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Novel finite point approach for solving time-fractional convection-dominated diffusion equations

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Abstract

In this paper, a stabilized numerical method with high accuracy is proposed to solve time-fractional singularly perturbed convection-diffusion equation with variable coefficients. The tailored finite point method (TFPM) is adopted to discrete equation in the spatial direction, while the time direction is discreted by the G-L approximation and the L1 approximation. It can effectively eliminate non-physical oscillation or excessive numerical dispersion caused by convection dominant. The stability of the scheme is verified by theoretical analysis. Finally, one-dimensional and two-dimensional numerical examples are presented to verify the efficiency of the method.

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1 Introduction

Fractional calculus is a generalization of traditional integer-order calculus to noninteger order (fractional order). The fractional integrals and derivatives are of nonlocal property because they are quasi-differential operators. So, they provide valuable tools for describing the memory and genetic properties of different materials and processes, as well as the dynamics of complex systems controlled by anomalous diffusion [1, 2]. The fractional calculus has a long history of rapid development and widespread application [3–5], and it is involved in nonlinear oscillating earthquakes [6], hydrodynamic models [7], continuous statistical mechanics [8], physical phenomena modeling [9], colored noise [10], solid mechanics [11], economics [12], anomalous transport [13], bioengineering [14] and many other aspects. Fractional partial differential equations (FPDEs) are characterized by noninteger-order derivatives, so they can effectively describe the memory and genetic properties of matter and play important roles in engineering, physics, fluid mechanics, mathematical biology, electrochemistry, and other science. As part of the fractional dynamic equation, fractional convection-diffusion equation is a powerful tool to simulate various anomalous diffusion phenomena. Time-fractional convection-diffusion equation

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can be utilized to simulate the time-related abnormal diffusion process. It is a generalization of the classical convection-diffusion equation by replacing the integer-order time derivative with a fractional-order time derivative, which is widely used in oil reservoir simulations, transport of mass and energy, dispersion of chemicals in reactors, etc. In recent decades, scholars in different fields have pointed out and confirmed that fractional model is more suitable than integer model to simulate the process of memory, genetic heterogeneity, and the abnormal power transmission. Therefore, it is of theoretical and practical significance to find the numerical solutions of FPDE. Several numerical methods have been introduced to solve FPDE, such as the finite difference method [15, 16], finite element method [17], variational iteration method [18, 19], operational method [20], Sinc–Legendre collocation method [21], generalized differential transform method [22], etc.

In the recent years, several numerical methods have been proposed for solving the time-fractional convection-diffusion equation (TFCDE). Saadmandi et al. [21] used the Sinc–Legendre collocation method for the solution of one-dimensional equation with homogeneous boundary conditions. Uddin and Haq [23] applied radial basis functions for the numerical solution of equation with constant coefficients, and Mohammad and Jafar [24] proposed a spectral method based on Gegenbauer collocation for solving this problem. In [25], the mixed generalized Jacobi and Chebyshev collocation methods were used to solve one-dimensional equations with variable coefficient. In addition, for a class of equations with variable coefficients, the third type of Chebyshev wavelet method was discussed in [26], and Cui [27] derived a compact difference scheme to solve this problem numerically. Furthermore, Wang et al. [28] proposed high-order exponential ADI format for solving two-dimensional TFCDE. Deng [29] proposed numerical algorithm for the time-fractional Fokker–Planck equation. Gorenflo [30] studied time-fractional diffusion equation using discrete random walk approach. There have been extensive works of high-order accurate schemes for Caputo derivative [31–35]. Kumar et al. [36–46] gave several methods for models with fractional derivative. Agarwal [47–50] proposed some methods for other kinds of equations with fractional derivative recently. Although the above numerical methods have solved TFCDE with various conditions to some extent, few of them could consider the effect of convection dominant, which means that the diffusion coefficient ε is extremely small. When the equation is convection dominant, the use of traditional numerical methods (central difference method or Galerkin method) will produce non-physical shock or excessive numerical diffusion (upwind difference method). Therefore, it is of significant importance in developing effective numerical method for the solution of convection dominant problem.

In this paper, we use the tailored finite point method (TFPM) to solve the time-fractional convection-dominant diffusion problem with variable coefficient, and we find that this algorithm is very effective. TFPM is based on the local exponential basis function, which was first proposed by Han et al. [51] for solving the Hemker problem numerically. In many cases, TFPM can preserve the important local properties of the problem. Subsequently, Han et al. used TFPM to solve the second-order singularly perturbed elliptic equation in [52]. In [53], TFPM was proposed for solving the parabolic problem. Han and Huang applied the method to solving the fourth-order singular perturbation elliptic equation in [54]. Huang et al. [55, 56] used TFPM to solve the surface layer problem and the first-order wave equation. Moreover, Tsai et al. [57] applied it to the numerical solution

of one-dimensional Burgers’ equation. Motivated by the advantages of TFPM, we adopt TFPM discrete in the spatial direction and use the G-L and L1 approximations of Caputo derivative discrete in the time direction in this paper to solve one-dimensional and two-dimensional time-fractional convection-dominated diffusion equations numerically. The research shows that TFPM is an efficient method to solve the convection-dominated problem.

The paper is organized as follows. In Sect. 2, we adopt TFPM for the steady problems, and then we use the G-L approximation and L1 approximation for the time-fractional derivative to give a highly efficient discrete scheme for one-dimensional time-fractional convection-dominated diffusion equation. In Sect. 3, we solve two-dimensional time-fractional convection-dominated diffusion equation numerically. In Sect. 4, we theoretically analyze the stability of the method in this paper. Finally, numerical examples of one dimension and two dimensions are given respectively in Sect. 5 to verify the high efficiency of the proposed algorithm. This paper closes with a short summary in Sect. 6.

2 One-dimensional time fractional convection-dominated diffusion equation

Let us consider the following time-fractional convection-diffusion equation:

$$\frac{\partial^\gamma u(x, t)}{\partial t^\gamma} - \varepsilon \frac{\partial^2 u(x, t)}{\partial x^2} + p(x, t) \frac{\partial u(x, t)}{\partial x} = f(x, t), \quad x \in I, t \in (0, T], \tag{2.1}$$

where ε is the diffusion coefficient, $0 < \varepsilon \ll 1$, $p(x, t) \neq 0$ is a continuous function; f is the source term; $I = (L_1, L_2)$ is the calculation interval, ∂I is the boundary; the fractional derivative $\frac{\partial^\gamma u(x, t)}{\partial t^\gamma}$ is the Caputo fractional derivative ${}_0^c D_t^\gamma u$ ($0 < \gamma < 1$) of the function $u(x, t)$. The derivative is ${}_0^c D_t^\gamma u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \cdot \frac{1}{(t-\tau)^\gamma} d\tau$.

Corresponding boundary conditions and initial conditions of equation (2.1) are:

$$u(L_1, t) = \mu_1(t), \quad u(L_2, t) = \mu_2(t), \tag{2.2}$$

$$u(x, 0) = v(x). \tag{2.3}$$

We assume that $\Omega = (L_1, L_2) \times (0, T]$, and we take a uniform partition, i.e., let $\tau = T/NT$ be the time step and $h = (L_2 - L_1)/(NX + 1)$ be the mesh size for some positive integers $NT, NX \in \mathbf{N}$. Take

$$x_j = L_1 + jh \quad (j = 0, 1, \dots, NX), \quad t^n = n\tau \quad (n = 0, 1, \dots, NT).$$

Then $\{P_j^n = (x_j, t^n), 0 \leq j \leq NX, 0 \leq n \leq NT\}$ is the set of mesh points.

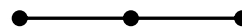
2.1 TFPM discretization for second-order derivative

We take the TFPM discrete scheme on the cell I_j (see Fig. 1)

$$u_{xx}|_{x=x_j} = \alpha_{j-1}u_{j-1} + \alpha_j u_j + \alpha_{j+1}u_{j+1}, \tag{2.4}$$

where $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ are satisfied with a relationship, as described below.

Figure 1 Stencil for constructing the second-order derivative



Assume that $u(x)$ can be linearly expressed by basis function on the stencil I_j , and let

$$V = \text{span}\{e^{-x/\varepsilon}, e^{x/\varepsilon}\}, \tag{2.5}$$

with $u_j = u(x_j, t)$, such that it holds for all $u \in V$ on I_j . Thus we obtain

$$u(x)|_{I_j} = c_{1j}e^{-(x-x_j)/\varepsilon} + c_{2j}e^{(x-x_j)/\varepsilon}. \tag{2.6}$$

Taking (2.6) in $\alpha_{j-1}u_{j-1} + \alpha_j u_j + \alpha_{j+1}u_{j+1} = 0$, we obtain

$$\alpha_{j-1}[c_{1j}e^{-h/\varepsilon} + c_{2j}e^{h/\varepsilon}] + \alpha_j[c_{1j} + c_{2j}] + \alpha_{j+1}[c_{1j}e^{h/\varepsilon} + c_{2j}e^{-h/\varepsilon}] = 0. \tag{2.7}$$

Then we obtain

$$\begin{cases} \alpha_{j-1}e^{-h/\varepsilon} + \alpha_j + \alpha_{j+1}e^{h/\varepsilon} = 0, \\ \alpha_{j-1}e^{h/\varepsilon} + \alpha_j + \alpha_{j+1}e^{-h/\varepsilon} = 0. \end{cases} \tag{2.8}$$

Solving the above linear system, we get

$$\alpha_{j-1} = \alpha_{j+1} = -\frac{\alpha_j}{e^{h/\varepsilon} + e^{-h/\varepsilon}}. \tag{2.9}$$

Then we obtain the TFPM scheme as follows:

$$u_{xx}|_{x=x_j} = -\frac{\alpha_j}{e^{h/\varepsilon} + e^{-h/\varepsilon}}u_{j-1} + \alpha_j u_j - \frac{\alpha_j}{e^{h/\varepsilon} + e^{-h/\varepsilon}}u_{j+1}, \tag{2.10}$$

where α_j satisfies the discrete maximum principle.

2.2 TFPM for one-dimensional time-fractional convection-dominated diffusion equation

2.2.1 TFPM based on G-L approximation

For equation (2.1), we apply the TFPM discrete in the spatial direction and adopt the G-L approximation discrete in the temporal direction. First, we give the definition of shifted G-L derivative as follows:

$${}_0^G D_t^\gamma u(t) = \tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} u_j^{n-k}, \tag{2.11}$$

where $w_k^{(\gamma)} = (-1)^j \binom{\gamma}{j}$, $j = 0, 1, 2, \dots$. The discretization scheme of TFPM based on the G-L approximation for equation (2.1) is

$$\begin{cases} \tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} u_j^{n-k} = \varepsilon \alpha_{j-1} u_{j-1}^n + \varepsilon \alpha_j u_j^n + \varepsilon \alpha_{j+1} u_{j+1}^n + p_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h} + f_j^n, \\ u_j^0 = v(x), \quad 0 \leq j \leq NX, \\ u_0^n = \mu_1(t), \quad u_{NX}^n = \mu_2(t), \quad 0 \leq n \leq NT. \end{cases} \tag{2.12}$$

Let $u^n = (u_1^n, u_2^n, \dots, u_{NX-1}^n)^T$, then (2.12) can be rewritten in the following matrix form:

$$\tau^{-\gamma} \sum_{k=0}^{\infty} w_k^{(\gamma)} u_j^{n-k} = Au^n + F^n, \tag{2.13}$$

where $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ are determined by (2.9), and

$$A = \text{diag} \left(\varepsilon \alpha_{j-1} - \frac{p_j^n}{2h}, \varepsilon \alpha_j, \varepsilon \alpha_{j+1} + \frac{p_j^n}{2h} \right)_{(NX-1) \times (NX-1)},$$

$$F^n = \left(f_1^n + \frac{u_0^n}{\varepsilon \alpha_{j-1}}, \dots, f_{NX}^n + \frac{u_{NX}^n}{\varepsilon \alpha_{j+1}} \right)^T. \tag{2.14}$$

2.2.2 TFPM based on L1 approximation

For equation (2.1), we apply the TFPM discrete in the spatial direction and adopt the L1 approximation discrete in the temporal direction.

According to the definition of Caputo fractional derivative,

$${}^c_0D_t^\gamma u(t)|_{t=t_n} = \frac{1}{\Gamma(1-\gamma)} \int_0^{t_n} \frac{u'(s)}{(t_n-s)^\gamma} ds = \frac{1}{\Gamma(1-\gamma)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{u'(s)}{(t_n-s)^\gamma} ds. \tag{2.15}$$

The linear interpolation of $u(s)$ on the $[t_{k-1}, t_k]$ interval is obtained.

$$L_{1,k}(s) = \frac{t_k - s}{\tau} u(t_{k-1}) + \frac{s - t_{k-1}}{\tau} u(t_k), \tag{2.16}$$

$$u(s) - L_{1,k}(s) = \frac{1}{2} u''(\xi_k)(s - t_{k-1})(s - t_k), \quad s \in [t_{k-1}, t_k], \tag{2.17}$$

where

$$\xi_k = \xi_k(s) \in (t_{k-1}, t_k). \tag{2.18}$$

$L_{1,k}u(s)$ approximate $u(s)$ to get

$$u(s) \approx \frac{t_k - s}{\tau} u(t_{k-1}) + \frac{s - t_{k-1}}{\tau} u(t_k) \tag{2.19}$$

$$\Rightarrow u'(s) \approx \frac{u(t_k) - u(t_{k-1})}{\tau}.$$

And then

$${}^c_0D_t^\gamma u(t)|_{t=t_n}$$

$$\approx \frac{1}{\Gamma(1-\gamma)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{u(t_k) - u(t_{k-1})}{\tau} \cdot \frac{1}{(t_n-s)^\gamma} ds$$

$$= \frac{1}{\Gamma(1-\gamma)} \sum_{k=1}^n \frac{u(t_k) - u(t_{k-1})}{\tau} \cdot \int_{t_{k-1}}^{t_k} \frac{1}{(t_n-s)^\gamma} ds$$

$$= \frac{1}{\Gamma(1-\gamma)} \sum_{k=1}^n \frac{u(t_k) - u(t_{k-1})}{\tau} \cdot \frac{1}{1-\gamma}$$

$$\cdot [(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}] \tag{2.20}$$

$$\begin{aligned}
 &= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=1}^n a_{n-k}^{(\gamma)} [u(t_k) - u(t_{k-1})] \\
 &= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[a_0^{(\gamma)} u(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u(t_k) - a_{n-1}^{(\gamma)} u(t_0) \right].
 \end{aligned}$$

Then the discretization scheme of TFFPM based on the L1 approximation for equation (2.1) is

$$\begin{cases} \frac{\tau^\gamma}{\Gamma(2-\gamma)} [a_0^{(\gamma)} u_j^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u_j^k - a_{n-1}^{(\gamma)} u_j^0] \\ \quad = \varepsilon \alpha_{j-1} u_{j-1}^n + \varepsilon \alpha_j u_j^n + \varepsilon \alpha_{j+1} u_{j+1}^n + p_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h} + f_j^n, \\ u_j^0 = v(x), \quad 0 \leq j \leq NX, \\ u_0^n = \mu_1(t), \quad u_{NX}^n = \mu_2(t), \quad 0 \leq n \leq NT. \end{cases} \tag{2.21}$$

Rewrite (2.21) as the following matrix form:

$$\frac{\tau^\gamma}{\Gamma(2-\gamma)} \left[a_0^{(\gamma)} u_j^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u_j^k - a_{n-1}^{(\gamma)} u_j^0 \right] = Au^n + F^n, \tag{2.22}$$

where A, F^n are defined in (2.22) and $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ are determined by (2.9).

3 Two-dimensional time-fractional convection-dominated diffusion equation

Let us consider the following time-fractional convection-diffusion equation:

$$\begin{cases} {}_0^c D_t^\gamma u(x, y, t) - \varepsilon^2 \Delta u + p(x, y, t) u_x + q(x, y, t) u_y \\ \quad = \hat{f}(x, y, t), \quad (x, y) \in \Omega, t > 0, \\ u(x, y, 0) = \mu_0(t), \quad (x, y) \in \Omega, \\ u(x, y, t) = \mu_1(t), \quad (x, y) \in \partial\Omega, t > 0, \end{cases} \tag{3.1}$$

where ε is the diffusion coefficient. Ω is a bounded area, $\partial\Omega$ is a smooth boundary; $p(x, y, t) \neq 0, q(x, y, t) \neq 0$ are continuous functions; \hat{f} is the source term.

Rewrite the above equation as follows:

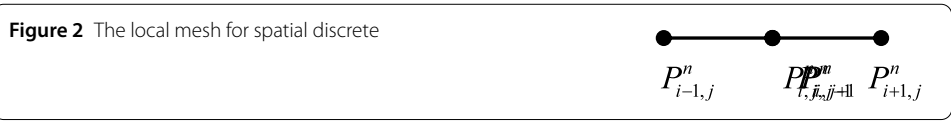
$${}_0^c D_t^\gamma u(x, y, t) - \varepsilon^2 \Delta u + p(x, y, t) u_x + q(x, y, t) u_y = \bar{f} + f = \hat{f}(x, y, t), \tag{3.2}$$

where $\bar{f}(x, y, t)$ is the source term for the time variable and $f(x, y, t)$ is the source term for the spatial variable.

We assume that $\Omega = [x_L, x_R] \times [y_L, y_R], t \in (0, T]$, and we take a uniform partition, i.e., let $h_x = h_y = h$ be the mesh size and $\tau = \Delta t$ be the time step, and

$$x_i = ih_x, \quad y_j = jh_y, \quad t_n = n\tau, \quad 0 \leq i \leq NX, 0 \leq j \leq NY, 0 \leq n \leq NT. \tag{3.3}$$

Then $\{P_{i,j}^k = (x_i, y_j, t_n), 0 \leq i \leq NX, 0 \leq j \leq NY, 0 \leq n \leq NT\}$ is the set of mesh points.



3.1 TFPM discretization in spatial direction

For equation (3.2), let

$$p_{i,j}^n = p(x_i, y_j, t_n), \quad q_{i,j}^n = q(x_i, y_j, t_n), \quad f_{i,j}^n = f(x_i, y_j, t_n). \tag{3.4}$$

Then the equation corresponding to the spatial direction of equation (3.2) is

$$-\varepsilon^2 \Delta u + p_{i,j}^n u_x + q_{i,j}^n u_y = f_{i,j}^n. \tag{3.5}$$

We now construct our tailored finite point scheme for (3.5) on cell Ω_0 (see Fig. 2).

Let

$$u(x, y, t) = v(x, y, t) e^{\frac{p_{i,j}^n x + q_{i,j}^n y}{2\varepsilon^2}} + \frac{p_{i,j}^n x + q_{i,j}^n y}{p_{i,j}^{n2} + q_{i,j}^{n2}} f_{i,j}^n. \tag{3.6}$$

Substituting the above formula to (3.5), we can obtain the following equation:

$$-\varepsilon^2 \Delta u + d_{i,j}^{n2} u = 0, \tag{3.7}$$

where $d_{i,j}^{n2} = \frac{p_{i,j}^{n2} + q_{i,j}^{n2}}{4\varepsilon^2}$.

Let $\mu_{i,j}^n = \frac{d_{i,j}^n}{\varepsilon}$, and let the base function space be as follows:

$$H_4 = \{ v(x, y, t) | v = c_1 e^{-\mu_{i,j}^n x} + c_2 e^{\mu_{i,j}^n x} + c_3 e^{-\mu_{i,j}^n y} + c_4 e^{\mu_{i,j}^n y}, \forall c_i \in R \}. \tag{3.8}$$

We take the scheme as follows (see Fig. 2):

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_0 V_0 = 0, \tag{3.9}$$

where $V_0 = v(x_i, y_j, t_n)$, $V_1 = v(x_{i+1}, y_j, t_n)$, $V_2 = v(x_i, y_{j+1}, t_n)$, $V_3 = v(x_{i-1}, y_j, t_n)$, $V_4 = v(x_i, y_{j-1}, t_n)$. Due to $\alpha_k \in R$ ($k = 0, 1, 2, 3, 4$), so that it holds for all $v \in H_4$. Thus we obtain

$$\alpha_1 e^{-\mu_{i,j}^n h} + \alpha_2 + \alpha_3 e^{\mu_{i,j}^n h} + \alpha_4 + \alpha_0 = 0, \tag{3.10}$$

$$\alpha_1 e^{\mu_{i,j}^n h} + \alpha_2 + \alpha_3 e^{-\mu_{i,j}^n h} + \alpha_4 + \alpha_0 = 0, \tag{3.11}$$

$$\alpha_1 + \alpha_2 e^{-\mu_{i,j}^n h} + \alpha_3 + \alpha_4 e^{\mu_{i,j}^n h} + \alpha_0 = 0, \tag{3.12}$$

$$\alpha_1 + \alpha_2 e^{\mu_{i,j}^n h} + \alpha_3 + \alpha_4 e^{-\mu_{i,j}^n h} + \alpha_0 = 0. \tag{3.13}$$

Solving the above linear system (3.10)–(3.13), we get

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{-\alpha_0}{e^{\mu_{i,j}^n h} + e^{-\mu_{i,j}^n h} + 2} = \frac{-\alpha_0}{4 \cosh^2(\frac{\mu_{i,j}^n h}{2})}. \tag{3.14}$$

Let

$$\alpha_0 = \frac{e^{\mu_{ij}^n h} + e^{-\mu_{ij}^n h} + 2}{e^{\mu_{ij}^n h} + e^{-\mu_{ij}^n h} - 2} = \frac{\cosh^2\left(\frac{\mu_{ij}^n h}{2}\right)}{\sinh^2\left(\frac{\mu_{ij}^n h}{2}\right)}. \tag{3.15}$$

We can obtain

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{4 \sinh^2\left(\frac{\mu_{ij}^n h}{2}\right)}. \tag{3.16}$$

Finally, we obtain the discrete scheme of equation (3.5) as follows:

$$\begin{aligned} u_{ij}^n &= \frac{e^{-\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i+1,j}^n + e^{-\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j+1}^n + e^{\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i-1,j}^n + e^{\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j-1}^n}{4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right)} \\ &= \frac{f_{ij}^n h}{8 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right)} \left(\frac{e^{\frac{p_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{p_{ij}^n h}{2\varepsilon^2}}}{p_{ij}^{n2}} + \frac{e^{\frac{q_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{q_{ij}^n h}{2\varepsilon^2}}}{q_{ij}^{n2}} \right). \end{aligned} \tag{3.17}$$

3.2 TFPM for two-dimensional time-fractional convection-dominated diffusion equation

For equation (3.1), the spatial direction is discreted by TFPM, and the time direction is discreted by the G-L approximation of the Caputo fractional derivative and the L1 approximation, respectively.

3.2.1 TFPM based on G-L approximation

Combining equations (3.2) and (3.17), we can get the following GL-TFPM discrete scheme:

$$\begin{aligned} \tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} u_{ij}^{n-k} + u_{ij}^n &= \frac{e^{-\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i+1,j}^n + e^{-\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j+1}^n + e^{\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i-1,j}^n + e^{\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j-1}^n}{4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right)} \\ &= \frac{f_{ij}^n h}{8 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right)} \left(\frac{e^{\frac{p_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{p_{ij}^n h}{2\varepsilon^2}}}{p_{ij}^{n2}} + \frac{e^{\frac{q_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{q_{ij}^n h}{2\varepsilon^2}}}{q_{ij}^{n2}} \right) + \bar{f}_{ij}^n = \hat{f}_{ij}^n, \end{aligned} \tag{3.18}$$

where $\mu_{ij}^n = \frac{d_{ij}^n}{\varepsilon}$, $d_{ij}^{n2} = \frac{p_{ij}^{n2} + q_{ij}^{n2}}{4\varepsilon^2}$.

3.2.2 TFPM based on L1 approximation

Combining equations (3.2) and (3.17), we can get the following L1-TFPM discrete scheme:

$$\begin{aligned} & \frac{\tau^\gamma}{\Gamma(2-\gamma)} \left[a_0^{(\gamma)} u_{ij}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u_{ij}^k - a_{n-1}^{(\gamma)} u_{ij}^0 \right] \\ & + u_{ij}^n - \frac{e^{-\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i+1,j}^n + e^{-\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j+1}^n + e^{\frac{p_{ij}^n h}{2\varepsilon^2}} u_{i-1,j}^n + e^{\frac{q_{ij}^n h}{2\varepsilon^2}} u_{i,j-1}^n}{4 \cosh^2\left(\frac{u_{ij}^n h}{2}\right)} \\ & = \frac{f_{ij}^n h}{8 \cosh^2\left(\frac{u_{ij}^n h}{2}\right)} \left(\frac{e^{\frac{p_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{p_{ij}^n h}{2\varepsilon^2}}}{p_{ij}^{n2}} + \frac{e^{\frac{q_{ij}^n h}{2\varepsilon^2}} - e^{-\frac{q_{ij}^n h}{2\varepsilon^2}}}{q_{ij}^{n2}} \right) + \bar{f}_{ij}^n = \hat{f}_{ij}^n, \end{aligned} \tag{3.19}$$

where $\mu_{ij}^n = \frac{d_{ij}^n}{\varepsilon}$, $d_{ij}^{n2} = \frac{p_{ij}^{n2} + q_{ij}^{n2}}{4\varepsilon^2}$.

4 Stability analysis

4.1 Stability analysis for one-dimensional time-fractional convection-dominated diffusion equation

4.1.1 Stability analysis of TFPM based on G-L approximation

Theorem 1 Assume that $\{v_j^n | 0 \leq j \leq NX, 0 \leq n \leq NT\}$ is the solution of the GL-TFPM discrete scheme of (2.12) as $v_0^n = 0, v_{NX}^n = 0, 0 \leq n \leq NT$. Then we have

$$\|v^n\|_\infty \leq \frac{5}{1-\gamma} \|v^0\|_\infty + \frac{5}{(1-\gamma)2^\gamma} t_n^\gamma \max_{1 \leq m \leq n} \|f^m\|_\infty, \quad 1 \leq n \leq NT, \tag{4.1}$$

where $\|f^m\|_\infty = \max_{1 \leq j \leq NX-1} |f_j^m|$.

Proof Rewrite equation (4.1) as follows:

$$\begin{aligned} (1-\alpha_j)v_j^n &= \sum_{k=1}^n (-w_k^{(\gamma)})v_j^{n-k} + \left(\alpha_{j-1} - \frac{1}{2h}\right)v_{j-1}^n + \left(\alpha_{j+1} + \frac{1}{2h}\right)v_{j+1}^n + \tau^\gamma f_j^n, \\ 1 \leq j &\leq NX-1, \quad 1 \leq n \leq NT. \end{aligned} \tag{4.2}$$

Assume that $\|v^n\|_\infty = |v_{j_n}^n|$, where $j_n \in \{1, 2, \dots, NX-1\}$. Let $j = j_n$ in (4.2), and take the absolute value in the above formula. Then the triangular inequality is used. We have

$$(1-\alpha_j)\|v^n\|_\infty \leq \sum_{k=1}^n (-w_k^{(\gamma)})\|v^{n-k}\|_\infty + (\alpha_{j-1}\|v^n\|_\infty + \alpha_{j+1}\|v^n\|_\infty) + \tau^\gamma \|f^n\|_\infty.$$

Due to $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ being defined by (2.9), and applying the triangular inequality, we have

$$\|v^n\|_\infty \leq \sum_{k=1}^n (-w_k^{(\gamma)})\|v^{n-k}\|_\infty + \tau^\gamma \|f^n\|_\infty, \quad 1 \leq n \leq NT. \tag{4.3}$$

Starting from formula (4.3), the mathematical induction method is used to prove (4.1). Let

$$A_n = \frac{5}{1-\gamma} \|v^0\|_\infty + \frac{5}{(1-\gamma)2^\gamma} n^\gamma \tau^\gamma \max_{1 \leq m \leq n} \|f^m\|_\infty, \quad 1 \leq n \leq NT.$$

From (4.3), when $n = 1$, we have

$$\|v^1\|_\infty \leq (-w_1^{(\gamma)})\|v^0\|_\infty + \tau^\gamma \|f^1\|_\infty = \gamma \|v^0\|_\infty + \tau^\gamma \|f^1\|_\infty \leq A_1.$$

That is, (4.1) is set up for $k = 1$. Assume that (4.1) is also set up for $k = 1, 2, \dots, n - 1$ ($n \geq 2$), then it can be obtained by (4.3)

$$\begin{aligned} \|v^n\|_\infty &\leq \sum_{k=1}^{n-1} (-w_k^{(\gamma)})\|v^{n-k}\|_\infty + (-w_n^{(\gamma)})\|v^0\|_\infty + \tau^\gamma \|f^n\|_\infty \\ &\leq \sum_{k=1}^{n-1} (-w_k^{(\gamma)})A_{n-k} + \gamma \left(\frac{2}{n+1}\right)^{\gamma+1} \|v^0\|_\infty + \tau^\gamma \|f^n\|_\infty \\ &\leq \sum_{k=1}^{n-1} (-w_k^{(\gamma)})A_n + \gamma \left(\frac{2}{n}\right)^\gamma \|v^0\|_\infty + \tau^\gamma \|f^n\|_\infty \\ &\leq \left[\sum_{k=1}^\infty (-w_k^{(\gamma)}) - \sum_{k=n}^\infty (-w_k^{(\gamma)}) \right] A_n + \gamma \left(\frac{2}{n}\right)^\gamma \|v^0\|_\infty + \tau^\gamma \|f^n\|_\infty \\ &\leq \left[1 - \frac{1-\gamma}{5} \left(\frac{2}{n}\right)^\gamma \right] A_n + \gamma \left(\frac{2}{n}\right)^\gamma \|v^0\|_\infty + \tau^\gamma \|f^n\|_\infty \\ &\leq A_n - \frac{1-\gamma}{5} \left(\frac{2}{n}\right)^\gamma \left[A_n - \frac{5\gamma}{1-\gamma} \|v^0\|_\infty - \frac{5}{1-\gamma} \left(\frac{n}{2}\right)^\gamma \tau^\gamma \|f^n\|_\infty \right] \\ &\leq A_n. \end{aligned}$$

The above proof process is applied to $(\frac{2}{n+1})^{\gamma+1} < (\frac{2}{n})^{\gamma+1} \leq (\frac{2}{n})^\gamma$, ($n \geq 2$). Therefore, it can be concluded that (4.1) is also established for $k = n$. □

4.1.2 Stability analysis of TFPM based on L1 approximation

Theorem 2 Assume that $\{v_j^n | 0 \leq j \leq NX, 0 \leq n \leq NT\}$ is the solution of L1-TFPM scheme (2.21) as $v^0 = v(x_j) = 0, 1 \leq j \leq NX - 1, v_0^n = 0, v_{NX}^n = 0, 0 \leq n \leq NT$.

Then we have

$$\|v^n\|_\infty \leq \|v^0\|_\infty + t_n^\gamma \Gamma(1 - \gamma) \max_{1 \leq m \leq n} \|f^m\|_\infty, \quad 1 \leq n \leq NT, \tag{4.4}$$

where $\|f^m\|_\infty = \max_{1 \leq j \leq NX-1} |f_j^m|$.

Proof Rewrite equation (2.21) as follows:

$$\begin{aligned} a_0^{(\gamma)} v_j^n &= \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) v_j^k + a_{n-1}^{(\gamma)} v_j^0 + \tau^\gamma \Gamma(2 - \gamma) (\alpha_{j-1} v_{j-1}^n + \alpha_j v_j^n + \alpha_{j+1} v_{j+1}^n) \\ &\quad + \tau^\gamma \Gamma(2 - \gamma) p_j^n \left(\frac{1}{2h} v_{j+1}^n - \frac{1}{2h} v_{j-1}^n \right) \\ &\quad + \tau^\gamma \Gamma(2 - \gamma) f_j^n, \quad 1 \leq j \leq NX - 1, 1 \leq n \leq NT. \end{aligned}$$

That is,

$$\begin{aligned}
 [1 - \tau^\gamma \Gamma(2 - \gamma)\alpha_j]v_j^n &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})v_j^k + a_{n-1}^{(\gamma)}v_j^0 \\
 &\quad + \tau^\gamma \Gamma(2 - \gamma)\left(\alpha_{j-1} - \frac{1}{2h}p_j^n\right)v_{j-1}^n \\
 &\quad + \tau^\gamma \Gamma(2 - \gamma)\left(\alpha_{j+1} + \frac{1}{2h}p_j^n\right)v_{j+1}^n \\
 &\quad + \tau^\gamma \Gamma(2 - \gamma)f_j^n, \quad 1 \leq j \leq NX - 1, 1 \leq n \leq NT.
 \end{aligned}$$

Assume that $\|v^n\|_\infty = |v_{j_n}^n|$, where $j_n \in \{1, 2, \dots, NX - 1\}$. Let $j = j_n$, and take the absolute value in the above formula. Then the triangular inequality is used. We have

$$\begin{aligned}
 &\| [1 - \tau^\gamma \Gamma(2 - \gamma)\alpha_j]v^n \|_\infty \\
 &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\|v^k\|_\infty + a_{n-1}^{(\gamma)}\|v^0\|_\infty \\
 &\quad + 2\tau^\gamma \Gamma(2 - \gamma)\alpha_{j+1}\|v\|_\infty \\
 &\quad + \tau^\gamma \Gamma(2 - \gamma)f_j^n, \quad 1 \leq j \leq NX - 1, 1 \leq n \leq NT.
 \end{aligned}$$

Due to $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$ being defined by (2.9), and applying the triangular inequality, we obtain

$$\begin{aligned}
 \|v^n\|_\infty &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\|v^k\|_\infty \\
 &\quad + a_{n-1}^{(\gamma)}\left[\|v^0\|_\infty + \frac{\tau^\gamma \Gamma(2 - \gamma)}{a_{n-1}^{(\gamma)}}\|f^n\|_\infty\right], \quad 1 \leq n \leq NT.
 \end{aligned}$$

Notice that

$$\frac{\tau^\gamma \Gamma(2 - \gamma)}{a_{n-1}^{(\gamma)}} \leq \frac{\tau^\gamma \Gamma(2 - \gamma)}{(1 - \gamma)n^{-\gamma}} = (n\tau)^\gamma \Gamma(1 - \gamma), \tag{4.5}$$

then we have

$$\begin{aligned}
 \|v^n\|_\infty &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\|v^k\|_\infty \\
 &\quad + a_{n-1}^{(\gamma)}(\|v^0\|_\infty + t_n^\gamma \Gamma(1 - \gamma)\|f^n\|_\infty), \quad 1 \leq n \leq NT.
 \end{aligned} \tag{4.6}$$

We adopt the mathematical induction with equation (4.9) to prove the conclusion.

When $n = 1$, we obtain

$$a_0^{(\gamma)}|v^1| \leq |v^0| + \tau^\gamma \Gamma(1 - \gamma)|f(t_1)|.$$

It is easy to know that the conclusion is established for $n = 1$. Assume that it is established for $k = 1, 2, \dots, n - 1$, then

$$\begin{aligned} \|v^n\|_\infty &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) \left[\|v^0\|_\infty + (k\tau)^\gamma \Gamma(1-\gamma) \max_{1 \leq m \leq k} \|f^m\|_\infty \right] \\ &\quad + a_{n-1}^{(\gamma)} \left[\|v^0\|_\infty + t_n^\gamma \Gamma(1-\gamma) \|f^n\|_\infty \right] \\ &\leq \left\{ \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) + a_{n-1}^{(\gamma)} \right\} \left[\|v^0\|_\infty + (n\tau)^\gamma \Gamma(1-\gamma) \max_{1 \leq m \leq n} \|f^m\|_\infty \right], \\ &= \|v^0\|_\infty + t_n^\gamma \Gamma(1-\gamma) \max_{1 \leq m \leq n} \|f^m\|_\infty. \end{aligned}$$

Therefore

$$\|v^n\|_\infty \leq \|v^0\|_\infty + t_n^\gamma \Gamma(1-\gamma) \max_{1 \leq m \leq n} \|f^m\|_\infty, \quad 1 \leq n \leq NT.$$

That is, the conclusion is established for $k = n$. □

4.2 Stability analysis for two-dimensional time-fractional convection-dominated diffusion equation

Let $\omega = \{(i, j) | (x_i, y_j) \in \Omega\}$, $\partial\omega = \{(i, j) | (x_i, y_j) \in \partial\Omega\}$, $\bar{\omega} = \omega \cup \partial\omega$, and define the mesh function as follows:

$$V_h = \{u | u = \{u_{ij} | (i, j) \in \bar{\omega}\}\}, \quad u \text{ is the mesh function on } \Omega,$$

$$V_h^\circ = \{u | u \in V_h; \text{ if } (i, j) \in \partial\omega, \text{ then } u_{ij} = 0\}.$$

For mesh function $v \in V_h$, let $\kappa = 1/2 \cosh(\frac{\mu_{ij}^n h}{2})$, and we introduce the following notation:

$$\delta_x v_{i-\frac{1}{2}, j} = \kappa \left(\cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) v_{ij}^n - e^{\frac{p_{ij}^n h}{2\varepsilon^2}} v_{i-1, j}^n \right),$$

$$\delta_x v_{i+\frac{1}{2}, j} = \kappa \left(e^{-\frac{p_{ij}^n h}{2\varepsilon^2}} v_{i+1, j}^n - \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) v_{ij}^n \right),$$

$$\delta_y v_{i, j-\frac{1}{2}} = \kappa \left(\cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) v_{ij}^n - e^{\frac{q_{ij}^n h}{2\varepsilon^2}} v_{i, j-1}^n \right),$$

$$\delta_y v_{i, j+\frac{1}{2}} = \kappa \left(e^{-\frac{q_{ij}^n h}{2\varepsilon^2}} v_{i, j+1}^n - \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) v_{ij}^n \right),$$

$$\delta_x^2 v_{i, j} = \kappa (\delta_x v_{i+\frac{1}{2}, j} - \delta_x v_{i-\frac{1}{2}, j}),$$

$$\delta_y^2 v_{i, j} = \kappa (\delta_y v_{i, j+\frac{1}{2}} - \delta_y v_{i, j-\frac{1}{2}}),$$

$$\delta_x \delta_y v_{i-\frac{1}{2}, j-\frac{1}{2}} = \kappa (\delta_y v_{i, j-\frac{1}{2}} - \delta_y v_{i-1, j-\frac{1}{2}}),$$

$$\delta_x^2 \delta_y^2 v_{i, j} = \kappa^2 (\delta_y^2 v_{i-1, j} - 2\delta_y^2 v_{i, j} + \delta_y^2 v_{i+1, j}).$$

For the mesh function $v \in V_h^\circ$, define that

$$\begin{aligned}
 (u, v) &= 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} u_{ij} v_{ij}, \quad \|u\| = \sqrt{(u, u)}, \\
 (u, v)_{1,x} &= 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (\delta_x u_{i-\frac{1}{2},j}) \delta_x v_{i-\frac{1}{2},j}, \quad \|\delta_x u\| = \sqrt{(u, u)_{1,x}}, \\
 (u, v)_{1,y} &= 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (\delta_y u_{i,j-\frac{1}{2}}) \delta_y v_{i,j-\frac{1}{2}}, \quad \|\delta_y u\| = \sqrt{(u, u)_{1,y}}, \\
 (u, v)_{x,y} &= 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (\delta_x \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}) (\delta_x \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}), \\
 \|\delta_x \delta_y u\| &= \sqrt{(u, u)_{x,y}}, \\
 \|\nabla_h u\| &= \sqrt{\|\delta_x u\|^2 + \|\delta_y u\|^2}, \quad \|u\|_\infty = \max_{\substack{1 \leq i \leq NX \\ 1 \leq j \leq NY}} |u_{ij}|.
 \end{aligned}$$

It is easy to verify, for any mesh function $u, v \in V_h^\circ$, we have

$$(-\delta_x^2 u, v) := 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (-\delta_x^2 u_{ij}) v_{ij} = (u, v)_{1,x}, \tag{4.7}$$

$$(-\delta_y^2 u, v) := 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (-\delta_y^2 u_{ij}) v_{ij} = (u, v)_{1,y}, \tag{4.8}$$

$$(\delta_x^2 \delta_y^2 u, v) := 4 \cosh^2\left(\frac{\mu_{ij}^n h}{2}\right) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (\delta_x^2 \delta_y^2 u_{ij}) v_{ij} = (u, v)_{x,y}. \tag{4.9}$$

Lemma 1 ([58]) *For any $u \in V_h^\circ$, let $L_x = x_R - x_L, L_y = y_R - y_L$, then*

$$\|u\|^2 \leq \frac{1}{\frac{6}{L_x^2} + \frac{6}{L_y^2}} \|\nabla_h u\|^2.$$

Here, $\|\cdot\|$ indicates the L_2 norm.

4.2.1 Stability analysis of TFPM based on G-L approximation

Theorem 3 *Assume that $\{v_{ij}^n(i, j) \in \bar{\omega}, 0 \leq n \leq NT\}$ is the solution of GL-TFPM scheme (3.18) or as below:*

$$\tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} v_{ij}^{n-k} = \delta_x^2 v_{ij}^n + \delta_y^2 v_{ij}^n + \hat{f}_{ij}^n, \quad (i, j) \in \omega, 1 \leq n \leq NT, \tag{4.10}$$

$$v_{ij}^0 = \mu_0(x_i, y_j), \quad (i, j) \in \omega, \tag{4.11}$$

$$v_{ij}^n = 0, (i, j) \in \partial\omega, \quad 0 \leq n \leq NT. \tag{4.12}$$

Then we have

$$\begin{aligned} \|v^n\|^2 &\leq \frac{5}{1-\gamma} \|v^0\|^2 \\ &\quad + \frac{L_x^2 L_y^2}{12(L_x^2 + L_y^2)} \frac{5}{(1-\gamma)2^\gamma} t_n^\gamma \max_{1 \leq m \leq n} \|f^m\|^2, \quad 1 \leq n \leq NT, \end{aligned} \tag{4.13}$$

where $\|f^m\|^2 = 4 \cosh^2(\frac{\mu_y^m h}{2}) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (f_{ij}^m)^2$.

Proof We use the inner product simultaneously on both sides of equation (4.10). Noticing (4.12) and applying (4.7)–(4.9), we can obtain

$$\begin{aligned} \tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} (v^{n-k}, v^n) &= -(v^n, v^n)_{1,x} - (v^n, v^n)_{1,y} + (\hat{f}^n, v^n) \\ &= -\|\nabla_h v^n\|^2 + (\hat{f}^n, v^n), \quad 1 \leq n \leq NT. \end{aligned} \tag{4.14}$$

By the Cauchy–Schwarz inequality, and noting Lemma 1, we have

$$\begin{aligned} (\hat{f}^n, v^n) &\leq \|\hat{f}^n\| \|v^n\| \\ &\leq 6 \left(\frac{1}{L_x^2} + \frac{1}{L_y^2} \right) \|v^n\|^2 + \frac{1}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2 \\ &\leq \|\nabla_h v^n\|^2 + \frac{1}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2, \quad 1 \leq n \leq NT. \end{aligned} \tag{4.15}$$

Combining (4.15) and (4.14), we can get

$$\tau^{-\gamma} \sum_{k=0}^n w_k^{(\gamma)} (v^{n-k}, v^n) \leq \frac{1}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2, \quad 1 \leq n \leq NT.$$

Reorganizing the above formula and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|v^n\|^2 &\leq \sum_{k=1}^n (-w_k^{(\gamma)}) (v^{n-k}, v^n) + \frac{L_x^2 L_y^2}{24(L_x^2 + L_y^2)} \tau^\gamma \|\hat{f}^n\|^2 \\ &\leq \sum_{k=1}^n (-w_k^{(\gamma)}) \left[\frac{1}{2} (\|v^{n-k}\|^2 + \|v^n\|^2) \right] \\ &\quad + \frac{L_x^2 L_y^2}{24(L_x^2 + L_y^2)} \tau^\gamma \|\hat{f}^n\|^2, \quad 1 \leq n \leq NT. \end{aligned}$$

Notice that $\sum_{k=1}^n (-w_k^{(\gamma)}) \leq w_0^{(\gamma)} = 1$, and multiply by 2 on both sides of the above formula, then

$$\|v^n\|^2 \leq \sum_{k=1}^n (-w_k^{(\gamma)}) \|v^{n-k}\|^2 + \frac{L_x^2 L_y^2}{12(L_x^2 + L_y^2)} \tau^\gamma \|\hat{f}^n\|^2, \quad 1 \leq n \leq NT. \tag{4.16}$$

From inequality (4.16), it is easy to verify (4.13) using the mathematical induction method (similar to the induction process in Theorem 1, omitted here). □

4.2.2 Stability analysis of TFFPM based on L1 approximation

Theorem 4 Assume that $\{v_{ij}^n | (i, j) \in \bar{\omega}, 0 \leq n \leq NT\}$ is the solution of L1-TFFPM discrete scheme (3.19) or as below:

$$\begin{aligned} & \frac{\tau^\gamma}{\Gamma(2-\gamma)} \left[a_0^{(\gamma)} u_{i,j}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u_{i,j}^k - a_{n-1}^{(\gamma)} u_{i,j}^0 \right] \\ & = \delta_x^2 v_{ij}^n + \delta_y^2 v_{ij}^n + \hat{f}_{ij}^n, \quad (i, j) \in \omega, 1 \leq n \leq NT, \end{aligned} \tag{4.17}$$

where

$$v_{ij}^0 = \mu_0(x_i, y_j), \quad (i, j) \in \omega, \tag{4.18}$$

and

$$v_{ij}^n = 0, \quad (i, j) \in \partial\omega, 0 \leq n \leq NT. \tag{4.19}$$

Then we have

$$\|v^n\|^2 \leq \|v^0\|^2 + \frac{L_x^2 L_y^2}{12(L_x^2 + L_y^2)} \Gamma(1-\gamma) t_n^\gamma \max_{1 \leq m \leq n} \|f^m\|^2, \quad 1 \leq n \leq NT, \tag{4.20}$$

where $\|f^m\|^2 = 4 \cosh^2(\frac{\mu_{ij}^n h}{2}) \sum_{i=1}^{NX-1} \sum_{j=1}^{NY-1} (f_{ij}^m)^2$.

Proof Taking both sides of equation (4.17) with v^n for the inner product (\cdot, \cdot) simultaneously, we have

$$\begin{aligned} & \frac{\tau^\gamma}{\Gamma(2-\gamma)} \left(\left[a_0^{(\gamma)} v^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) v^k - a_{n-1}^{(\gamma)} v^0 \right], v^n \right) \\ & = (\delta_x^2 v^n, v^n) + (\delta_y^2 v^n, v^n) + (\hat{f}^n, v^n), \quad 1 \leq n \leq NT. \end{aligned} \tag{4.21}$$

Noting (4.19) and applying (4.7)–(4.9), we can get

$$(\delta_x^2 v^n, v^n) + (\delta_y^2 v^n, v^n) = -(v^n, v^n)_{1,x} - (v^n, v^n)_{1,y} = -\|\nabla_h v^n\|^2. \tag{4.22}$$

By the Cauchy–Schwarz inequality and noting Lemma 1, there are

$$\begin{aligned} (\hat{f}^n, v^n) & \leq \|\hat{f}^n\| \|v^n\| \\ & \leq 6(1/L_x^2 + 1/L_y^2) \|v^n\|^2 + \frac{1}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2 \\ & \leq \|\nabla_h v^n\|^2 + \frac{1}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2, \quad 1 \leq n \leq NT. \end{aligned} \tag{4.23}$$

Substituting (4.22), (4.23) into (4.21) and then applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} a_0^{(\gamma)}(v^n, v^n) &\leq \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})(v^k, v^n) + a_{n-1}^{(\gamma)}(v^0, v^n) + \frac{\tau^\gamma \Gamma(2-\gamma)}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2 \\ &\leq \frac{1}{2} \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) [(v^k, v^k) + (v^n, v^n)] \\ &\quad + \frac{1}{2} a_{n-1}^{(\gamma)} [(v^0, v^0) + (v^n, v^n)] + \frac{\tau^\gamma \Gamma(2-\gamma)}{24(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2. \end{aligned}$$

Multiplying by 2 on both sides of the above formula, we get

$$\begin{aligned} a_0^{(\gamma)}(v^n, v^n) &\leq \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k})(v^k, v^k) + a_{n-1}(v^0, v^0) \\ &\quad + \frac{\tau^\gamma \Gamma(2-\gamma)}{12(1/L_x^2 + 1/L_y^2)} \|\hat{f}^n\|^2. \end{aligned} \tag{4.24}$$

Notice that

$$\frac{\tau^\gamma \Gamma(2-\gamma)}{a_{n-1}^{(\gamma)}} \leq \frac{\tau^\gamma \Gamma(2-\gamma)}{(1-\gamma)n^{-\gamma}} = (n\tau)^\gamma \Gamma(1-\gamma).$$

Then we obtain

$$\begin{aligned} \|v^n\|^2 &\leq \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \|v^k\|^2 \\ &\quad + a_{n-1}^{(\gamma)} \left[\|v^0\|^2 + \frac{L_x^2 L_y^2}{12(L_x^2 + L_y^2)} t_n^\gamma \Gamma(1-\gamma) \|\hat{f}^n\|^2 \right]. \end{aligned} \tag{4.25}$$

From inequality (4.25), using the mathematical induction method (similar to the induction process in Theorem 2) can lead to

$$\|v^n\|^2 \leq \|v^0\|^2 + \frac{L_x^2 L_y^2}{12(L_x^2 + L_y^2)} \Gamma(1-\gamma) t_n^\gamma \max_{1 \leq m \leq n} \|\hat{f}^m\|^2, \quad 1 \leq n \leq NT. \quad \square$$

5 Numerical examples

Example 1 We consider the following one-dimensional time-fractional convection-dominated diffusion equation:

$$\begin{cases} {}_0^c D_t^\gamma u + p(x, t)u_x - \varepsilon u_{xx} = f(x, t), & 0 < x < 1, 0 < t < 1, \\ u(0, t) = \mu_1(t), \quad u(1, t) = \mu_2(t), & 0 < t < 1, \\ u(x, 0) = v(x) = 0, & 0 < x < 1, \end{cases} \tag{5.1}$$

where ε is a nonnegative small parameter, let $\varepsilon = 10^{-5}$, $p(x, t) = 1$. The exact solution of the equation is

$$u(x, t) = t^{2+\gamma} (x + (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)).$$

Here, the corresponding right term, the initial conditions, and the boundary conditions can be obtained directly from the exact solution. Let the time step $\Delta t = 0.01$.

There is the boundary layer on $x = 1$ near the exact solution of the equation. We adopt the TFPM scheme based on G-L approximation (2.11) and the TFPM scheme based on L1 approximation (2.13) and use the classical difference scheme (DM) for numerical solution of the equation.

In order to compare the advantages and disadvantages of the algorithm in our paper, the difference scheme is calculated and the error is estimated by L_2 norm. We use the L1-TFPM discrete scheme for the equation. The results are shown in Table 3.

It can be seen from Figs. 3–5 that the method used in this paper can effectively eliminate the numerical oscillations at the boundary layer. It can be also seen from Tables 1 and 2 that the algorithm has achieved a perfect error accuracy. The convergence rate of the L1-TFPM method is presented in Table 3.

Example 2 Here we consider the following two-dimensional time-fractional convection-dominated diffusion equation:

$$\begin{cases} {}^c_0D_t^\gamma u(x, y, t) - \varepsilon^2 \Delta u + p(x, y, t)u_x + q(x, y, t)u_y \\ = \hat{f}(x, y, t), & (x, y) \in \Omega, t > 0, \\ u(x, y, 0) = \mu_0, & (x, y) \in \Omega, t > 0, \\ u(x, y, t) = \mu_1, & (x, y) \in \partial\Omega, t > 0, \end{cases} \tag{5.2}$$

where $(x, y) \in \Omega = [0, 1] \times [0, 1]$. Let $\varepsilon^2 = 3 \times 10^{-3}$, $p(x, y, t) = 1$, $q(x, y, t) = 1$, the exact solution of the equation is $u(x, y, t) = t^{2+\gamma}y(1-y)(e^{x-1/\varepsilon^2} + (x-1)e^{-1/\varepsilon^2} - x)$, and \hat{f}, μ_0, μ_1 are determined by the exact solution. Take the time step $\Delta t = 0.01$.

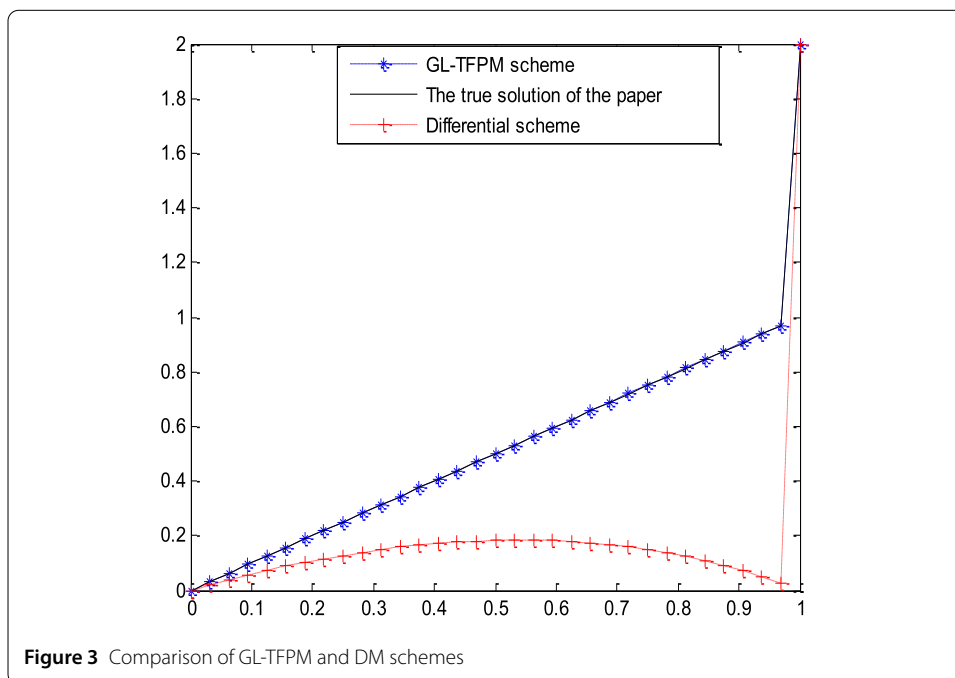
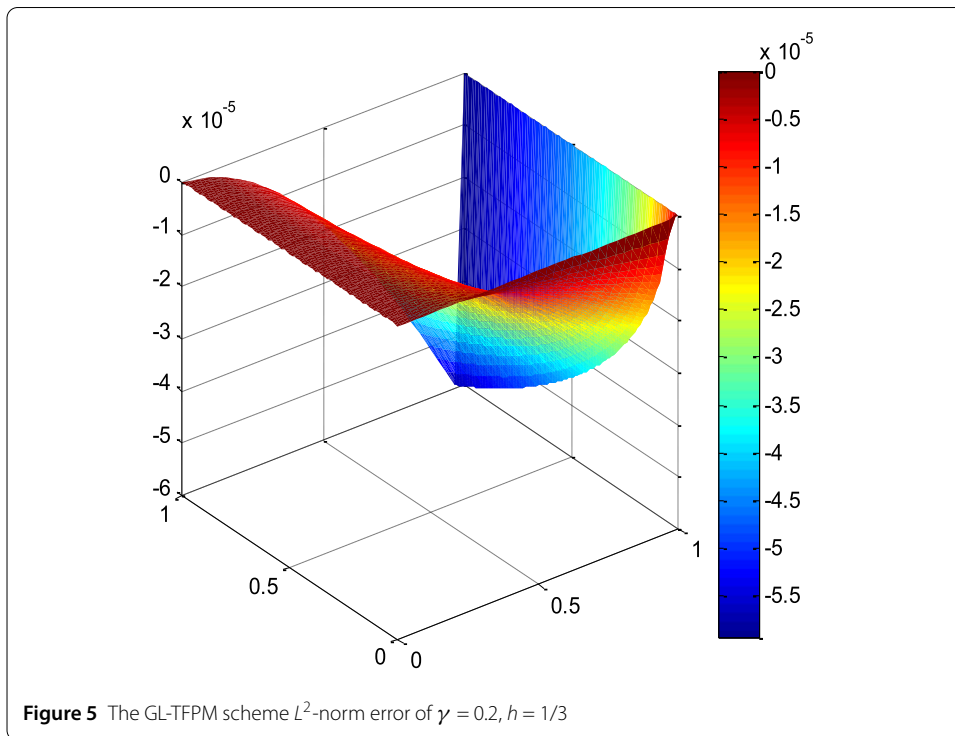
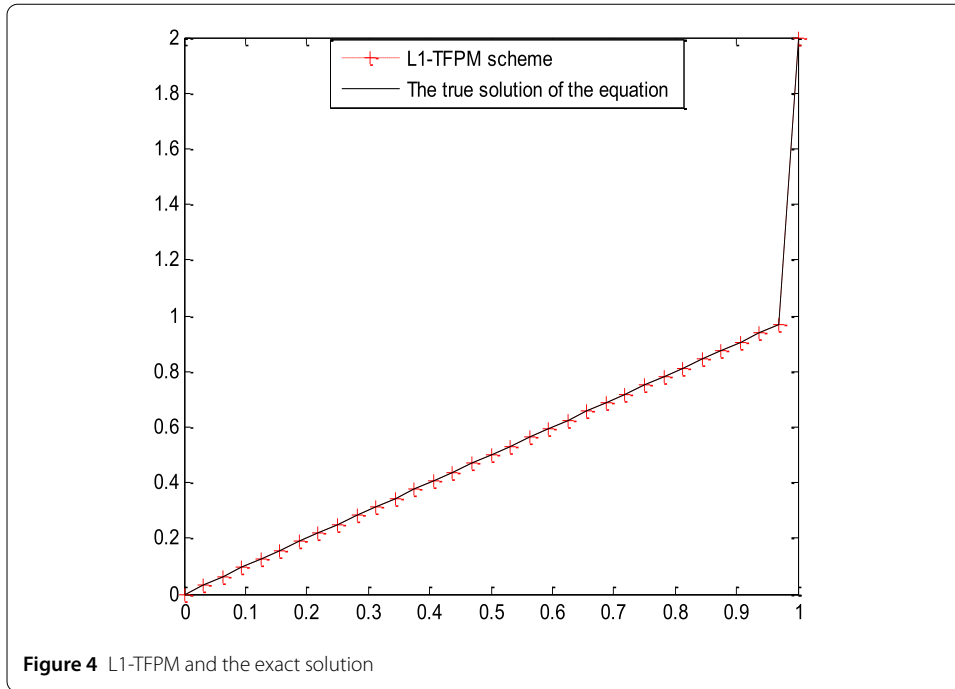


Figure 3 Comparison of GL-TFPM and DM schemes



We employ the TFPM scheme based on G-L approximation (3.18) and the TFPM scheme based on L1 approximation (3.19) and use the classical difference scheme (DM) for numerical solution of the equation. Figures 6 and 7 show the three-dimensional figures of the exact solution, numerical solution of the GL-TFPM and L1-TFPM, respectively. Ta-

Table 1 The L^2 -norm error of GL-TFPM

GL-TFPM	N	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.8$
L^2 -norm	16	1.8741e-004	6.7410e-004	1.3695e-003	3.0370e-004
	32	1.8671e-004	6.6632e-004	1.3575e-003	3.0239e-003
	64	1.8294e-004	6.5091e-004	1.3270e-003	2.9375e-003
	128	1.7628e-004	6.4179e-004	1.3228e-003	2.0456e-003

Table 2 Comparison of L^2 -norm error between the difference scheme and the method in the paper

	N	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.8$
L1-TFPM (L^2 -norm)	16	3.6293e-006	2.9616e-005	1.4826e-004	1.9736e-003
	32	3.5706e-006	2.8942e-005	1.4615e-004	1.2638e-003
	64	3.4309e-006	2.8541e-005	1.3479e-004	1.2415e-003
	128	3.2558e-006	2.2658e-005	1.2900e-004	2.9737e-004
DM (L^2 -norm)	16	4.5172e-001	7.8411e-001	4.3837e-001	4.1023e-001
	32	4.6780e-001	7.7903e-001	3.5481e-001	3.9876e-001
	64	2.7585e-001	7.7663e-001	4.6309e-001	4.7101e-001
	128	4.7989e-001	7.6865e-001	1.6723e-001	4.5263e-001

Table 3 The convergence order of the proposed method L1-TFPM as $\gamma = 0.8$

The number of nodes	The computation error	Convergence order
16	1.9736e-003	
32	1.2638e-003	0.64
64	1.2415e-003	0.03
128	2.9737e-004	2.06

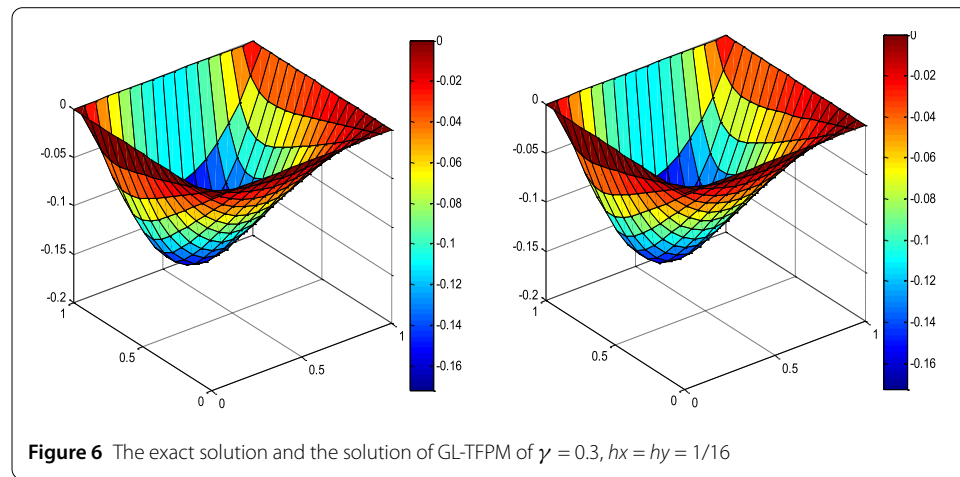


Figure 6 The exact solution and the solution of GL-TFPM of $\gamma = 0.3, hx = hy = 1/16$

ble 4 shows the error comparison between the GL-TFPM scheme and the DM scheme. Table 4 shows the error of the L1-TFPM.

As can be seen from Figs. 6–7, TFPM can effectively eliminate numerical oscillations. From Tables 4, 5 and 6, it can be also seen that the algorithm constructed in this paper is feasible and the error precision is perfectly high.

6 Conclusion

In this paper, the tailored finite point method to solve the time-fractional convection-dominant diffusion problem with variable coefficient is derived. And the stability based on L1 approximation and G-L approximation is also analyzed. At the same time, the 1D

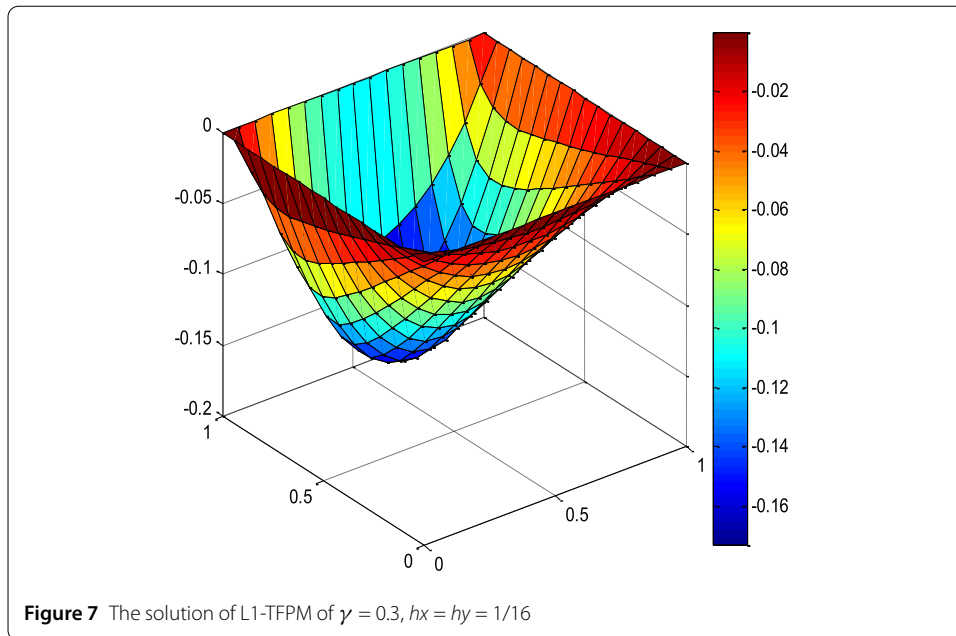


Table 4 The comparison of error between the difference scheme and GL-TFPM

	$N_x \times N_y$	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.8$
GL-TFPM	11 × 11	2.2629e-003	3.4251e-003	4.2396e-003	9.3615e-003
	21 × 21	1.0380e-003	3.3749e-003	6.1552e-003	9.3210e-003
	31 × 31	9.8423e-004	3.1869e-003	5.9977e-003	9.0921e-003
	51 × 51	7.6516e-004	1.0529e-003	4.0131e-003	7.6581e-003
DM	11 × 11	7.9724e-001	7.8411e-001	7.7611e-001	7.4650e-001
	21 × 21	7.9207e-001	7.7903e-001	7.6215e-001	7.4036e-001
	31 × 31	7.8944e-001	7.7663e-001	7.6186e-001	7.2850e-001
	51 × 51	7.8073e-001	7.6865e-001	7.5929e-001	7.0866e-001

Table 5 The error of L1-TFPM in the paper

	$N_x \times N_y$	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.8$
L1-TFPM	11 × 11	2.0811e-003	7.0324e-003	2.0845e-002	5.1065e-002
	21 × 21	1.2307e-003	6.9693e-003	3.6682e-003	4.8705e-002
	31 × 31	1.1760e-003	6.7579e-003	3.6122e-003	4.8311e-002
	51 × 51	9.6585e-004	4.3210e-003	1.8219e-003	5.0135e-003

Table 6 The convergence order of the proposed method L1-TFPM as $\gamma = 0.8$

The number of nodes	The computation error	Convergence order
11 × 11	5.1065e-002	
21 × 21	4.8705e-002	0.07
31 × 31	4.8311e-002	0.02
51 × 51	5.0135e-003	3.27

and 2D cases are also numerically simulated. We compare the errors between the proposed method and the finite difference method. The numerical results show that the calculation accuracy and convergence result of the proposed method exceed DM. Therefore the tailored finite point method is an effective numerical method that can be used to solve the time-fractional convection-dominant diffusion problem.

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Authors' contributions

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