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Degenerate Lah–Bell polynomials arising from degenerate Sheffer sequences

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Abstract

Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials. Recently, Kim–Kim (J. Math. Anal. Appl. 493(1):124521, 2021) introduced the λ-Sheffer sequence and the degenerate Sheffer sequence. They defined the λ-linear functionals and λ-differential operators, respectively, instead of the linear functionals and the differential operators of umbral calculus established by Rota. In this paper, the author gives various interesting identities related to the degenerate Lah–Bell polynomials and special polynomials and numbers by using degenerate Sheffer sequences, and at the same time derives the inversion formulas of these identities.

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1 Introduction

It is important to note that many academics in the field of mathematics have been researching various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [9, 12, 14, 16–23], beginning with Carlitz's degenerate Bernoulli polynomials and the degenerate Euler polynomials [2].

Moreover, umbral calculus, established by Rota in the 1970s, was based on modern concepts such as linear functionals, linear operators, and adjoints [28]. Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials [5, 6, 24, 25, 28]. Recently, Kim–Kim [11] introduced the λ-Sheffer sequence and the degenerate Sheffer sequence. They defined the λ- linear functionals and λ- differential operators, respectively, instead of the linear functionals and the differential operators used by Rota [28]. Also, Kim et al. introduced the Lah–Bell polynomials and studied some identities of Lah–Bell polynomials [10, 25]. The two papers mentioned above inspired me. So, I focus on finding the noble identities of degenerate Lah–Bell polynomials in terms of quite a few well-known special polynomials and numbers arising from the degenerate Sheffer sequence. In addition, the author derives the inversion formulas of the identities obtained in this paper. They include the degenerate and other special polynomials and numbers such as Lah numbers, the degenerate falling factorial, the degenerate Bernoulli polynomials and numbers, degenerate Frobenius–Euler polynomials and numbers of order r, the degenerate Deahee polynomials, the degenerate Bell polynomials, and degenerate Stirling numbers of the first and second kinds.

Now, we give some definitions and properties needed in this paper.

The unsigned Lah number $L(n,k)$ counts the number of ways of all distributions of n balls, labeled $1,2,\dots,n$, among k unlabeled, contents-ordered boxes, with no box left empty and have an explicit formula

$$\mathbf{L}(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}\quad \text{(see [10, 25]).} \tag{1}$$

From (1), the generating function of $L(n,k)$ is given by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\quad (k\geq 0)\ \text{(see [10, 25–27]).} \tag{2}$$

Recently, Lah–Bell polynomials were introduced by Kim–Kim to be

$$e^{x(\frac{1}{1-t}-1)}=\sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!}\quad \text{(see [10]).} \tag{3}$$

When $x=1$, $B_n^L=B_n^L(1)$ are called Lah–Bell numbers.

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\qquad e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}}\quad \text{(see [2, 9, 12, 14, 16, 17, 19–23]).} \tag{4}$$

By Taylor expansion, we get

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\quad \text{(see [9, 12, 14, 16, 17, 19–23]),} \tag{5}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ $(n\geq 1)$.

It is known that

$$(1-t)^{-m}=\sum_{l=0}^{\infty}\binom{-m}{l}(-1)^l t^l=\sum_{l=0}^{\infty}\langle m\rangle_l\frac{t^l}{l!}\quad \text{(see [1, 4]).} \tag{6}$$

where $\langle x\rangle_0=1$, $\langle x\rangle_n=x(x+1)(x+2)\cdots(x+n-1)$, $(n\geq 1)$.

The degenerate Bernoulli polynomials and degenerate Euler polynomials of order r, respectively, are given by the generating functions

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\quad \text{(see [2, 7, 21])} \tag{7}$$

and

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [2, 7, 15, 18])}. \tag{8}$$

We note that $\beta_{n,\lambda}^{(r)}=\beta_{n,\lambda}^{(r)}(0)$ and $E_{n,\lambda}^{(r)}=E_{n,\lambda}^{(r)}(0)$ $(n\geq 0)$ are called degenerate Bernoulli and degenerate Euler numbers of order r, respectively.

Kim et al. introduced the degenerate Frobenius–Euler polynomials of order r defined by

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}, \quad (u\neq 1, u\in\mathbb{C})\ (k\geq 0)\ \text{(see [15])}. \tag{9}$$

When $x=0$, $h_{n,\lambda}^{(r)}(u)=h_{n,\lambda}^{(r)}(0|u)$ are called degenerate Frobenius–Euler numbers of order r.

The degenerate Daehee polynomials are defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [9])}. \tag{10}$$

Here $\log_\lambda(1+t)=\frac{1}{\lambda}((1+t)^\lambda-1)$ and $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda(t))=t$.

When $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called degenerate Daehee numbers.

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}\mathrm{Bel}_n(x)\frac{t^n}{n!} \quad \text{(see [8, 19, 22])}. \tag{11}$$

Kim–Kim introduced the degenerate Bell polynomial given by

$$e_\lambda^x\bigl(e_\lambda(t)-1\bigr)=\sum_{l=0}^{\infty}\mathrm{Bel}_{l,\lambda}(x)\frac{t^l}{l!} \quad \text{(see [19])}. \tag{12}$$

For $n\geq 0$, it is well known that the Stirling numbers of the first and second kind, respectively, are given by

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l \quad \text{and} \quad \frac{1}{k!}\bigl(\log(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!} \quad \text{(see [2, 3])} \tag{13}$$

and

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l \quad \text{and} \quad \frac{1}{k!}\bigl(e^t-1\bigr)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!} \quad \text{(see [2, 3])}, \tag{14}$$

where $(x)_0=1$, $(x)_n=x(x-1)\cdots(x-n+1)$ $(n\geq 1)$.

Moreover, the degenerate Stirling numbers of the first and second kind, respectively, are given by

$$\begin{aligned} &(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad \text{and} \\ &\frac{1}{k!}\bigl(\log_\lambda(1+t)\bigr)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \text{ (see [12, 14])}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_{l} \quad \text{and} \\ &\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \text{ (see [12, 14])}. \end{aligned} \tag{16}$$

For $k \in \mathbb{Z}$, Kim–Kim introduced the modified polyexponential function as

$$\mathrm{Ei}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^s} \quad \text{(see [18, 21])}. \tag{17}$$

By (17), we see that $\mathrm{Ei}_1(x) = e^x - 1$.

Kim–Jang considered the type 2 degenerate poly-Euler polynomials which are given by the generating function to be

$$\frac{\mathrm{Ei}_s(\log(1+2t))}{t(e_\lambda(t)+1)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(s)}(x)\frac{t^n}{n!} \quad \text{(see [13])}. \tag{18}$$

When $x = 0$, $\mathcal{E}_{n,\lambda}^{(s)} = \mathcal{E}_{n,\lambda}^{(s)}(0)$ are called type 2 degenerate poly-Euler numbers.

Since $\mathrm{Ei}_1(\log(1+2t)) = 2t$, we see that $\mathcal{E}_{n,\lambda}^{(1)}(x) = E_{n,\lambda}(x)$ $(n \geq 0)$ are the degenerate Euler polynomials.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{19}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n = \bigl\{ P(x) \in \mathbb{C}[x] \,|\, \deg P(x) \leq n \bigr\} \quad (n \geq 0). \tag{20}$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

Recently, Kim–Kim [11] considered the λ-linear functional and λ-differential operator as follows:

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t)|\cdot\rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n, \quad \text{for all } n \geq 0 \text{ (see [11]).} \tag{21}$$

and in particular $\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k}$, for all $n,k \geq 0$, where $\delta_{n,k}$ is Kronecker's symbol.

For $\lambda = 0$, we observe that the linear functional $\langle f(t)|\cdot\rangle_0$ agrees with the one in $\langle f(t)|x^n\rangle = a_k$, $(k \geq 0)$.

For each $\lambda \in \mathbb{R}$, and each nonnegative integer k, they also defined the differential operator on $\mathbb{P}$ by

$$\left(t^k\right)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0 & \text{if } k \geq n \text{ (see [11]).} \end{cases} \tag{22}$$

and for any power series $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, $(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}$ $(n \geq 0)$.

Note that different λ give rise to different linear functionals on $\mathbb{P}$ (see [11] p. 5, p. 8).

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t)) = 0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t)) = 1$ and it has a compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exist unique sequences $s_{n,\lambda}(x)$ such that we have the orthogonality conditions

$$\left\langle g(t)\left(f(t)\right)^k | s_{n,\lambda}(x)\right\rangle_\lambda = n!\delta_{n,k} \quad (n,k \geq 0) \text{ (see [11]).} \tag{23}$$

The sequences $s_{n,\lambda}(x)$ are called the λ-Sheffer sequences for $(g(t), f(t))$, which are denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g(\bar{f}(t))} e_\lambda^x\left(\bar{f}(t)\right) = \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \quad (n,k \geq 0) \text{ (see [11]).} \tag{24}$$

Assume that, for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \bar{f}_\lambda(t) = \bar{f}(t)$, where is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$. Let $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$. In this case, Kim–Kim called this the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathcal{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), g(t))_\lambda$ $(n \geq 0)$. Then

$$\begin{aligned} s_{n,\lambda}(x) &= \sum_{k=0}^{n} \mu_{n,k} r_{k,\lambda}(x) \quad (n \geq 0), \\ &\text{where } \mu_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}\left(l(\bar{f}(t))\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \quad (n,k \geq 0) \text{ (see [11]).} \end{aligned} \tag{25}$$

2 Degenerate Lah–Bell polynomials arising from degenerate Sheffer sequences

In this section, we derive several identities between the degenerate Lah–Bell polynomials and some other polynomials arising from degenerate Sheffer sequences.

Kim–Kim introduced the degenerate Lah–Bell polynomials given by

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^\infty \mathbf{B}_{n,\lambda}^L(x)\frac{t^n}{n!}\quad (n,k\geq 0)\text{ (see [10])}. \tag{26}$$

When $x=1$, $\mathbf{B}_{n,\lambda}^L := \mathbf{B}_{n,\lambda}^L(1)$ are called the nth degenerate Lah–Bell numbers.

When $\lambda\to 0$, $\lim_{\lambda\to 0}\mathbf{B}_{n,\lambda}^L=\mathbf{B}_n^L$ are the nth Lah–Bell numbers.

For $n\in\mathbb{N}\cup\{0\}$ and $P(x)=\sum_{k=0}^n Z_k\mathbf{B}_{k,\lambda}^L(x)\in\mathbb{P}_n$,

by using (23), we observe that

$$\left\langle\left(\frac{t}{1+t}\right)^k\middle|P(x)\right\rangle_\lambda=\sum_{l=0}^n Z_l\left\langle\left(\frac{t}{1+t}\right)^k\middle|\mathbf{B}_{l,\lambda}^L(x)\right\rangle_\lambda=\sum_{l=0}^n Z_l l!\delta_{k,l}=k!Z_k. \tag{27}$$

From (27), we have

$$Z_k=\frac{1}{k!}\left\langle\left(\frac{t}{1+t}\right)^k\middle|P(x)\right\rangle_\lambda. \tag{28}$$

Thus, we have

$$P(x)=\sum_{k=0}^n Z_k\mathbf{B}_{k,\lambda}^L(x)\quad\text{where } Z_k=\frac{1}{k!}\left\langle\left(\frac{t}{1+t}\right)^k\middle|P(x)\right\rangle_\lambda. \tag{29}$$

Theorem 1 *For $n\in\mathbb{N}\cup\{0\}$, we have*

$$\mathbf{B}_{n,\lambda}^L(x)=\sum_{k=0}^n L(n,k)(x)_{k,\lambda}=\sum_{k=0}^n\left(\frac{1}{k!}\sum_{l=0}^k\binom{k}{l}(-1)^{k-l}\langle l\rangle_n\right)(x)_{k,\lambda}. \tag{30}$$

As the inversion formula of (30), *we have*

$$(x)_{n,\lambda}=\sum_{k=0}^n(-1)^{n-k}L(n,k)\mathbf{B}_{k,\lambda}^L(x)=\sum_{k=0}^n\left(\frac{1}{k!}\sum_{l=0}^k\binom{k}{l}(-1)^{l+n}\langle l\rangle_n\right)\mathbf{B}_{k,\lambda}^L(x). \tag{31}$$

Proof From (5), (24) and (26), we consider the following two Sheffer sequences:

$$\mathbf{B}_{n,\lambda}^L(x)\sim\left(1,\frac{t}{1+t}\right)_\lambda\quad\text{and}\quad(x)_{n,\lambda}\sim(1,t)_\lambda. \tag{32}$$

From (6), (25) and (32), we have

$$\mathbf{B}_{n,\lambda}^L(x)=\sum_{k=0}^n\mu_{n,k}(x)_{k,\lambda}, \tag{33}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{t}{1-t}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda = L(n,k),\\ \text{or}\quad \mu_{n,k} &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\langle \left(\frac{1}{1-t}\right)^l \middle| (x)_{n,\lambda}\right\rangle_\lambda = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\langle l\rangle_n.\end{aligned} \tag{34}$$

Therefore, we have the identity (30).

To find the inversion formula of (30), let $P(x) = (x)_{n,\lambda}$. From (29),

$$(x)_{n,\lambda} = \sum_{k=0}^{n} Z_k \mathbf{B}^L_{k,\lambda}(x), \tag{35}$$

where

$$\begin{aligned}Z_k &= \frac{1}{k!}\left\langle \left(\frac{t}{1+t}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda = (-1)^{n-k}L(n,k),\\ \text{or}\quad Z_k &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l}\left\langle \left(\frac{1}{1+t}\right)^l \middle| (x)_{n,\lambda}\right\rangle_\lambda = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{l+n}\langle l\rangle_n.\end{aligned} \tag{36}$$

Therefore, from (35) and (36), we have the identity (31). □

Theorem 2 *For $n \in \mathbb{N}\cup\{0\}$ and $r \in \mathbb{N}$, we have*

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\frac{(1)_{m+1,\lambda}}{(m+1)}L(l,k)L(n-l,m)\right)\beta_{k,\lambda}(x). \tag{37}$$

As the inversion formula of (37), *we have*

$$\begin{aligned}\beta_{n,\lambda}(x) &= \sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{l=0}^{k}\sum_{m=0}^{n}\binom{k}{l}\binom{l+n-m-1}{n-m}\binom{n}{m}(-1)^{l+n-m}(n-m)!\beta_{m,\lambda}\right)\mathbf{B}^L_{k,\lambda}(x)\\ &= \sum_{k=0}^{n}\left(\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m-k}L(n-m,k)\beta_{m,\lambda}\right)\mathbf{B}^L_{k,\lambda}(x),\end{aligned} \tag{38}$$

where $\beta_{n,\lambda}(x)$ are the degenerate Bernoulli polynomials.

Proof From (7), (24) and (26), we consider the following two degenerate Sheffer sequences:

$$\mathbf{B}^L_{n,\lambda}(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda \quad\text{and}\quad \beta_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t\right)_\lambda. \tag{39}$$

From (2), (5) and (25), we have

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\mu_{n,k}\beta_{k,\lambda}(x), \tag{40}$$

where

$$\begin{aligned}
\mu_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(\frac{t}{1-t})-1}{\frac{t}{1-t}}\right)\left(\frac{t}{1-t}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \left(\frac{e_\lambda(\frac{t}{1-t})-1}{\frac{t}{1-t}}\right) \middle| \left(\frac{1}{k!}\left(\frac{t}{1-t}\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle \left(\frac{e_\lambda(\frac{t}{1-t})-1}{\frac{t}{1-t}}\right) \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m!}\left(\frac{t}{1-t}\right)^{m-1} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}L(l,k)\left\langle \sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)!}\left(\frac{t}{1-t}\right)^{m} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}L(l,k)\sum_{m=0}^{n-l}\frac{(1)_{m+1,\lambda}}{(m+1)}\left\langle \frac{1}{m!}\left(\frac{t}{1-t}\right)^{m} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}L(l,k)\sum_{m=0}^{n-l}\frac{(1)_{m+1,\lambda}}{(m+1)}L(n-l,m).
\end{aligned} \tag{41}$$

To find the inversion formula of (37), let $P(x) = \beta_{n,\lambda}(x)$. From (29), we have

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k \mathbf{B}^L_{k,\lambda}(x) \quad (n \geq 0).$$

where

$$\begin{aligned}
Z_k &= \frac{1}{k!}\left\langle \left(\frac{t}{1+t}\right)^k \middle| \beta_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\left\langle \left(\frac{1}{1+t}\right)^l \middle| \beta_{n,\lambda}(x)\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{\nu=0}^{n}\binom{l+\nu-1}{\nu}(-1)^\nu\left\langle t^\nu \middle| \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{\nu=0}^{n}\binom{l+\nu-1}{\nu}(-1)^\nu\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}\langle t^\nu | (x)_{n-m,\lambda}\rangle_\lambda \\
&= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\binom{l+n-m-1}{n-m}(-1)^{l+n-m}\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(n-m)!.
\end{aligned} \tag{42}$$

Stated differently, we get

$$\begin{aligned}
Z_k &= \frac{1}{k!}\left\langle \left(\frac{t}{1+t}\right)^k \middle| \beta_{n,\lambda}(x)\right\rangle_\lambda \\
&= \left\langle \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \middle| \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)_{n-m,\lambda}\right\rangle_\lambda
\end{aligned} \tag{43}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \beta_{m,\lambda} \left\langle \frac{1}{k!} \left(\frac{t}{1+t} \right)^k \middle| (x)_{n-m,\lambda} \right\rangle_{\lambda} = \sum_{m=0}^{n} \binom{n}{m} \beta_{m,\lambda} (-1)^{n-m-k} L(n-m,k).$$

Therefore, from (42) and (43) we have the identity (38). □

Theorem 3 *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned} &\mathbf{B}_{n,\lambda}^{L}(x) \\ &= \frac{1}{(1-u)^r} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{j=0}^{r} \sum_{m=0}^{n-l} \binom{n}{l} \binom{r}{j} (-u)^{r-j} (j)_{m,\lambda} L(l,k) L(n-l,m) \right) h_{k,\lambda}^{(r)}(x|u). \end{aligned} \tag{44}$$

As the inversion formula of (44), *we have*

$$\begin{aligned} &h_{n,\lambda}^{(r)}(x|u) \\ &= \sum_{k=0}^{n} \left(\frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{n} \binom{k}{l} \binom{n}{m} \binom{l+n-m-1}{n-m} (-1)^{l+n-m} (n-m)! h_{m,\lambda}^{(r)}(u) \right) \mathbf{B}_{k,\lambda}^{L}(x) \\ &= \sum_{k=0}^{n} \left(\sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m-k} L(n-m,k) h_{m,\lambda}^{(r)}(u) \right) \mathbf{B}_{k,\lambda}^{L}(x), \end{aligned} \tag{45}$$

where $h_{n,\lambda}^{(r)}(x|u)$ are the degenerate Frobenius–Euler polynomials of order r.

Proof From (9), (24) and (26), we consider the following two degenerate Sheffer sequences:

$$\mathbf{B}_{n,\lambda}^{L}(x) \sim \left(1, \frac{t}{1+t} \right)_{\lambda} \quad \text{and} \quad h_{n,\lambda}^{(r)}(x|u) \sim \left(\left(\frac{e_\lambda(t)-u}{1-u} \right)^r, t \right)_{\lambda}. \tag{46}$$

By using (2), (5), (25) and (46), we have

$$\mathbf{B}_{n,\lambda}^{L}(x) = \sum_{k=0}^{n} \mu_{n,k} h_{k,\lambda}^{(r)}(x|u), \tag{47}$$

Here

$$\begin{aligned} \mu_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{(e_\lambda(\frac{t}{1-t})-u)}{1-u} \right)^r \left(\frac{t}{1-t} \right)^k \middle| (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= \left\langle \left(\frac{(e_\lambda(\frac{t}{1-t})-u)}{1-u} \right)^r \middle| \left(\frac{1}{k!} \left(\frac{t}{1-t} \right)^k \right)_{\lambda} (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= \frac{1}{(1-u)^r} \sum_{l=k}^{n} \binom{n}{l} L(l,k) \left\langle \left(e_\lambda \left(\frac{t}{1-t} \right) - u \right)^r \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda} \\ &= \frac{1}{(1-u)^r} \sum_{l=k}^{n} \binom{n}{l} L(l,k) \sum_{j=0}^{r} \binom{r}{j} (-u)^{r-j} \left\langle e_\lambda^j \left(\frac{t}{1-t} \right) \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda} \end{aligned} \tag{48}$$

$$= \frac{1}{(1-u)^r} \sum_{l=k}^{n} \binom{n}{l} L(l,k) \sum_{j=0}^{r} \binom{r}{j} (-u)^{r-j} \sum_{m=0}^{n-l} (j)_{m,\lambda} \left\langle \frac{1}{m!} \left(\frac{t}{1-t} \right)^m \middle| (x)_{n-l,\lambda} \right\rangle_\lambda$$

$$= \frac{1}{(1-u)^r} \sum_{l=k}^{n} \binom{n}{l} L(l,k) \sum_{j=0}^{r} \binom{r}{j} (-u)^{r-j} \sum_{m=0}^{n-l} (j)_{m,\lambda} L(n-l,m).$$

Therefore, from (47) and (48), we get the identity (44).

To find the inversion formula of (44), by (29), we have

$$h_{n,\lambda}^{(r)}(x|u) = \sum_{k=0}^{n} Z_k \mathbf{B}_{k,\lambda}^{L}(x).$$

In the same way as (42) and (43), we have

$$\begin{aligned} Z_k &= \frac{1}{k!} \left\langle \left(\frac{t}{1+t} \right)^k \middle| h_{n,\lambda}^{(r)}(x|u) \right\rangle_\lambda \\ &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \left\langle \left(\frac{1}{1+t} \right)^l \middle| h_{n,\lambda}^{(r)}(x|u) \right\rangle_\lambda \\ &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \sum_{\nu=0}^{n} \binom{l+\nu-1}{\nu} (-1)^\nu \left\langle t^\nu \middle| \sum_{m=0}^{n} \binom{n}{m} h_{m,\lambda}^{(r)}(u) (x)_{n-m,\lambda} \right\rangle_\lambda \\ &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \sum_{\nu=0}^{n} \binom{l+\nu-1}{\nu} (-1)^\nu \sum_{m=0}^{n} \binom{n}{m} h_{m,\lambda}^{(r)}(u) \langle t^\nu | (x)_{n-m,\lambda} \rangle_\lambda \\ &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} \binom{l+n-m-1}{n-m} (-1)^{l+n-m} \sum_{m=0}^{n} \binom{n}{m} h_{m,\lambda}^{(r)}(u) (n-m)!. \end{aligned} \tag{49}$$

In another way, we can get

$$Z_k = \frac{1}{k!} \left\langle \left(\frac{t}{1+t} \right)^k \middle| h_{n,\lambda}^{(r)}(x|u) \right\rangle_\lambda = \sum_{m=0}^{n} \binom{n}{m} h_{m,\lambda}^{(r)}(u) (-1)^{n-m-k} L(n-m,k). \tag{50}$$

Therefore, from (49) and (50), we have the identity (45). □

When $u = -1$ in Theorem 3, we have the following corollary.

Corollary 4 *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$\mathbf{B}_{k,\lambda}^{L}(x) = \frac{1}{2^r} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{j=0}^{r} \sum_{m=0}^{n-l} \binom{n}{l} \binom{r}{j} (j)_{m,\lambda} L(l,k) L(n-l,m) \right) E_{k,\lambda}^{(r)}(x). \tag{51}$$

By the inversion formula of (51), we have

$$E_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \left(\frac{1}{k!} \sum_{l=0}^{k} \sum_{m=0}^{n} \binom{k}{l} \binom{n}{m} \binom{l+n-m-1}{n-m} (-1)^{l+n-m} (n-m)! E_{m,\lambda}^{(r)} \right) \mathbf{B}_{k,\lambda}^{L}(x),$$

where $E_{n,\lambda}^{(r)}(x)$ are the degenerate Euler polynomials of order r.

Theorem 5 *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$\mathbf{B}_{n,\lambda}^{L}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{n}{m}\frac{(1)_{j+1,\lambda}}{j+1}S_{2,\lambda}(l,k)L(m,l)L(n-m,j)\right)D_{k,\lambda}(x). \tag{52}$$

As the inversion formula of (52), we have

$$D_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{n}{m}(-1)^{n-m-k}S_{1,\lambda}(n-m,j)L(n-m,k)D_{m,\lambda}\right)\mathbf{B}_{k,\lambda}^{L}(x), \tag{53}$$

where $D_{n,\lambda}(x)$ are the degenerate Daehee polynomials.

Proof From (10) and (24), we consider the following two degenerate Sheffer sequences:

$$\mathbf{B}_{n,\lambda}^{L}(x) \sim \left(1, \frac{t}{1+t}\right)_{\lambda} \quad \text{and} \quad D_{n,\lambda}(x) \sim \left(\frac{e_{\lambda}(t)-1}{t}, e_{\lambda}(t)-1\right)_{\lambda}. \tag{54}$$

From (2), (16), (25) and (54), we have

$$\mathbf{B}_{n,\lambda}^{L}(x) = \sum_{k=0}^{n}\mu_{n,k}D_{k,\lambda}(x), \tag{55}$$

where

$$\begin{aligned}
\mu_{n,k} &= \frac{1}{k!}\left\langle \frac{e_{\lambda}(\frac{t}{1-t})-1}{\frac{t}{1-t}}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)^{k}\middle|(x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \left\langle \frac{1-t}{t}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)\frac{1}{k!}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)^{k}\middle|(x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\left\langle \frac{1-t}{t}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)\middle|\left(\frac{1}{l!}\left(\frac{t}{1-t}\right)^{l}\right)_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\sum_{m=0}^{n}\binom{n}{m}L(m,l)\left\langle \frac{1-t}{t}\left(e_{\lambda}\left(\frac{t}{1-t}\right)-1\right)\middle|(x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\sum_{m=0}^{n}\binom{n}{m}L(m,l)\left\langle \sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{1}{j!}\left(\frac{t}{1-t}\right)^{j-1}\middle|(x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\sum_{m=0}^{n}\binom{n}{m}L(m,l)\left\langle \sum_{j=0}^{\infty}(1)_{j+1,\lambda}\frac{1}{(j+1)!}\left(\frac{t}{1-t}\right)^{j}\middle|(x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\sum_{m=0}^{n}\binom{n}{m}L(m,l)\sum_{j=0}^{n-m}\frac{(1)_{j+1,\lambda}}{j+1}\left\langle \frac{1}{j!}\left(\frac{t}{1-t}\right)^{j}\middle|(x)_{n-m,\lambda}\right\rangle_{\lambda} \\
&= \sum_{l=k}^{n}S_{2,\lambda}(l,k)\sum_{m=0}^{n}\binom{n}{m}L(m,l)\sum_{j=0}^{n-m}\frac{(1)_{j+1,\lambda}}{j+1}L(n-m,j).
\end{aligned} \tag{56}$$

Therefore, from (55) and (56), we get the identity (52).

To find the inversion formula of (52), from (29), we have

$$D_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k B^L_{k,\lambda}(x). \tag{57}$$

By using $(1+t)^x = \sum_{n=0}^{\infty}(x)_n \frac{t^n}{n!}$ and the first equation of (15), we have

$$\begin{aligned}
Z_k &= \left\langle \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \middle| D_{n,\lambda}(x)\right\rangle_\lambda \\
&= \left\langle \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \middle| \sum_{m=0}^{n}\binom{n}{m} D_{m,\lambda}(x)_{n-m}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\binom{n}{m} D_{m,\lambda}\left\langle \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \middle| \sum_{j=0}^{n-m} S_{1,\lambda}(n-m,j)(x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\binom{n}{m} D_{m,\lambda}\sum_{j=0}^{n-m} S_{1,\lambda}(n-m,j)\left\langle \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \middle| (x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\binom{n}{m} D_{m,\lambda}\sum_{j=0}^{n-m} S_{1,\lambda}(n-m,j)(-1)^{n-m-k}L(n-m,k).
\end{aligned} \tag{58}$$

Therefore, from (57) and (58), we have the identity (53). □

Theorem 6 *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} S_{1,\lambda}(l,k)L(n,k)\right)\mathrm{Bel}_{k,\lambda}(x). \tag{59}$$

As the inversion formula of (59), *we have*

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=0}^{n}(-1)^{l-k} S_{2,\lambda}(n,l)L(l,k)\right)\mathbf{B}^L_{k,\lambda}(x), \tag{60}$$

where $\mathrm{Bel}_{n,\lambda}(x)$ are the degenerate Bell polynomials.

Proof From (12), (24) and (26), we consider two degenerate Sheffer sequences as follows:

$$\mathbf{B}^L_{n,\lambda}(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad \mathrm{Bel}_{n,\lambda}(x) \sim \big(1, \log_\lambda(1+t)\big)_\lambda. \tag{61}$$

By using (2), (15), (25) and (61), we have

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k}\mathrm{Bel}_{k,\lambda}(x), \tag{62}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle\left(\log_\lambda\left(1+\frac{t}{1-t}\right)\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n} S_{1,\lambda}(l,k)\left\langle\frac{1}{l!}\left(\frac{t}{1-t}\right)^l\middle|(x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=k}^{n} S_{1,\lambda}(l,k)L(n,l).\end{aligned} \tag{63}$$

Therefore from (62) and (63), we get the identity (59).

To find inversion formula of (59), from (29), we have

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k B^L_{k,\lambda}(x). \tag{64}$$

From (5) and (16), we observe that

$$\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!} = e_\lambda^x\big(e_\lambda(t)-1\big) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_{l,\lambda}\right)\frac{t^n}{n!}.$$

Thus, by using $\mathrm{Bel}_{n,\lambda}(x) = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_{l,\lambda}$ and (6), we have

$$\begin{aligned}Z_k &= \frac{1}{k!}\left\langle\left(\frac{t}{1+t}\right)^k\middle|\mathrm{Bel}_{n,\lambda}(x)\right\rangle_\lambda \\ &= \left\langle\frac{1}{k!}\left(\frac{t}{1+t}\right)^k\middle|\sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_{l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=0}^{n} S_{2,\lambda}(n,l)\left\langle\frac{1}{k!}\left(\frac{t}{1+t}\right)^k\middle|(x)_{l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=0}^{n} S_{2,\lambda}(n,l)(-1)^{l-k}L(l,k).\end{aligned} \tag{65}$$

Therefore, from (64) and (65), we have the identity (60). □

Theorem 7 *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n} S_{2,\lambda}(l,k)L(n,l)\right)(x)_n \quad (n \geq 0). \tag{66}$$

Proof Since $e_\lambda^x(\log(1+t)) = (1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$, we have $(x)_n \sim (1, e_\lambda(t)-1)_\lambda$.

Therefore, we consider the two degenerate Sheffer sequences as follows:

$$\mathbf{B}^L_{n,\lambda}(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad (x)_n \sim \big(1, e_\lambda(t)-1\big)_\lambda. \tag{67}$$

Thus, from (2), (16) and (67), we have

$$\mathbf{B}^L_{n,\lambda}(x) = \sum_{k=0}^{n}\mu_{n,k}(x)_k \quad (n \geq 0), \tag{68}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle \left(e_\lambda\left(\frac{t}{1-t}\right)-1\right)^k \Big| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n} S_{2,\lambda}(l,k)\left\langle \frac{1}{l!}\left(\frac{t}{1-t}\right)^l \Big| (x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=k}^{n} S_{2,\lambda}(l,k)L(n,l).\end{aligned} \tag{69}$$

Therefore, from (68) and (69), we have the identity (66). □

Theorem 8 *For $n \in \mathbb{N}\cup\{0\}$ and $s \in \mathbb{N}$, we have*

$$\mathcal{E}_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}(-1)^{l-k}L(l,k)\mathcal{E}_{n-l,\lambda}^{(s)}\right)\mathbf{B}_{k,\lambda}^{L}(x), \tag{70}$$

where $\mathcal{E}_{n,\lambda}^{(s)}(x)$ are type 2 degenerate poly-Euler polynomials.

Proof From (18), (24) and (26), we consider the following two degenerate Sheffer sequences as follows:

$$\mathbf{B}_{n,\lambda}^{L}(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad \mathcal{E}_{n,\lambda}^{(s)}(x) \sim \left(\frac{t(e_\lambda(t)+1)}{\mathrm{Ei}_s(\log(1+2t))}, t\right)_\lambda. \tag{71}$$

By using (2) and (25), we have

$$\mathcal{E}_{n,\lambda}^{(s)}(x) = \sum_{k=0}^{n}\mu_{n,k}\mathbf{B}_{k\lambda}^{L}(x), \tag{72}$$

where

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle \frac{\mathrm{Ei}_s(\log(1+2t))}{t(e_\lambda(t)+1)}\left(\frac{t}{1+t}\right)^k \Big| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \frac{\mathrm{Ei}_s(\log(1+2t))}{t(e_\lambda(t)+1)} \Big| \left(\frac{1}{k!}\left(\frac{t}{1+t}\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}(-1)^{l-k}L(l,k)\left\langle \frac{\mathrm{Ei}_s(\log(1+2t))}{t(e_\lambda(t)+1)} \Big| (x)_{n-l,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}\binom{n}{l}(-1)^{l-k}L(l,k)\mathcal{E}_{n-l,\lambda}^{(s)}.\end{aligned} \tag{73}$$

Therefore, from (72) and (73), we get the identity (70). □

3 Conclusion

The author represented the degenerate Lah–Bell polynomials in terms of quite a few well-known special polynomials and at the same time derived the inversion formulas of those identities by using the degenerate Sheffer sequences. We addressed the special polynomials and numbers: the degenerate falling factorial, the Lah numbers and the degenerate

Bernoulli polynomials; the Lah numbers and the degenerate Frobenius–Euler polynomials of order r; the Lah numbers and the degenerate Deahee polynomials; the Lah numbers and the degenerate Bell polynomials; the Lah numbers and the type 2 degenerate poly Euler polynomials. Therefore, the paper demonstrates that degenerate versions are not only applicable for number theory and combinatorics but also to symmetric identities, differential equations and probability theory. Building upon this, the author would like to further study into degenerate versions of certain special polynomials and numbers and their applications to physics, economics and engineering as well as mathematics.

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