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Existence of chaos for partial difference equations via tangent and cotangent functions

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Abstract

This paper is concerned with the existence of chaos for a type of partial difference equations. We establish four chaotification schemes for partial difference equations with tangent and cotangent functions, in which the systems are shown to be chaotic in the sense of Li–Yorke or of both Li–Yorke and Devaney. For illustration, we provide three examples are provided.

Keywords: Chaos; Partial difference equation; Li–Yorke chaos; Devaney chaos

1 Introduction

In this paper, we focus on the existence of chaos in the following partial difference equation:

$$x(n+1,m) = f\bigl(x(n,m), x(n,m+1)\bigr), \tag{1}$$

where $n \geq 0$ is the time step, m is the lattice point with $0 \leq m \leq k < +\infty$, $f : D \subset \mathbf{R}^2 \to \mathbf{R}$ is a map, and $k+1$ is the system size. In many engineering applications, such as imaging, digital filter, and spatial dynamical system, Eq. (1) plays an important role [1, 2].

In the past years, with the development of chaos theory, chaos has been applied in many fields, such as physics, chemistry, engineering, and mathematics. In mathematics, chaos has become a significant branch of dynamical systems [3]. Furthermore, anticontrol of chaos (chaotification) is an important branch of chaos, and many researchers devoted much effort to chaotification. The first important result was obtained by Chen and Liu [4] proved that Eq. (1) in $\mathbf{R}^3$ is chaotic in the Li–Yorke sense by constructing spatial periodic orbits of specified period. Later, Eq. (1) was reformulated into a discrete system [5]. By applying this method Shi [6] established some criteria of chaos by applying chaos in scalar ordinary difference equations and snap-back repeller theory. Recently, chaotification problems for Eq. (1) with general controllers, sawtooth functions, and mod operations were studied, respectively, and all the controlled systems were proved to be chaotic in the sense of both Devaney and Li–Yorke [7–9]. In [10], two chaotification schemes of Eq. (1)

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via sine functions,

$$x(n+1,m)=f\big(x(n,m),x(n,m+1)\big)+\varepsilon\sin\big(\mu x(n,m)\big),$$

were established for $\mu>1$. Furthermore, we proved that not only the above controlled system but also Eq. (1) with cosine functions are chaotic in the sense of both Li–Yorke and Devaney for $\mu=1$ [11].

As one of the main elements of basic elementary functions, trigonometric functions are of great importance. Sine, cosine, tangent, and cotangent functions are basic ones. It is known that sine and cosine are continuous and have a similar geometric shape with sawtooth functions and mod operations [6–8, 12, 13]. However, tangent and cotangent are piecewise continuous, and their geometric shapes are different from those of sine, cosine, sawtooth, and mod. Can tangent and cotangent functions be viewed as controllers to make the controlled Eq. (1) to be chaotic? In this paper, we attempt to address such an interesting question and try to establish chaotification schemes for the following controlled systems:

$$x(n+1,m)=f\big(x(n,m),x(n,m+1)\big)+\varepsilon\tan\big(x(n,m)\big),\tag{2}$$

$$x(n+1,m)=f\big(x(n,m),x(n,m+1)\big)+\varepsilon\cot\big(x(n,m)\big),\tag{3}$$

$$x(n+1,m)=f\big(x(n,m),x(n,m+1)\big)+\varepsilon\tan\big(x(n,m+1)\big),\tag{4}$$

$$x(n+1,m)=f\big(x(n,m),x(n,m+1)\big)+\varepsilon\cot\big(x(n,m+1)\big).\tag{5}$$

The rest of this paper is organized as follows. In Sect. 2, we list some basic concepts and lemmas about chaos. In Sects. 3, we consider anticontrol of chaos of Eq. (1) with tangent and cotangent functions, give four theorems, and prove that all the controlled systems are chaotic in the sense of Li–Yorke or of both Li–Yorke and Devaney by the coupled-expansion theory. Finally, in Sect. 4, we provide three illustrative examples.

2 Preliminaries

Now we introduce some basic concepts and lemmas.

Definition 1 ([14]) Let (X,d) be a metric space, and let $F:X\to X$ be a map. A subset S of X is called a scrambled set of F if for any two different points $x,y\in S$,

$$\liminf_{n\to\infty} d\big(F^n(x),F^n(y)\big)=0,\qquad \limsup_{n\to\infty} d\big(F^n(x),F^n(y)\big)>0.$$

The map F is said to be chaotic in the Li–Yorke sense if there exists an uncountable scrambled set S of F.

Definition 2 ([15]) A map $F:V\subset X\to V$ is said to be chaotic on V in the sense of Devaney if

(i) F is topologically transitive in V;
(ii) the periodic points of F in V are dense in V;
(iii) F has sensitive dependence on initial conditions in V.

By the result of [16], conditions (i) and (ii) imply (iii) if F is continuous in V that contains infinitely many points. Under some conditions, chaos in the sense of Devaney is stronger than that of Li–Yorke [17].

A nonperiodic boundary condition is given for Eq. (1) as

$$x(n,k+1) = \varphi\big(x(n,p)\big), \quad n \geq 0, 0 \leq p \leq k, \tag{6}$$

where p is an integer, and $\varphi : I \subset \mathbf{R} \to \mathbf{R}$ is a map. For any given initial condition $x(0,m) = \phi(m)$, $0 \leq m \leq k+1$, where ϕ satisfies (6), Eq. (1) obviously has a unique solution satisfying this condition. By setting

$$x_n = \big(x(n,0), x(n,1), \dots, x(n,k)\big)^T \in \mathbf{R}^{k+1}, \quad n \geq 0,$$

Equation (1) with (6) can be written as

$$x_{n+1} = F(x_n), \quad n \geq 0, \tag{7}$$

where

$$F(x_n) = \Big(f\big(x(n,0), x(n,1)\big), f\big(x(n,1), x(n,2)\big), \dots, f\big(x(n,k), \varphi\big(x(n,p)\big)\big)\Big)^T. \tag{8}$$

System (7) is called the system induced by Eq. (1) with (6).

Definition 3 ([8]) Equation (1) with (6) is said to be chaotic in the sense of Devaney (or Li–Yorke) on $V \subset \mathbf{R}^{k+1}$ if its induced system (7) is chaotic in the sense of Devaney (or Li–Yorke) on V.

Definition 4 ([18]) Let (X,d) be a metric space, and let $f : D \subset X \to X$ be a map. If there exist m (≥ 2) subsets V_i $(1 \leq i \leq m)$ of D with $V_i \cap V_j = \partial_D V_i \cap \partial_D V_j$ for each pair of (i,j), $1 \leq i \neq j \leq m$, such that

$$f(V_i) \supset \bigcup_{j=1}^{m} V_j, \quad 1 \leq i \leq m,$$

where $\partial_D V_i$ is the relative boundary of V_i with respect to D, then f is said to be a coupled-expanding map in V_i, $1 \leq i \leq m$. Further, the map f is said to be a strictly coupled-expanding map in V_i, $1 \leq i \leq m$, if $d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$.

Lemma 5 ([19]) *Let (X,d) be a metric space, and let V_j $(1 \leq j \leq m)$ be disjoint compact sets of X. If $f : D \equiv \bigcup_{j=1}^{m} V_j \to X$ is a strictly coupled-expanding continuous map in V_j, $1 \leq j \leq m$, then f is chaotic in the sense of Li–Yorke.*

Lemma 6 ([20, 21]) *Let (X,d) be a complete metric space, and let $f : D \subset X \to X$ be a map. Assume that there exist k disjoint bounded closed subsets V_i of D, $1 \leq i \leq k$, such that f is continuous in $\bigcup_{i=1}^{k} V_i$ and satisfies*

(i) *f is strictly coupled-expanding in V_i, $1 \leq i \leq k$;*

(ii) *there exists a constant* $\lambda > 1$ *such that*

$$d\bigl(f(x),f(y)\bigr) \geq \lambda d(x,y), \quad \forall x,y \in V_i, 1 \leq i \leq k.$$

Then f has an invariant Cantor set $V \subset \bigcup_{i=1}^{k} V_i$ such that $f : V \to V$ is topologically conjugate to the subshift $\Sigma_k^+ \to \Sigma_k^+$. Consequently, f is chaotic on V in the Devaney and Li–Yorke senses.

3 Main results

In this section, we establish four chaotification schemes for Eq. (1) with tangent and cotangent functions.

Theorem 1 *Consider the controlled system* (2), *that is,*

$$x(n+1,m) = f\bigl(x(n,m),x(n,m+1)\bigr) + \varepsilon \tan\bigl(x(n,m)\bigr), \quad n \geq 0, 0 \leq m \leq k < +\infty$$

with (6). *Suppose that*

(i) *there exist positive constants r and L such that*

$$\bigl|f(x_1,y_1) - f(x_2,y_2)\bigr| \leq L \max\bigl\{|x_1 - x_2|, |y_1 - y_2|\bigr\}, \quad \forall x_1,x_2,y_1,y_2 \in [-r,r]; \tag{9}$$

(ii) *$\varphi : [-r,r] \to [-r,r]$ is a map with $\varphi(0) = 0$, and there exists a constant $\lambda > 0$ such that*

$$\bigl|\varphi(x) - \varphi(y)\bigr| \leq \lambda |x - y|, \quad \forall x,y \in [-r,r]. \tag{10}$$

If $r > 5\pi/4$, then for each constant ε satisfying

$$\varepsilon > \varepsilon_0 := \max\left\{\frac{5\pi}{4}\bigl(1 + L\max\{1,\lambda\}\bigr) - f(0,0), \frac{\pi}{4}\bigl(1 + 5L\max\{1,\lambda\}\bigr) + f(0,0)\right\},$$

there exists a Cantor set $\Lambda_1 \subset [-\frac{\pi}{4},\frac{\pi}{4}]^{k+1} \cup [\frac{3\pi}{4},\frac{5\pi}{4}]^{k+1}$ such that system (2) *with* (6) *is chaotic on Λ_1 in the Li–Yorke sense. Further, for each constant ε satisfying*

$$\varepsilon > \max\bigl\{\varepsilon_0, 1 + L\max\{1,\lambda\}\bigr\},$$

there exists a Cantor set $\Lambda_2 \subset [-\frac{\pi}{4},\frac{\pi}{4}]^{k+1} \cup [\frac{3\pi}{4},\frac{5\pi}{4}]^{k+1}$ such that system (2) *with* (6) *is chaotic on Λ_2 in the Li–Yorke and Devaney senses.*

Proof We use Lemmas 5 and 6. Let

$$x_{n+1} = F(x_n) + \varepsilon \operatorname{Tan}(x_n) := G_\varepsilon(x_n), \quad n \geq 0,$$

be the induced system of the controlled system (2) with (6), where $F(x_n)$ is (8), and

$$\operatorname{Tan}(x_n) = \Bigl(\tan\bigl(x(n,0)\bigr), \tan\bigl(x(n,1)\bigr), \ldots, \tan\bigl(x(n,k)\bigr)\Bigr)^T.$$

Let

$$V_1 = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]^{k+1}, \qquad V_2 = \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]^{k+1}.$$

Then $V_1, V_2 \subset [-r, r]^{k+1}$ are nonempty, closed, and bounded, and

$$d(V_1, V_2) = \inf\left\{\|x-y\| : x \in V_1, y \in V_2\right\} = \frac{\pi}{2} > 0.$$

The whole proof is divided into two parts.

Step 1. System (2) with (6) is chaotic in the Li–Yorke sense.

By Lemma 5 we will show that G_ε is a strictly coupled-expanding map in V_1 and V_2.

For each $x = (x(0), x(1), \ldots, x(k))^T \in V_1$ with $x(j) = -\pi/4$, from (9) it follows that, for $0 \leq j \leq k-1$,

$$\begin{aligned} G_{\varepsilon,j}(x) &= f\bigl(x(j), x(j+1)\bigr) + \varepsilon \tan\bigl(x(j)\bigr) \\ &= f\left(-\frac{\pi}{4}, x(j+1)\right) + \varepsilon \tan\left(-\frac{\pi}{4}\right) \\ &\leq L \max\left\{\frac{\pi}{4}, \bigl|x(j+1)\bigr|\right\} - \varepsilon + f(0,0) \\ &= \frac{\pi}{4}L - \varepsilon + f(0,0) \leq -\frac{\pi}{4}, \end{aligned} \tag{11}$$

and for $j = k$, from (6), (9), and (10) it follows that

$$\begin{aligned} G_{\varepsilon,k}(x) &= f\bigl(x(k), \varphi\bigl(x(p)\bigr)\bigr) + \varepsilon \tan\bigl(x(k)\bigr) \\ &= f\left(-\frac{\pi}{4}, \varphi\bigl(x(p)\bigr)\right) + \varepsilon \tan\left(-\frac{\pi}{4}\right) \\ &\leq L \max\left\{\frac{\pi}{4}, \bigl|\varphi\bigl(x(p)\bigr)\bigr|\right\} - \varepsilon + f(0,0) \\ &\leq L \max\left\{\frac{\pi}{4}, \lambda\bigl|x(p)\bigr|\right\} - \varepsilon + f(0,0) \\ &\leq \frac{\pi}{4}L \max\{1, \lambda\} - \varepsilon + f(0,0) \leq -\frac{\pi}{4}. \end{aligned} \tag{12}$$

For each $x \in V_1$ with $x(j) = \pi/4$, it follows from (6), (9), and (10) that, for $0 \leq j \leq k-1$,

$$\begin{aligned} G_{\varepsilon,j}(x) &= f\left(\frac{\pi}{4}, x(j+1)\right) + \varepsilon \tan\left(\frac{\pi}{4}\right) \\ &\geq -L \max\left\{\frac{\pi}{4}, \bigl|x(j+1)\bigr|\right\} + \varepsilon + f(0,0) \\ &= -\frac{\pi}{4}L + \varepsilon + f(0,0) \geq \frac{5\pi}{4}, \end{aligned} \tag{13}$$

and for $j = k$,

$$\begin{aligned} G_{\varepsilon,k}(x) &= f\left(\frac{\pi}{4}, \varphi\bigl(x(p)\bigr)\right) + \varepsilon \tan\left(\frac{\pi}{4}\right) \\ &\geq -L \max\left\{\frac{\pi}{4}, \lambda\bigl|x(p)\bigr|\right\} + \varepsilon + f(0,0) \\ &\geq -\frac{\pi}{4} L \max\{1, \lambda\} + \varepsilon + f(0,0) \geq \frac{5\pi}{4}. \end{aligned} \tag{14}$$

By (9) and (10) G_ε is continuous in $[-r, r]^{k+1}$. By the intermediate value theorem and (11)–(14) we have $G_\varepsilon(V_1) \supset V_1 \cup V_2$.

For each $x \in V_2$ with $x(j) = 3\pi/4$, we have that for $0 \leq j \leq k-1$,

$$\begin{aligned} G_{\varepsilon,j}(x) &= f\left(\frac{3\pi}{4}, x(j+1)\right) + \varepsilon \tan\left(\frac{3\pi}{4}\right) \\ &\leq L \max\left\{\frac{3\pi}{4}, \bigl|x(j+1)\bigr|\right\} - \varepsilon + f(0,0) \\ &\leq \frac{5\pi}{4} L - \varepsilon + f(0,0) \leq -\frac{\pi}{4}, \end{aligned} \tag{15}$$

and for $j = k$,

$$\begin{aligned} G_{\varepsilon,k}(x) &= f\left(\frac{3\pi}{4}, \varphi\bigl(x(p)\bigr)\right) + \varepsilon \tan\left(\frac{3\pi}{4}\right) \\ &\leq L \max\left\{\frac{3\pi}{4}, \lambda\bigl|x(p)\bigr|\right\} - \varepsilon + f(0,0) \\ &\leq \frac{5\pi}{4} L \max\{1, \lambda\} - \varepsilon + f(0,0) \leq -\frac{\pi}{4}. \end{aligned} \tag{16}$$

For each $x \in V_2$ with $x(j) = 5\pi/4$, we have that for $0 \leq j \leq k-1$,

$$\begin{aligned} G_{\varepsilon,j}(x) &= f\left(\frac{5\pi}{4}, x(j+1)\right) + \varepsilon \tan\left(\frac{5\pi}{4}\right) \\ &\geq -L \max\left\{\frac{5\pi}{4}, \bigl|x(j+1)\bigr|\right\} + \varepsilon + f(0,0) \\ &= -\frac{5\pi}{4} L + \varepsilon + f(0,0) \geq \frac{5\pi}{4} \end{aligned} \tag{17}$$

and for $j = k$,

$$\begin{aligned} G_{\varepsilon,k}(x) &= f\left(\frac{5\pi}{4}, \varphi\bigl(x(p)\bigr)\right) + \varepsilon \tan\left(\frac{5\pi}{4}\right) \\ &\geq -L \max\left\{\frac{5\pi}{4}, \lambda\bigl|x(p)\bigr|\right\} + \varepsilon + f(0,0) \\ &\geq -\frac{5\pi}{4} L \max\{1, \lambda\} + \varepsilon + f(0,0) \geq \frac{5\pi}{4}. \end{aligned} \tag{18}$$

By the intermediate value theorem and (15)–(18) we have $G_\varepsilon(V_2) \supset V_1 \cup V_2$.

By the above discussion, G_ε is a strictly coupled-expanding map in V_1 and V_2. Therefore by Lemma 5 system (2) with (6) is chaotic in the Li-Yorke sense.

Step 2. System (2) with (6) is chaotic in both Li–Yorke and Devaney senses.

Since $V_1, V_2 \subset [-r,r]^{k+1}$, from (6), (9), and (10) it follows that for all $x, y \in V_1$ and $x, y \in V_2$,

$$\begin{aligned} \|F(x) - F(y)\| &= \max\{|f(x(j), x(j+1)) - f(y(j), y(j+1))|, 0 \le j \le k\} \\ &\le L \max\{|x(j) - y(j)|, |\varphi(x(p)) - \varphi(y(p))|, 0 \le j, p \le k\} \\ &\le L \max\{|x(j) - y(j)|, \lambda|x(p) - y(p)|, 0 \le j, p \le k\} \\ &\le L \max\{1, \lambda\}\|x - y\|. \end{aligned} \tag{19}$$

On the other hand, by Lagrange's mean value theorem, for all $x, y \in V_1$ and $x, y \in V_2$,

$$\begin{aligned} \|\mathrm{Tan}(x) - \mathrm{Tan}(y)\| &= \max\{|\tan(x(j)) - \tan(y(j))|, 0 \le j \le k\} \\ &= \max\{|\sec^2 \xi (x(j) - y(j))|, 0 \le j \le k\} \\ &\ge \|x - y\|, \end{aligned} \tag{20}$$

where $\xi \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$. Hence from (19) and (20) it follows that for all $x, y \in V_1$ and $x, y \in V_2$,

$$\|G_\varepsilon(x) - G_\varepsilon(y)\| \ge (\varepsilon - L \max\{1, \lambda\})\|x - y\|.$$

Since $\varepsilon - L\max\{1, \lambda\} > 1$, G_ε satisfies assumption (ii) of Lemma 6. Together with the result obtained in step 1, by Lemma 6 system (2) with (6) is chaotic in both Li–Yorke and Devaney senses. The proof is complete. □

Theorem 2 *Consider the controlled system* (3), *that is,*

$$x(n+1, m) = f(x(n, m), x(n, m+1)) + \varepsilon \cot(x(n, m)), \quad n \ge 0, 0 \le m \le k < +\infty$$

with (6). *Suppose that all the conditions in Theorem* 1 *hold. Then for all constants* ε, r *satisfying*

$$\varepsilon > \max\left\{\frac{3\pi}{4}(1 + L\max\{1, \lambda\}) - f(0,0), \frac{3\pi}{4}(1 + L\max\{1, \lambda\}) + f(0,0)\right\}$$

and $r > 3\pi/4$, *there exists a Cantor set* $\Lambda \subset [-\frac{3\pi}{4}, -\frac{\pi}{4}]^{k+1} \cup [\frac{\pi}{4}, \frac{3\pi}{4}]^{k+1}$ *such that system* (3) *with* (6) *is chaotic on* Λ *in both Li–Yorke and Devaney senses.*

Proof We use Lemmas 5 and 6. The induced system of (3) with (6) is

$$x_{n+1} = F(x_n) + \varepsilon \,\mathrm{Cot}(x_n) := H_\varepsilon(x_n), \quad n \ge 0,$$

where F is defined in (8), and

$$\mathrm{Cot}(x_n) = (\cot(x(n,0)), \cot(x(n,1)), \ldots, \cot(x(n,k)))^T.$$

Let

$$\widetilde{V}_1 = \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]^{k+1}, \qquad \widetilde{V}_2 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]^{k+1}.$$

Obviously, $\widetilde{V}_1, \widetilde{V}_2 \subset [-r, r]^{k+1}$ are nonempty, closed, and bounded sets, and $d(\widetilde{V}_1, \widetilde{V}_2) = \pi/2 > 0$.

First, we show that $H_\varepsilon(\widetilde{V}_i) \supset \widetilde{V}_1 \cup \widetilde{V}_2$ for $i = 1, 2$.

For each $x \in \widetilde{V}_1$ with $x(j) = -3\pi/4$, from (6), (9), and (10) it follows that for $0 \le j \le k-1$,

$$\begin{aligned} H_{\varepsilon,j}(x) &= f\left(-\frac{3\pi}{4}, x(j+1)\right) + \varepsilon \cot\left(-\frac{3\pi}{4}\right) \\ &\ge -L \max\left\{\frac{3\pi}{4}, \left|x(j+1)\right|\right\} + \varepsilon + f(0,0) \\ &= -\frac{3\pi}{4} L + \varepsilon + f(0,0) \ge \frac{3\pi}{4}, \end{aligned} \tag{21}$$

and for $j = k$,

$$\begin{aligned} H_{\varepsilon,k}(x) &= f\left(-\frac{3\pi}{4}, \varphi\left(x(p)\right)\right) + \varepsilon \cot\left(-\frac{3\pi}{4}\right) \\ &\ge -L \max\left\{\frac{3\pi}{4}, \lambda\left|x(p)\right|\right\} + \varepsilon + f(0,0) \\ &= -\frac{3\pi}{4} L \max\{1, \lambda\} + \varepsilon + f(0,0) \ge \frac{3\pi}{4}. \end{aligned} \tag{22}$$

For each $x \in \widetilde{V}_1$ with $x(j) = -\pi/4$, from (6), (9), and (10) it follows that for $0 \le j \le k-1$,

$$\begin{aligned} H_{\varepsilon,j}(x) &= f\left(-\frac{\pi}{4}, x(j+1)\right) + \varepsilon \cot\left(-\frac{\pi}{4}\right) \\ &\le L \max\left\{\frac{\pi}{4}, \left|x(j+1)\right|\right\} - \varepsilon + f(0,0) \\ &\le \frac{3\pi}{4} L - \varepsilon + f(0,0) \le -\frac{3\pi}{4}, \end{aligned} \tag{23}$$

and for $j = k$,

$$\begin{aligned} H_{\varepsilon,k}(x) &= f\left(-\frac{\pi}{4}, \varphi\left(x(p)\right)\right) + \varepsilon \cot\left(-\frac{\pi}{4}\right) \\ &\le L \max\left\{\frac{\pi}{4}, \lambda\left|x(p)\right|\right\} - \varepsilon + f(0,0) \\ &\le \frac{3\pi}{4} L \max\{1, \lambda\} - \varepsilon + f(0,0) \le -\frac{3\pi}{4}. \end{aligned} \tag{24}$$

For each $x \in \widetilde{V}_2$ with $x(j) = \pi/4$, for $0 \le j \le k-1$,

$$\begin{aligned} H_{\varepsilon,j}(x) &= f\left(\frac{\pi}{4}, x(j+1)\right) + \varepsilon \cot\left(\frac{\pi}{4}\right) \\ &\ge -L \max\left\{\frac{\pi}{4}, \left|x(j+1)\right|\right\} + \varepsilon + f(0,0) \\ &\ge -\frac{3\pi}{4} L + \varepsilon + f(0,0) \ge \frac{3\pi}{4}, \end{aligned} \tag{25}$$

and for $j = k$,

$$\begin{aligned} H_{\varepsilon,k}(x) &= f\left(\frac{\pi}{4}, \varphi\big(x(p)\big)\right) + \varepsilon \cot\left(\frac{\pi}{4}\right) \\ &\ge -L \max\left\{\frac{\pi}{4}, \lambda\left|x(p)\right|\right\} + \varepsilon + f(0,0) \\ &\ge -\frac{3\pi}{4} L \max\{1,\lambda\} + \varepsilon + f(0,0) \ge \frac{3\pi}{4}. \end{aligned} \tag{26}$$

For each $x \in \widetilde{V}_2$ with $x(j) = 3\pi/4$, for $0 \le j \le k-1$,

$$\begin{aligned} H_{\varepsilon,j}(x) &= f\left(\frac{3\pi}{4}, x(j+1)\right) + \varepsilon \cot\left(\frac{3\pi}{4}\right) \\ &\le L \max\left\{\frac{3\pi}{4}, \left|x(j+1)\right|\right\} - \varepsilon + f(0,0) \\ &= \frac{3\pi}{4} L - \varepsilon + f(0,0) \le -\frac{3\pi}{4}, \end{aligned} \tag{27}$$

and for $j = k$,

$$\begin{aligned} H_{\varepsilon,k}(x) &= f\left(\frac{3\pi}{4}, \varphi\big(x(p)\big)\right) + \varepsilon \cot\left(\frac{3\pi}{4}\right) \\ &\le L \max\left\{\frac{3\pi}{4}, \lambda\left|x(p)\right|\right\} - \varepsilon + f(0,0) \\ &\le \frac{3\pi}{4} L \max\{1,\lambda\} - \varepsilon + f(0,0) \le -\frac{3\pi}{4}. \end{aligned} \tag{28}$$

By the intermediate value theorem and (21)–(28), we have $H_\varepsilon(\widetilde{V}_i) \supset \widetilde{V}_1 \cup \widetilde{V}_2$, $i = 1,2$. So by Lemma 5 system (3) with (6) is chaotic in the Li–Yorke sense.

Next, we show that H_ε satisfies assumption (ii) in Lemma 6.

By Lagrange's mean value theorem we can verify that for all $x, y \in \widetilde{V}_1$ and $x, y \in \widetilde{V}_2$,

$$\begin{aligned} \left\|\operatorname{Cot}(x) - \operatorname{Cot}(y)\right\| &= \max\left\{\left|\cot\big(x(j)\big) - \cot\big(y(j)\big)\right|, 0 \le j \le k\right\} \\ &= \max\left\{\left|-\csc^2\theta\big(x(j) - y(j)\big)\right|, 0 \le j \le k\right\} \\ &\ge \|x - y\|, \end{aligned}$$

where $\theta \in (-\frac{3\pi}{4}, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{3\pi}{4})$. Hence, by (19), for all $x, y \in \widetilde{V}_1$ and $x, y \in \widetilde{V}_2$,

$$\left\|H_\varepsilon(x) - H_\varepsilon(y)\right\| \ge \big(\varepsilon - L\max\{1,\lambda\}\big)\|x - y\|.$$

Since $\varepsilon > \frac{3}{4}\pi(1 + L\max\{1,\lambda\})$, we have $\varepsilon - L\max\{1,\lambda\} > 1$. Thus H_ε satisfies assumption (ii) in Lemma 6. By Lemma 6 system (3) with (6) is chaotic in both Li–Yorke and Devaney senses. This completes the proof. □

Now we consider the controlled systems (4) and (5). For convenience, we give a periodic boundary condition for Eq. (1):

$$x(n,k+1) = x(n,0), \quad n \geq 0. \tag{29}$$

We have the following two results.

Theorem 3 *Consider the controlled system* (4), *that is*,

$$x(n+1,m) = f\big(x(n,m),x(n,m+1)\big) + \varepsilon\tan\big(x(n,m+1)\big), \quad n \geq 0, 0 \leq m \leq k < +\infty,$$

with (29). *Suppose that condition* (*i*) *in Theorem* 1 *holds. Then all the results in Theorem* 1 *hold for system* (4) *with* (29), *except that* $\max\{1,\lambda\}$ *in Theorem* 1 *is replaced by* 1.

Proof The induced system of (4) with (29) can be written as

$$x_{n+1} = \widetilde{F}(x_n) + \varepsilon\,\mathrm{Tan}(\widehat{x}_n) := \widetilde{G}_\varepsilon(x_n), \quad n \geq 0,$$

where

$$\begin{aligned} \widetilde{F}(x_n) &= \Big(f\big(x(n,0),x(n,1)\big), f\big(x(n,1),x(n,2)\big), \ldots, f\big(x(n,k),x(n,0)\big)\Big)^T, \\ \mathrm{Tan}(\widehat{x}_n) &= \Big(\tan\big(x(n,1)\big), \tan\big(x(n,2)\big), \ldots, \tan\big(x(n,k)\big), \tan\big(x(n,0)\big)\Big)^T. \end{aligned} \tag{30}$$

Let V_1 and V_2 be the same as in Theorem 1. We divide the proof into two parts.

Step 1. System (4) with (29) is chaotic in the Li–Yorke sense.

For each $x \in V_1$ with $x(j+1) = -\pi/4$, from (9) it follows that for $0 \leq j \leq k-1$,

$$\begin{aligned} \widetilde{G}_{\varepsilon,j}(x) &= f\big(x(j),x(j+1)\big) + \varepsilon\tan\big(x(j+1)\big) \\ &= f\left(x(j),-\frac{\pi}{4}\right) + \varepsilon\tan\left(-\frac{\pi}{4}\right) \\ &\leq L\max\left\{\big|x(j)\big|,\frac{\pi}{4}\right\} - \varepsilon + f(0,0) \\ &= \frac{\pi}{4}L - \varepsilon + f(0,0) \leq -\frac{\pi}{4}, \end{aligned} \tag{31}$$

and for $j = k$, from (9) and (29) it follows that $x(k+1) = x(0) = -\pi/4$, so that

$$\begin{aligned} \widetilde{G}_{\varepsilon,k}(x) &= f\big(x(k),x(0)\big) + \varepsilon\tan\big(x(0)\big) \\ &= f\left(x(k),-\frac{\pi}{4}\right) + \varepsilon\tan\left(-\frac{\pi}{4}\right) \\ &\leq L\max\left\{\big|x(k)\big|,\frac{\pi}{4}\right\} - \varepsilon + f(0,0) \\ &= \frac{\pi}{4}L - \varepsilon + f(0,0) \leq -\frac{\pi}{4}. \end{aligned} \tag{32}$$

For each $x \in V_1$ with $x(j+1) = \pi/4$, by (9) and (29), $x(k+1) = x(0) = \pi/4$ for $j = k$. Therefore for $0 \le j \le k$,

$$\begin{aligned} \widetilde{G}_{\varepsilon,j}(x) &= f\left(x(j), \frac{\pi}{4}\right) + \varepsilon \tan\left(\frac{\pi}{4}\right) \\ &\ge -L \max\left\{\left|x(j)\right|, \frac{\pi}{4}\right\} + \varepsilon + f(0,0) \\ &= -\frac{\pi}{4}L + \varepsilon + f(0,0) \ge \frac{5\pi}{4}. \end{aligned} \tag{33}$$

For each $x \in V_2$ with $x(j+1) = 3\pi/4$, $0 \le j \le k$, from (9) and (29) it follows that

$$\begin{aligned} \widetilde{G}_{\varepsilon,j}(x) &= f\left(x(j), \frac{3\pi}{4}\right) + \varepsilon \tan\left(\frac{3\pi}{4}\right) \\ &\le L \max\left\{\left|x(j)\right|, \frac{3\pi}{4}\right\} - \varepsilon + f(0,0) \\ &\le \frac{5\pi}{4}L - \varepsilon + f(0,0) \le -\frac{\pi}{4}, \end{aligned} \tag{34}$$

and for each $x \in V_2$ with $x(j+1) = 5\pi/4$, $0 \le j \le k$, we have

$$\begin{aligned} \widetilde{G}_{\varepsilon,j}(x) &= f\left(x(j), \frac{5\pi}{4}\right) + \varepsilon \tan\left(\frac{5\pi}{4}\right) \\ &\ge -L \max\left\{\left|x(j)\right|, \frac{5\pi}{4}\right\} + \varepsilon + f(0,0) \\ &= -\frac{5\pi}{4}L + \varepsilon + f(0,0) \ge \frac{5\pi}{4}. \end{aligned} \tag{35}$$

By the intermediate value theorem and (31)–(35) we have $\widetilde{G}_\varepsilon(V_i) \supset V_1 \cup V_2$, $i = 1, 2$. Therefore by Lemma 5 system (4) with (29) is chaotic in the Li–Yorke sense.

Step 2. System (4) with (29) is chaotic in both Li–Yorke and Devaney senses.

Since $V_1, V_2 \subset [-r, r]^{k+1}$, from (9) and (29) it follows that for all $x, y \in V_1$ and $x, y \in V_2$,

$$\begin{aligned} \left\|\widetilde{F}(x) - \widetilde{F}(y)\right\| &= \max\left\{\left|f\left(x(j), x(j+1)\right) - f\left(y(j), y(j+1)\right)\right|, 0 \le j \le k\right\} \\ &\le L \max\left\{\left|x(j) - y(j)\right|, 0 \le j \le k\right\} \\ &= L\|x - y\|. \end{aligned} \tag{36}$$

On the other hand, by Lagrange's mean value theorem, for all $x, y \in V_1$ and $x, y \in V_2$,

$$\begin{aligned} \left\|\mathrm{Tan}(\widehat{x}) - \mathrm{Tan}(\widehat{y})\right\| &= \max\left\{\left|\tan\left(x(j)\right) - \tan\left(y(j)\right)\right|, 0 \le j \le k\right\} \\ &= \max\left\{\left|\sec^2 \eta\left(x(j) - y(j)\right)\right|, 0 \le j \le k\right\} \\ &\ge \|x - y\|, \end{aligned} \tag{37}$$

where $\eta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$. Hence from (36) and (37) it follows that

$$\left\|\widetilde{G}_\varepsilon(x) - \widetilde{G}_\varepsilon(y)\right\| \ge (\varepsilon - L)\|x - y\|, \quad \forall x, y \in V_1 \text{ or } x, y \in V_2.$$

Since $\varepsilon - L > 1$, $\widetilde{G}_\varepsilon$ satisfies assumption (ii) of Lemma 6. Together with the result obtained in step 1, by Lemma 6 system (4) with (29) is chaotic in both Li–Yorke and Devaney senses. This completes the proof. □

Remark 1 The boundary conditions imposed on systems (2)–(3) and (4)–(5) are different. If (6) is imposed on system (4), then in (32), $x(k+1) = \varphi(x(p)) = -\pi/4$, $0 \le p \le k$, but we cannot ensure that $x(p) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Thus $x \in V_1$ may not hold. Therefore (29) is imposed on systems (4) and (5).

Theorem 4 *Consider the controlled system* (5), *that is,*

$$x(n+1,m) = f\big(x(n,m), x(n,m+1)\big) + \varepsilon \cot\big(x(n,m+1)\big), \quad n \ge 0, 0 \le m \le k < +\infty,$$

with (29). *Suppose that condition* (*i*) *in Theorem* 1 *holds. Then all the results in Theorem* 2 *hold for system* (5) *with* (29), *where* $\max\{1, \lambda\} = 1$.

Proof We use Lemmas 5 and 6. Let

$$x_{n+1} = \widetilde{F}(x_n) + \varepsilon \operatorname{Cot}(\widehat{x}_n) := \widetilde{H}_\varepsilon(x_n), \quad n \ge 0,$$

be the induced system of system (5) with (29), where $\widetilde{F}$ is defined in (30), and

$$\operatorname{Cot}(\widehat{x}_n) = \big(\cot\big(x(n,1)\big), \cot\big(x(n,2)\big), \dots, \cot\big(x(n,k), \cot\big(x(n,0)\big)\big)^T.$$

Let $\widetilde{V}_1$ and $\widetilde{V}_2$ be the same as in Theorem 2.

For each $x \in \widetilde{V}_1$ with $x(j+1) = -3\pi/4$, $0 \le j \le k$, from (9) and (29) it follows that

$$\begin{aligned} \widetilde{H}_{\varepsilon,j}(x) &= f\left(x(j), -\frac{3\pi}{4}\right) + \varepsilon \cot\left(-\frac{3\pi}{4}\right) \\ &\ge -L \max\left\{\big|x(j)\big|, \frac{3\pi}{4}\right\} + \varepsilon + f(0,0) \\ &= -\frac{3\pi}{4} L + \varepsilon + f(0,0) \ge \frac{3\pi}{4}, \end{aligned} \tag{38}$$

and for each $x \in \widetilde{V}_1$ with $x(j+1) = -\pi/4$, $0 \le j \le k$, from (9) and (29) it follows that

$$\begin{aligned} \widetilde{H}_{\varepsilon,j}(x) &= f\left(x(j), -\frac{\pi}{4}\right) + \varepsilon \cot\left(-\frac{\pi}{4}\right) \\ &\le L \max\left\{\big|x(j)\big|, \frac{\pi}{4}\right\} - \varepsilon + f(0,0) \\ &\le \frac{3\pi}{4} L - \varepsilon + f(0,0) \le -\frac{3\pi}{4}. \end{aligned} \tag{39}$$

By the intermediate value theorem and (38)–(39) we have $\widetilde{H}_\varepsilon(\widetilde{V}_1) \supset \widetilde{V}_1 \cup \widetilde{V}_2$. Similarly, we can prove that $\widetilde{H}_\varepsilon(\widetilde{V}_2) \supset \widetilde{V}_1 \cup \widetilde{V}_2$.

By Lagrange's mean value theorem we can verify that for all $x, y \in \widetilde{V}_1$ and $x, y \in \widetilde{V}_2$,

$$\begin{aligned} \|\mathrm{Cot}(\widehat{x}) - \mathrm{Cot}(\widehat{y})\| &= \max\{|\cot(x(j)) - \cot(y(j))|, 0 \le j \le k\} \\ &= \max\{|-\csc^2\theta(x(j) - y(j))|, 0 \le j \le k\} \\ &\ge \|x - y\|, \end{aligned}$$

where $\theta \in (-\frac{3\pi}{4}, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{3\pi}{4})$. Together with (36), for all $x, y \in \widetilde{V}_1$ and $x, y \in \widetilde{V}_2$, we have

$$\|\widetilde{H}_\varepsilon(x) - \widetilde{H}_\varepsilon(y)\| \ge (\varepsilon - L)\|x - y\|,$$

where $\varepsilon > \frac{3\pi}{4}(1 + L) > 1 + L$. By Lemma 6 system (5) with (29) is chaotic in both Li–Yorke and Devaney senses. This completes the proof. □

4 Examples

In this section, we discuss three examples with computer simulations.

Example 1 Consider the controlled system (2) with (6), where

$$f(x, y) = \begin{cases} \frac{1}{32}xy + \frac{1}{2}\pi, & x, y \in [-4, 4], \\ \frac{1}{2}^{|xy|}, & \text{else}, \end{cases}$$

and

$$\varphi(x) = \frac{1}{2}x, \quad \forall x \in \mathbf{R}.$$

It is evident that $|f_x(x, y)| + |f_y(x, y)| \le 1/4$ for all $x, y \in [-4, 4]$, that is,

$$|f(x_1, y_1) - f(x_2, y_2)| \le \frac{1}{4}\max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad \forall x_1, x_2, y_1, y_2 \in [-4, 4].$$

Thus f and φ satisfy all the assumptions in Theorem 1 with $r = 4$, $L = 1/4$, $\lambda = 1/2$, and $f(0, 0) = \pi/2$. By Theorem 1, for any $\varepsilon > 17\pi/16$, there exists a Cantor set $\Lambda \subset [-\frac{1}{4}\pi, \frac{1}{4}\pi]^{k+1} \cup [\frac{3}{4}\pi, \frac{5}{4}\pi]^{k+1}$ such that the controlled system is chaotic on Λ in both Li–Yorke and Devaney senses. Two simulation results on two-dimensional plane $(x(\cdot, 0), x(\cdot, 1))$ and three-dimensional space $(x(\cdot, 0), x(\cdot, 1), x(\cdot, 2))$ are given in Fig. 1 for $p = 1$, $k = 1, 2$, and $\varepsilon = 9\pi/8$, which exhibit complicated dynamical behaviors of the controlled system on Λ.

Example 2 Consider the controlled system (3) with (6), where

$$f(x, y) = \begin{cases} \frac{1}{9}x^2 + \frac{1}{3}y, & x, y \in [-3, 3], \\ \cos(x + y), & \text{else}, \end{cases} \tag{40}$$

and

$$\varphi(x) = \frac{4}{3}x, \quad \forall x \in \mathbf{R}.$$

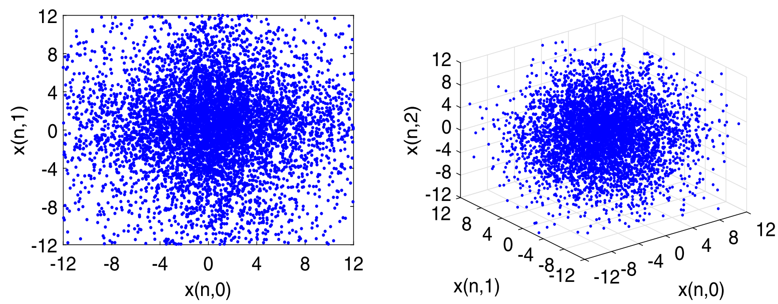

Figure 1 Simulations for system (2) with (6), where $n = 0, 1, \ldots, 10{,}000$ and $p = 0$. In the 2-D graph, the initial values are taken as $x(0,0) = 0.1$ and $x(0,1) = -0.1$. The initial values are $x(0,0) = 0.1$, $x(0,1) = -0.1$, and $x(0,2) = 0.1$ in the 3-D graph

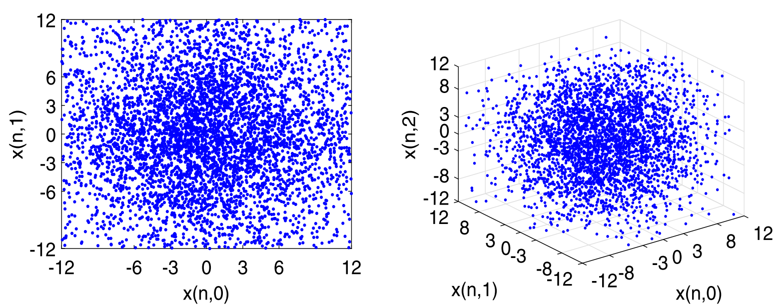

Figure 2 Simulations for system (3) with (6), where $n = 0, 1, \ldots, 10{,}000$ and $p = 1$. In the 2-D graph, $k = 1$, and the initial values are $x(0,0) = 1$, $x(0,1) = -0.1$. In the 3-D graph, $k = 2$, and the initial value are $x(0,0) = 0.1$, $x(0,1) = -0.1$, and $x(0,2) = 0.1$

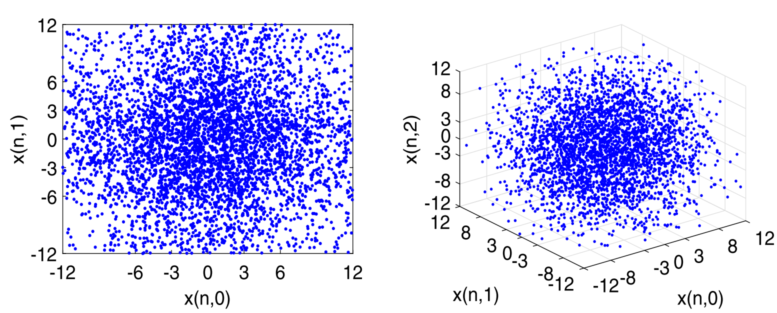

Figure 3 Simulations for system (5) with (29), where $n = 0, 1, \ldots, 10{,}000$. In the 2-D graph, $k = 1$, and the initial values are $x(0,0) = 1$, $x(0,1) = -0.1$. In the 3-D graph, $k = 2$, and the initial values are $x(0,0) = 0.1$, $x(0,1) = -0.1$, and $x(0,2) = 0.1$

Obviously, $f(0,0) = 0$ and $|f_x(x,y)| + |f_y(x,y)| \leq 1$ for all $x, y \in [-3,3]$, which implies that

$$\bigl|f(x_1,y_1) - f(x_2,y_2)\bigr| \leq \max\bigl\{|x_1 - x_2|, |y_1 - y_2|\bigr\}, \quad \forall x_1, x_2, y_1, y_2 \in [-3,3].$$

Hence f and φ satisfy all the assumptions in Theorem 2 with $r = 3$, $L = 1$, $\lambda = 4/3$. Thus, by Theorem 2, for any constant $\varepsilon > 7\pi/4$, there exists a Cantor set $\Lambda \subset [-\frac{3}{4}\pi, -\frac{1}{4}\pi]^{k+1} \cup [\frac{1}{4}\pi, \frac{3}{4}\pi]^{k+1}$ such that the controlled system (3) with (6) is chaotic on Λ in both Li–Yorke and Devaney senses. Two simulation results are shown in Fig. 2 for $p = 1$ and $\varepsilon = 2\pi$, which indicate that the controlled system has very complicated dynamical behaviors on Λ.

Example 3 Consider the controlled system (5) with (29), where $f(x, y)$ is (40). By the previous discussion, f satisfies all the assumptions in Theorem 4 with $r = 3$ and $L = 1$. Thus, by Theorem 4, for any constant $\varepsilon > 3\pi/2$, there exists a Cantor set $\Lambda \subset [-\frac{3}{4}\pi, -\frac{1}{4}\pi]^{k+1} \cup [\frac{1}{4}\pi, \frac{3}{4}\pi]^{k+1}$ such that the controlled system is chaotic on Λ in both Li–Yorke and Devaney senses. Simulation results are shown in Fig. 3 for $\varepsilon = 2\pi$, which show that the controlled system has very complicated dynamical behaviors on Λ.

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Authors' contributions
WL contributed to the idea of this paper, wrote the manuscript, and revised it. HG proved the theorems and wrote this paper. Both authors read and approved the final manuscript.

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