langle \int_0^T X(s) dW(s) \rangle$, where W is a Wiener process. The following properties will be useful [13].

Proposition 2.2 *Let $X \in L^2(I \times \Omega, \mathcal{N}; \mathbb{R})$, then $\{\langle \int_0^t X(s) dW(s) \rangle\}_{t \in I}$ is a fuzzy stochastic process and we have $\langle \int_0^t X(s) dW(s) \rangle \in L^2(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$.*

Proposition 2.3 *Let $X \in L^2(I \times \Omega, \mathcal{N}; \mathbb{R})$, then $\{\langle \int_0^t X(s) dW(s) \rangle\}_{t \in I}$ is d_∞ -continuous.*

3 Application to fuzzy stochastic differential equation

In this section we consider a class of FSDEs driven by the fBm which is given as follows:

$$X(t) = X_0 + \int_0^t f(s, X(s)) ds + \left\langle \int_0^t g(s, X(s)) dB_H(s) \right\rangle, \quad X_0 = X(0), \tag{3.1}$$

where B_H is a Liouville form fBm with $H \in (\frac{1}{2}, 1)$, $X_0 : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is an FRV, $f : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$, and $g : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$. The corresponding approximation equation (3.1) is

$$X^\epsilon(t) = X_0 + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle. \tag{3.2}$$

Assumptions 3.1 Consider the following assumptions on the equation coefficients:

- A1) The mappings $f : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$ and $g : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$ are $\mathcal{N} \otimes \mathcal{B}_{d_s} | \mathcal{B}_{d_s}$ -measurable and $\mathcal{N} \otimes \mathcal{B}_{d_s} | \mathcal{B}(\mathbb{R})$ -measurable, respectively.
- A2) For every $u, v \in \mathcal{F}(\mathbb{R})$ and every $t \in I$, there exists a constant $L > 0$ such that

$$\max\{d_\infty(f(t, \omega, u), f(t, \omega, v)), |g(t, \omega, u) - g(t, \omega, v)|\} \leq Ld_\infty(u, v).$$

- A3) For every $u, v \in \mathcal{F}(\mathbb{R})$ and every $t \in I$, there exists a constant $C > 0$ such that

$$\max\{d_\infty(f(t, \omega, u), \langle 0 \rangle), |g(t, \omega, u)|\} \leq C(1 + d_\infty(u, \langle 0 \rangle)).$$

Proposition 3.1 ([13]) Consider $X, Y \in L^2(I \times \Omega, \mathcal{N}; \mathbb{R})$, then

$$\mathbb{E} \sup_{u \in [0, t]} d_\infty^2 \left(\left\langle \int_0^u X(s) dW(s) \right\rangle, \left\langle \int_0^u Y(s) dW(s) \right\rangle \right) \leq 4\mathbb{E} \int_0^t d_\infty^2(\langle X(s) \rangle, \langle Y(s) \rangle) ds, \tag{3.3}$$

for every $t \in I$.

Theorem 3.2 Suppose that $f : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$ and $g : I \times \Omega \times \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$ as mappings satisfy assumptions (A1)–(A3) and $X_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{F}(\mathbb{R}))$. Then Eq. (3.2) has a strong unique solution.

Proof Consider the SDE (3.2),

$$X^\epsilon(t) = X_0 + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle. \tag{3.4}$$

By Eq. (2.3), we can write

$$\begin{aligned} X^\epsilon(t) = X_0 + \int_0^t f(s, X^\epsilon(s)) ds \\ + \left\langle \int_0^t \alpha \varphi^\epsilon(s) g(s, X^\epsilon(s)) ds + \int_0^t \epsilon^\alpha g(s, X^\epsilon(s)) dW(s) \right\rangle. \end{aligned} \tag{3.5}$$

Let us consider the Picard iterations

$$X_n^\epsilon(t) = X_0 + \int_0^t f(s, X_{n-1}^\epsilon(s)) ds$$

$$+ \left\langle \int_0^t \alpha \varphi^\epsilon(s)g(s, X_{n-1}^\epsilon(s)) ds + \int_0^t \epsilon^\alpha g(s, X_{n-1}^\epsilon(s)) dW(s) \right\rangle, \quad \text{a.e.,} \tag{3.6}$$

for $n = 1, 2, \dots$, and for every $t \in I$, and $X_0(t) = X_0$. For $t \in I$ and $n \in \mathbb{N}$ we denote

$$j_n(t) = \mathbb{E} \sup_{u \in [0,t]} d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)).$$

Then, by Propositions 2.1, 3.1, and assumption A3, it can be written as

$$\begin{aligned} j_1(t) &= \mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\int_0^u f(s, X_0^\epsilon(s)) ds \right. \\ &\quad \left. + \left\langle \int_0^u \alpha \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds + \int_0^u \epsilon^\alpha g(s, X_0^\epsilon(s)) dW(s) \right\rangle, \langle 0 \rangle \right) \\ &\leq 3\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\int_0^u f(s, X_0^\epsilon(s)) ds, \langle 0 \rangle \right) \\ &\quad + 3\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \alpha \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\rangle, \langle 0 \rangle \right) \\ &\quad + 3\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \epsilon^\alpha g(s, X_0^\epsilon(s)) dW(s) \right\rangle, \langle 0 \rangle \right) \\ &\leq 3t\mathbb{E} \int_0^t d_\infty^2(f(s, X_0^\epsilon(s)), \langle 0 \rangle) ds + 3\alpha^2\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\rangle, \langle 0 \rangle \right) \\ &\quad + 12\epsilon^{2\alpha}\mathbb{E} \int_0^t d_\infty^2(g(s, X_0^\epsilon(s)), \langle 0 \rangle) ds \\ &\leq 6C^2(T + 4\epsilon^{2\alpha})(1 + \mathbb{E} \| [X^\epsilon]^0 \|^2)t \\ &\quad + 3\alpha^2\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\rangle, \langle 0 \rangle \right), \end{aligned}$$

for $\alpha = H - \frac{1}{2} > 0$. Hence

$$\begin{aligned} j_1(t) &\leq 6C^2(T + 4\epsilon^{2\alpha})(1 + \mathbb{E} \| [X^\epsilon]^0 \|^2)t \\ &\quad + 3\alpha^2\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\rangle, \langle 0 \rangle \right). \end{aligned} \tag{3.7}$$

We have

$$\begin{aligned} &\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\rangle, \langle 0 \rangle \right) \\ &\leq \mathbb{E} \sup_{u \in [0,t]} d_H^2 \left(\left\{ \int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right\}, \{0\} \right) \\ &\leq \mathbb{E} \sup_{u \in [0,t]} \left(\int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right)^2. \end{aligned} \tag{3.8}$$

By applying (2.4) to (3.8), and the Hölder inequality we get

$$\mathbb{E} \sup_{u \in [0,t]} \left(\int_0^u \varphi^\epsilon(s)g(s, X_0^\epsilon(s)) ds \right)^2$$

$$\begin{aligned}
 &= \mathbb{E} \sup_{u \in [0,t]} \left(\int_0^u \left(\int_0^s (s-r+\epsilon)^{\alpha-1} dW(r) \right) g(s, X_0^\epsilon(s)) ds \right)^2 \\
 &= \mathbb{E} \sup_{u \in [0,t]} \left(\int_0^u \int_s^u g(r, X_0^\epsilon(r)) (r-s+\epsilon)^{\alpha-1} dr dW(s) \right)^2 \\
 &\leq 4\mathbb{E} \int_0^t \left(\int_s^t g(r, X_0^\epsilon(r)) (r-s+\epsilon)^{\alpha-1} dr \right)^2 ds \\
 &\leq 4\mathbb{E} \int_0^t \left(\int_s^t g^2(r, X_0^\epsilon(r)) (r-s+\epsilon)^{\alpha-1} dr \right) \left(\int_s^t (r-s+\epsilon)^{\alpha-1} dr \right) ds \\
 &\leq \frac{4}{\alpha^2} (t+\epsilon)^\alpha \mathbb{E} \int_0^t g^2(r, X_0^\epsilon(r)) (r+\epsilon)^\alpha dr \\
 &\leq \frac{4}{\alpha^2} (t+\epsilon)^{2\alpha} \mathbb{E} \int_0^t g^2(r, X_0^\epsilon(r)) dr \\
 &\leq \frac{8C^2}{\alpha^2} (T+\epsilon)^{2\alpha} (1 + \mathbb{E} \|\| [X_0^\epsilon]^0 \|\|^2) t.
 \end{aligned} \tag{3.9}$$

Hence, from (3.7) and (3.9) we obtain

$$j_1(t) \leq 6C^2 (T + 4\epsilon^{2\alpha} + 4(T + \epsilon)^{2\alpha}) (1 + \mathbb{E} \|\| [X_0^\epsilon]^0 \|\|^2) t, \tag{3.10}$$

for every $t \in I$. Then, similarly,

$$\begin{aligned}
 j_{n+1}(t) &\leq 3(t + 4\epsilon^{2\alpha} + 4(t + \epsilon)^{2\alpha}) L^2 \mathbb{E} \int_0^t d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)) ds \\
 &\leq 3(t + 4\epsilon^{2\alpha} + 4(t + \epsilon)^{2\alpha}) L^2 \int_0^t \mathbb{E} \sup_{u \in [0,s]} d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)) ds \\
 &\leq 3(t + 4\epsilon^{2\alpha} + 4(t + \epsilon)^{2\alpha}) L^2 \int_0^t j_n(s) ds.
 \end{aligned}$$

Therefore

$$j_n(t) \leq 2C^2 3^n (t + 4\epsilon^{2\alpha} + 4(t + \epsilon)^{2\alpha})^n (1 + \mathbb{E} \|\| [X_0^\epsilon]^0 \|\|^2) L^{2(n-1)} \frac{t^n}{n!}, \quad t \in I, n \in \mathbb{N}.$$

Apply the Chebyshev inequality, it follows that

$$P\left(\sup_{u \in I} d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)) > \frac{1}{2^n}\right) \leq 2^n j_n(T).$$

The series $\sum_{n=1}^\infty 2^n j_n(T)$ is convergent. From the Borel–Cantelli lemma, we derive

$$P\left(\sup_{u \in I} d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)) > \frac{1}{(\sqrt{2})^n} \text{ infinitely often} \right) = 0.$$

For almost all $\omega \in \Omega$, there exists $n_0(\omega)$ such that

$$\sup_{u \in I} d_\infty^2(X_n^\epsilon(u), X_{n-1}^\epsilon(u)) \leq \frac{1}{(\sqrt{2})^n}, \quad \text{if } n \geq n_0.$$

The sequence $\{X_n^\epsilon(\cdot, \omega)\}$ is uniformly convergent to a d_∞^2 -continuous fuzzy process $\tilde{X}^\epsilon(\cdot, \omega)$ for every $\omega \in \Omega_c$, in which $\Omega_c \in \mathcal{A}$ and $P(\Omega_c) = 1$. We can define the mapping $X^\epsilon : I \times \Omega \rightarrow \mathcal{F}(\mathbb{R})$, as $X^\epsilon(\cdot, \omega) = \tilde{X}^\epsilon(\cdot, \omega)$ if $\omega \in \Omega_c$ and $X^\epsilon(\cdot, \omega)$ as freely chosen fuzzy function when $\omega \in \Omega \setminus \Omega_c$. For every $\alpha \in [0, 1]$ and every $t \in I$ with a.e., we have

$$d_H([X_n^\epsilon(t)]^\alpha, [X^\epsilon(t)]^\alpha) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, X^ϵ will be a continuous fuzzy stochastic process. Then, by $X_n^\epsilon \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$, we get $X^\epsilon \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}))$. Hence, as n goes to infinity, we can verify that

$$\mathbb{E} \sup_{t \in I} \left[d_\infty^2(X_n^\epsilon(t), X^\epsilon(t)) + d_\infty^2 \left(X_n^\epsilon(t), X_0^\epsilon + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle \right) \right]^2$$

tends to zero. Then

$$\mathbb{E} \sup_{t \in I} d_\infty^2 \left[\left(X^\epsilon(t), X_0^\epsilon + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle \right) \right] = 0.$$

Therefore

$$\sup_{t \in I} d_\infty^2 \left[\left(X^\epsilon(t), X_0^\epsilon + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle \right) \right] = 0,$$

which shows the existence of the strong solution.

Now, $X^\epsilon, Y^\epsilon : I \times \Omega \rightarrow \mathcal{F}(\mathbb{R})$ are assumed to be strong solutions. Consider

$$j(t) = \mathbb{E} \sup_{u \in [0, t]} d_\infty^2(X^\epsilon(u), Y^\epsilon(u)),$$

then, by computations similar to the existence case, we have

$$j(t) \leq 3(t + 4\epsilon^{2\alpha} + 4(t + \epsilon)^{2\alpha})L^2 \mathbb{E} \int_0^t d_\infty^2(X^\epsilon(s), Y^\epsilon(s)) ds \leq 3(T + 4\epsilon^{2\alpha} + \alpha^2)L^2 \int_0^t j(s) ds.$$

The implementation of the Gronwall inequality leads to $j(t) = 0$ for $t \in I$. Then

$$\sup_{t \in I} d_\infty^2(X^\epsilon(t), Y^\epsilon(t)) = 0, \quad \text{a.e.,}$$

by which the proof of uniqueness is completed. □

Lemma 3.1 For every $\epsilon > 0$ and $0 < \alpha < \frac{1}{2}$ we have

$$\int_s^t ((r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}) dr \leq \frac{\alpha + 1}{\alpha} \epsilon^\alpha. \tag{3.11}$$

Proof We apply the finite-increments formula to the function $f(x) = x^{\alpha-1}$ to obtain

$$(x + \epsilon)^{\alpha-1} - x^{\alpha-1} = (\alpha - 1)(x + \theta\epsilon)^{\alpha-2}\epsilon, \quad 0 < \theta < 1, \tag{3.12}$$

then

$$|(r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}| \leq |\alpha - 1| |r - s|^{\alpha-2} \epsilon. \tag{3.13}$$

Hence

$$\begin{aligned} \int_s^t |(r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}| dr &= \int_s^{s+\epsilon} |(r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}| dr \\ &\quad + \int_{s+\epsilon}^t |(r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}| dr \\ &\leq \int_s^{s+\epsilon} |(2(r - s))^{\alpha-1} - (r - s)^{\alpha-1}| dr \\ &\quad + |\alpha - 1| \epsilon \int_{s+\epsilon}^t |r - s|^{\alpha-2} dr. \end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned} \int_s^t |(r - s + \epsilon)^{\alpha-1} - (r - s)^{\alpha-1}| dr &\leq \int_s^{s+\epsilon} (r - s)^{\alpha-1} dr + |\alpha - 1| \epsilon \int_{s+\epsilon}^t (r - s)^{\alpha-2} dr \\ &\leq \frac{1}{\alpha} \epsilon^\alpha + |\alpha - 1| \epsilon \left(\frac{1}{\alpha - 1} \epsilon^{\alpha-1} \right) \\ &= \frac{\alpha + 1}{\alpha} \epsilon^\alpha. \end{aligned} \tag{3.15}$$

□

Proposition 3.2 *The solution $X^\epsilon(t)$ of Eq. (3.2) converges to the solution $X(t)$ of Eq. (3.1) in $\mathcal{L}^2(I \times \Omega)$ as $\epsilon \rightarrow 0$ uniformly with respect to $t \in [0, T]$.*

Proof Consider Eq. (3.1) and the corresponding approximation of equations as follows:

$$X(t) = X_0 + \int_0^t f(s, X(s)) ds + \left\langle \int_0^t g(s, X(s)) dB_H(s) \right\rangle, \tag{3.16}$$

$$X^\epsilon(t) = X_0 + \int_0^t f(s, X^\epsilon(s)) ds + \left\langle \int_0^t g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle. \tag{3.17}$$

We can write

$$\begin{aligned} &\mathbb{E} \sup_{u \in [0, t]} d_\infty^2(X(u), X^\epsilon(u)) \\ &\leq 2 \mathbb{E} \sup_{u \in [0, t]} \int_0^u d_\infty^2(f(s, X(s)), f(s, X^\epsilon(s))) ds \\ &\quad + 2 \mathbb{E} \sup_{u \in [0, t]} d_\infty^2 \left(\left\langle \int_0^u g(s, X(s)) dB_H(s) \right\rangle, \left\langle \int_0^u g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle \right). \end{aligned}$$

Then

$$\mathbb{E} \sup_{u \in [0, t]} d_\infty^2(X(u), X^\epsilon(u))$$

$$\begin{aligned}
 &\leq 2\mathbb{E} \sup_{u \in [0,t]} \int_0^u d_\infty^2(f(s, X(s)), f(s, X^\epsilon(s))) ds \\
 &\quad + 4\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u g(s, X(s)) dB_H(s), \int_0^u g(s, X(s)) dB_H^\epsilon(s) \right\rangle \right) \\
 &\quad + 4\mathbb{E} \sup_{u \in [0,t]} d_\infty^2 \left(\left\langle \int_0^u g(s, X(s)) dB_H^\epsilon(s), \int_0^u g(s, X^\epsilon(s)) dB_H^\epsilon(s) \right\rangle \right) \\
 &= 2\mathbb{E} \int_0^t d_\infty^2(f(s, X(s)), f(s, X^\epsilon(s))) ds \\
 &\quad + 4\mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u g(s, X^\epsilon(s)) (dB_H^\epsilon(s) - dB_H(s)) \right|^2 \\
 &\quad + 4\mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u (g(s, X(s)) - g(s, X^\epsilon(s))) dB_H^\epsilon(s) \right|^2.
 \end{aligned}$$

Apply Eq. (2.3) to get

$$\begin{aligned}
 &\mathbb{E} \sup_{u \in [0,t]} d_\infty^2(X(u), X^\epsilon(u)) \\
 &\leq 2\mathbb{E} \int_0^t d_\infty^2(f(s, X(s)), f(s, X^\epsilon(s))) ds \\
 &\quad + 8\epsilon^{2\alpha} \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u g(s, X^\epsilon(s)) dW(s) \right|^2 \\
 &\quad + 8\alpha^2 \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u g(s, X^\epsilon(s)) \left(\int_0^s ((s-r+\epsilon)^{\alpha-1} - (s-r)^{\alpha-1}) dW(r) \right) ds \right|^2 \\
 &\quad + 8\alpha^2 \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u (g(s, X(s)) - g(s, X^\epsilon(s))) \left(\int_0^s (s-r+\epsilon)^{\alpha-1} dW(r) \right) ds \right|^2 \\
 &\quad + 8\epsilon^{2\alpha} \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u (g(s, X(s)) - g(s, X^\epsilon(s))) dW(s) \right|^2. \tag{3.18}
 \end{aligned}$$

Then

$$\begin{aligned}
 &\mathbb{E} \sup_{u \in [0,t]} d_\infty^2(X(u), X^\epsilon(u)) \\
 &\leq 2\mathbb{E} \int_0^t d_\infty^2(f(s, X(s)), f(s, X^\epsilon(s))) ds \\
 &\quad + 8\epsilon^{2\alpha} \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u g(s, X^\epsilon(s)) dW(s) \right|^2 \\
 &\quad + 8\alpha^2 \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u \int_s^u g(r, X^\epsilon(r)) ((r-s+\epsilon)^{\alpha-1} - (r-s)^{\alpha-1}) dr dW(s) \right|^2 \\
 &\quad + 8\alpha^2 \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u \int_s^u (g(r, X(r)) - g(s, X^\epsilon(s))) (r-s+\epsilon)^{\alpha-1} dr dW(s) \right|^2 \\
 &\quad + 8\epsilon^{2\alpha} \mathbb{E} \sup_{u \in [0,t]} \left| \int_0^u (g(s, X(s)) - g(s, X^\epsilon(s))) dW(s) \right|^2. \tag{3.19}
 \end{aligned}$$

Apply the Doob inequality, the Hölder inequality and the Itô isometry property to obtain

$$\begin{aligned}
 & \mathbb{E} \sup_{u \in [0,t]} d_{\infty}^2(X(u), X^{\epsilon}(u)) \\
 & \leq 2\mathbb{E} \int_0^t d_{\infty}^2(f(s, X(s)), f(s, X^{\epsilon}(s))) ds + 32\epsilon^{2\alpha} \mathbb{E} \int_0^t g^2(s, X^{\epsilon}(s)) ds \\
 & \quad + 32\alpha^2 \mathbb{E} \int_0^t \left[\int_s^t g^2(r, X^{\epsilon}(r)) ((r-s+\epsilon)^{\alpha-1} - (r-s)^{\alpha-1}) dr \right. \\
 & \quad \times \left. \int_s^t ((r-s+\epsilon)^{\alpha-1} - (r-s)^{\alpha-1}) dr \right] ds \\
 & \quad + 32\alpha^2 \mathbb{E} \int_0^t \left(\int_s^t (g(r, X(r)) - g(s, X^{\epsilon}(s)))^2 (r-s+\epsilon)^{\alpha-1} dr \right) \\
 & \quad \times \left(\int_s^t (r-s+\epsilon)^{\alpha-1} dr \right) ds \\
 & \quad + 32\epsilon^{2\alpha} \mathbb{E} \int_0^t (g(s, X(s)) - g(s, X^{\epsilon}(s)))^2 ds. \tag{3.20}
 \end{aligned}$$

By similar arguments to (3.9), from (3.11) and assumptions (A1)–(A3), we infer that

$$\begin{aligned}
 & \mathbb{E} \sup_{u \in [0,t]} d_{\infty}^2(X(u), X^{\epsilon}(u)) \\
 & \leq 2L^2 \mathbb{E} \int_0^t d_{\infty}^2(X(s), X^{\epsilon}(s)) ds + 64\epsilon^{2\alpha} C^2 \int_0^t (1 + \mathbb{E} \|\| [X^{\epsilon}(s)]^0 \|\|^2) ds \\
 & \quad + 64C^2 (T + \epsilon)^{2\alpha} \frac{\alpha + 1}{\alpha} \epsilon^{\alpha} \int_0^t (1 + \mathbb{E} \|\| [X^{\epsilon}(s)]^0 \|\|^2) ds \\
 & \quad + 32L^2 (T + \epsilon)^{2\alpha} \mathbb{E} \int_0^t d_{\infty}^2(X(s), X^{\epsilon}(s)) ds \\
 & \quad + 32L^2 \epsilon^{2\alpha} \mathbb{E} \int_0^t d_{\infty}^2(X(s), X^{\epsilon}(s)) ds. \tag{3.21}
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \mathbb{E} \sup_{u \in [0,t]} d_{\infty}^2(X(u), X^{\epsilon}(u)) \\
 & \leq (2L^2 + 32L^2 (T + \epsilon)^{2\alpha} + 32L^2 \epsilon^{2\alpha}) \int_0^t \mathbb{E} \sup_{u \in [0,s]} d_{\infty}^2(X(u), X^{\epsilon}(u)) ds \\
 & \quad + \left(64\epsilon^{2\alpha} C^2 + 64C^2 (T + \epsilon)^{2\alpha} \frac{\alpha + 1}{\alpha} \epsilon^{\alpha} \right) \int_0^t (1 + \mathbb{E} \|\| [X^{\epsilon}(s)]^0 \|\|^2) ds. \tag{3.22}
 \end{aligned}$$

By Gronwall’s lemma, $\mathbb{E} \sup_{u \in [0,t]} d_{\infty}^2(X(u), X^{\epsilon}(u)) \rightarrow 0$ as $\epsilon \rightarrow 0$, which completes the proof. □

3.1 Example in finance

The following crisp SFDE is typically used in financial modeling:

$$X(t) = X_0 + \int_0^t \mu X(s) ds + \int_0^t \sigma X(s) dB_H(s), \quad X_0 = X(0), \tag{3.23}$$

where the underlying stochastic process is the fBm. The long-range dependence and self-similarity property of fBm make this process suitable to describe the financial quantities. On the other hand, we can model the price dynamics by the equation which involves the uncertainties. This leads to modeling with fuzzy processes in equations. In the case of linear coefficients, we obtain an explicit solution of Eq. (3.1). Thus, consider the fractional FSDE which satisfy assumptions of Theorem 3.2 as follows:

$$X(t) = X_0 + \int_0^t \mu X(s) ds + \left\langle \int_0^t \frac{\sigma}{2} (X_l^1(s) + X_u^1(s)) dB_H(s) \right\rangle, \tag{3.24}$$

where $X : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{F}(\mathbb{R})$, B_H is an fBm, $X_l^1, X_u^1 : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that $[X(t)]^1 = [X_l^1(t), X_u^1(t)]$, $X_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{F}(\mathbb{R}))$, and $\mu, \sigma \in \mathbb{R}$. In order to find a closed explicit form of a solution to (3.24), for $\mu \geq 0$, we obtain the following system of equations:

$$\begin{cases} X_l^1(t) = X_l^1(0) + \int_0^t \mu X_l^1(s) ds + \int_0^t \frac{\sigma}{2} (X_l^1(s) + X_u^1(s)) dB_H(s), \\ X_u^1(t) = X_u^1(0) + \int_0^t \mu X_u^1(s) ds + \int_0^t \frac{\sigma}{2} (X_l^1(s) + X_u^1(s)) dB_H(s), \end{cases}$$

then

$$\begin{aligned} X_l^1(t) + X_u^1(t) &= X_l^1(0) + X_u^1(0) + \int_0^t \mu (X_l^1(s) + X_u^1(s)) ds \\ &\quad + \int_0^t \sigma (X_l^1(s) + X_u^1(s)) dB_H(s). \end{aligned} \tag{3.25}$$

By equation (2.3), the approximation form of Eq. (3.25) is

$$\begin{aligned} X_l^{\epsilon 1}(t) + X_u^{\epsilon 1}(t) &= X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0) + \int_0^t (\mu + \sigma \alpha \varphi^\epsilon(s)) (X_l^{\epsilon 1}(s) + X_u^{\epsilon 1}(s)) ds \\ &\quad + \int_0^t \sigma \epsilon^\alpha (X_l^{\epsilon 1}(s) + X_u^{\epsilon 1}(s)) dW(s). \end{aligned} \tag{3.26}$$

Hence, from the explicit solution of the crisp linear SDEs, one has a unique solution,

$$\begin{aligned} X_l^{\epsilon 1}(t) + X_u^{\epsilon 1}(t) &= (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \exp\left(\mu t + \sigma \alpha \int_0^t \varphi^\epsilon(s) ds - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} t + \sigma \epsilon^\alpha W(t)\right) \\ &= (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \exp\left(\mu t + \sigma B_H^\epsilon(t) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} t\right). \end{aligned} \tag{3.27}$$

Now, for every $\alpha \in [0, 1]$, we apply a similar procedure to obtain the following systems:

$$\begin{cases} X_l^{\epsilon \alpha}(t) = X_l^{\epsilon \alpha}(0) + \int_0^t \mu X_l^{\epsilon \alpha}(s) ds + \int_0^t \frac{\sigma}{2} (X_l^{\epsilon 1}(s) + X_u^{\epsilon 1}(s)) dB_H^\epsilon(s), \\ X_u^{\epsilon \alpha}(t) = X_u^{\epsilon \alpha}(0) + \int_0^t \mu X_u^{\epsilon \alpha}(s) ds + \int_0^t \frac{\sigma}{2} (X_l^{\epsilon 1}(s) + X_u^{\epsilon 1}(s)) dB_H^\epsilon(s). \end{cases}$$

For $\mu \geq 0$, we apply the solution (3.27) to get the following system:

$$X_l^{\epsilon \alpha}(t) = X_l^{\epsilon \alpha}(0) + \int_0^t \mu X_l^{\epsilon \alpha}(s) ds$$

$$\begin{aligned}
 & + \int_0^t \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \exp\left(\mu s + \sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s), \\
 X_u^{\epsilon\alpha}(t) & = X_u^{\epsilon\alpha}(0) + \int_0^t \mu X_u^{\epsilon\alpha}(s) ds \\
 & + \int_0^t \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \exp\left(\mu s + \sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s),
 \end{aligned}$$

or in terms of the Wiener process W , we have

$$\begin{aligned}
 X_l^{\epsilon\alpha}(t) & = X_l^{\epsilon\alpha}(0) + \int_0^t \mu X_l^{\epsilon\alpha}(s) ds \\
 & + \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \varphi^\epsilon(s) \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 & + \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s), \\
 X_u^{\epsilon\alpha}(t) & = X_u^{\epsilon\alpha}(0) + \int_0^t \mu X_u^{\epsilon\alpha}(s) ds \\
 & + \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \varphi^\epsilon(s) \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 & + \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s). \tag{3.28}
 \end{aligned}$$

Apply Theorem 8.5.2 in [1] to obtain the unique solution to (3.28) in a form:

$$\begin{aligned}
 X_l^{\epsilon\alpha}(t) & = e^{\mu t} X_l^{\epsilon\alpha}(0) \\
 & + e^{\mu t} \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \varphi^\epsilon(s) \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 & + e^{\mu t} \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s), \\
 X_u^{\epsilon\alpha}(t) & = e^{\mu t} X_u^{\epsilon\alpha}(0) \\
 & + e^{\mu t} \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 & \times \int_0^t \varphi^\epsilon(s) \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds
 \end{aligned}$$

$$\begin{aligned}
 &+ e^{\mu t} \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s).
 \end{aligned}$$

Then

$$\begin{aligned}
 X_l^{\epsilon\alpha}(t) &= e^{\mu t} \left[X_l^{\epsilon\alpha}(0) + \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \right. \\
 &\quad \left. \times \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s) \right], \\
 X_u^{\epsilon\alpha}(t) &= e^{\mu t} \left[X_u^{\epsilon\alpha}(0) + \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \right. \\
 &\quad \left. \times \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s) \right].
 \end{aligned}$$

Therefore, the approximation fuzzy solution for $\mu \geq 0$ is

$$\begin{aligned}
 X^\epsilon(t) &= e^{\mu t} X(0) \\
 &+ \left\langle \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) e^{\mu t} \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s) \right\rangle. \tag{3.29}
 \end{aligned}$$

For $\mu < 0$, we can show that

$$\begin{aligned}
 X_l^{\epsilon\alpha}(t) &= X_l^{\epsilon\alpha}(0) + \int_0^t \mu X_u^{\epsilon\alpha}(s) ds \\
 &+ \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \varphi^\epsilon(s) \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 &+ \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s), \\
 X_u^{\epsilon\alpha}(t) &= X_u^{\epsilon\alpha}(0) + \int_0^t \mu X_l^{\epsilon\alpha}(s) ds \\
 &+ \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \varphi^\epsilon(s) \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 &+ \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \exp\left(\mu s + \sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s). \tag{3.30}
 \end{aligned}$$

The unique solution to (3.30) is of the following matrix form:

$$X_l^{\epsilon\alpha}(t) = X_l^{\epsilon\alpha}(0) \cosh(\mu t) + X_u^{\epsilon\alpha}(0) \sinh(\mu t)$$

$$\begin{aligned}
 &+ e^{\mu t} \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \varphi^\epsilon(s) \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 &+ e^{\mu t} \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s), \\
 X_u^{\epsilon\alpha}(t) &= X_l^{\epsilon\alpha}(0) \sinh(\mu t) + X_u^{\epsilon\alpha}(0) \cosh(\mu t) \\
 &+ e^{\mu t} \alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \varphi^\epsilon(s) \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) ds \\
 &+ e^{\mu t} \epsilon^\alpha \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \\
 &\times \int_0^t \exp\left(\sigma \alpha \int_0^s \varphi^\epsilon(u) du - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s + \sigma \epsilon^\alpha W(s)\right) dW(s).
 \end{aligned}$$

Then

$$\begin{aligned}
 X_l^{\epsilon\alpha}(t) &= X_l^{\epsilon\alpha}(0) \cosh(\mu t) + X_u^{\epsilon\alpha}(0) \sinh(\mu t) \\
 &+ e^{\mu t} \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s), \\
 X_u^{\epsilon\alpha}(t) &= X_l^{\epsilon\alpha}(0) \sinh(\mu t) + X_u^{\epsilon\alpha}(0) \cosh(\mu t) \\
 &+ e^{\mu t} \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s).
 \end{aligned}$$

Consequently, the approximation fuzzy solution for $\mu < 0$ is

$$\begin{aligned}
 X^\epsilon(t) &= X(0) \cosh(\mu t) + X(0) \sinh(\mu t) \\
 &+ \left\{ \frac{\sigma}{2} (X_l^{\epsilon 1}(0) + X_u^{\epsilon 1}(0)) e^{\mu t} \int_0^t \exp\left(\sigma B_H^\epsilon(s) - \frac{1}{2} \sigma^2 \epsilon^{2\alpha} s\right) dB_H^\epsilon(s) \right\}. \tag{3.31}
 \end{aligned}$$

4 Conclusions

We introduced a fuzzy stochastic differential equation with respect to the Liouville form fBm, which has many properties like a long-range dependence. We applied an approximation approach to fractional stochastic integrals and the embedding of classical Itô integral into fuzzy set space. Using the Picard iteration method we studied the existence and uniqueness of the solutions. We proved that the approximate solution converges uniformly to the exact solution.

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Authors' contributions

The main idea of this paper was proposed by MTM. HJ prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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