Al-Omari *Advances in Difference Equations* (2021) 2021:10
https://doi.org/10.1186/s13662-020-03147-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The q-Sumudu transform and its certain properties in a generalized q-calculus theory

Shrideh Khalaf Al-Omari[1*]

*Correspondence:
s.k.q.alomari@fet.edu.jo;
shridehalomari@bau.edu.jo
[1]Faculty of Engineering Technology, Department of Physics and Basic Sciences, Al-Balqa Applied University, 11134 Amman, Jordan

Abstract

In this paper we consider a generalization to the q-calculus theory in the space of q-integrable functions. We introduce q-delta sequences and develop q-convolution products to derive certain q-convolution theorem. By using the concept of q-delta sequences, we establish various axioms and set up q-spaces of generalized functions named q-Boehmian spaces. The new assigned spaces of q-generalized functions are acceptable and compatible with the classical spaces of the ordinary functions. Consequently, we extend the generalized q-Sumudu transform to the sets of q-Boehmian spaces. On top of that, we nominate the canonical q-embeddings between the q-integrable sets of functions and the q-integrable sets of q-Boehmians. Furthermore, we address the general properties of the generalized q-Sumudu transform and its inversion formula in some detail.

MSC: Primary 54C40; 14E20; secondary 46E25; 20C20

Keywords: q-delta sequences; q-Sumudu transform; q-convolution; q-Boehmian; q-calculus

1 Preliminaries

The subject of fractional calculus has attained an eminent concern during the past decades due to various applications of this subject in various fields of science and engineering. Recently, an increase of interest in this area has duly been implemented and utilized in the theory of ordinary fractional calculus, optimal control problems, q-transform analysis, statistics, mathematical physics, q-difference equations, and q-integral equations (see, e.g., [1, 2]). By fixing a real number q such that $0 < q < 1$, the q-derivative of a differentiable function ϑ is defined by

$$D_q\vartheta(x) = \frac{\vartheta(x) - \vartheta(qx)}{(1-q)x} \quad (x \neq 0).$$

The q-integrals from 0 to y and from 0 to ∞ have been respectively defined by Jackson [3] as follows:

$$I_q\vartheta(y) = \int_0^y \vartheta(x)\, d_qx = (1-q)y\sum_{n=0}^{\infty}\vartheta\left(yq^n\right)q^n$$

and

$$\int_0^\infty \vartheta(x)\,d_qx = (1-q)\sum_{n=-\infty}^{\infty}\vartheta\left(q^n\right)q^n,$$

provided the two sums converge absolutely. The q-integration by parts has been defined by

$$\int_a^b g(x)D_q\vartheta(x)\,d_qx = \vartheta(b)g(b)-\vartheta(a)g(a)-\int_a^b \vartheta(qx)D_qg(x)\,d_qx.$$

Ever after Jackson [3] presented the q-integral concept, various q-analogues of various types of integral transforms were given in a classical way (see, e.g., [4–19]). Consisting of the notion of regular operators [20], the theory of Boehmians was first introduced by [21] to generalize distributions and regular operators [22] (see, e.g., [21, 23–31]). Boehmians with their abstract nature are equivalence classes of quotients of sequences obtained from an integral domain, where the operations are addition and convolution. Complying with the q-calculus concept, we introduce the q-Boehmian concept as follows: Let A be a complex linear space and B be a subspace of A. Let $\bullet^q : A\times B\to A$ be a binary operation such that, for all $\alpha\in\mathbb{C}$, $\hat{\delta},\check{\delta},\tilde{\delta}\in A$ and $\gamma_1,\gamma_2,\gamma\in B$, we have $(\hat{\delta}+\check{\delta})\bullet^q\gamma=\hat{\delta}\bullet^q\gamma+\check{\delta}\bullet^q\gamma$, $(\alpha\hat{\delta})\bullet^q\gamma=\alpha(\hat{\delta}\bullet^q\gamma)$, $\hat{\delta}\bullet^q(\gamma_1\bullet^q\gamma_2)=(\hat{\delta}\bullet^q\gamma_1)\bullet^q\gamma_2$, $\tilde{\delta}_n\bullet^q\gamma\to\tilde{\delta}\bullet^q\gamma$ as $\tilde{\delta}_n\to\tilde{\delta}$ in A as $n\to\infty$ and, for all $(x_n),(y_n)\in\Delta_q$, we have $x_n\bullet^q y_n\in\Delta_q$, where Δ_q is a collection of sequences in B and $x_n\bullet^q\tilde{\delta}_n\to\tilde{\delta}$ as $n\to\infty$ provided $\tilde{\delta}_n\to\tilde{\delta}$ in A as $n\to\infty$. The name of q-Boehmian is proposed to mean the equivalence class $\frac{\tilde{\delta}_n}{x_n}$ obtained from the equivalence relation

$$\tilde{\delta}_n\bullet^q x_n=\tilde{\delta}_m\bullet^q x_n\quad(m,n\in\mathbb{N}),$$

where $(\tilde{\delta}_n)\in A$ and $(x_n)\in\Delta_q$. The collection of all q-Boehmians is claimed to form a q-Boehmian space denoted by B_q. The linear space A is identified as a subspace of the q-Boehmian space B_q justified by the identification formula

$$\check{\delta}\to\frac{\check{\delta}\bullet^q x_n}{x_n},\quad\text{where }\check{\delta}\in A\text{ and }(x_n)\in\Delta_q.$$

Two q-Boehmians $\frac{\hat{\delta}_n}{x_n}$ and $\frac{\tilde{\delta}_n}{y_n}$ are said to be equal in B_q if $\hat{\delta}_n\bullet^q x_m=\tilde{\delta}_m\bullet^q x_m$, $\forall m,n\in\mathbb{N}$. The addition is defined in B_q as

$$\frac{\hat{\delta}_n}{x_n}+\frac{\tilde{\delta}_n}{y_n}=\frac{\hat{\delta}_n\bullet^q y_n+\tilde{\delta}_n\bullet^q x_n}{x_n\bullet^q y_n}.$$

The scalar multiplication is defined in B_q as

$$\alpha\frac{\tilde{\delta}_n}{x_n}=\frac{\alpha\tilde{\delta}_n}{x_n}\quad\text{for all }\alpha\in\mathbb{C}.$$

For every $(x_n)\in\Delta_q$, convergence of type δ_q, $\beta_n\overset{\delta_q}{\to}\beta$, is defined in B_q when

$$\beta_n\bullet^q x_k\in A,\forall k,n\in\mathbb{N},\beta\bullet^q x_k\in A,\forall k\in\mathbb{N},$$

and for each $k \in \mathbb{N}$, $\beta_n \bullet^q x_k \to \beta \bullet^q x_k$ as $n \to \infty$ in A, whereas, convergence of type Δ_q, $\beta_n \stackrel{\Delta_q}{\to} \beta$, is defined in B_q if for some $(x_n) \in \Delta_q$, $(\beta_n - \beta) \bullet^q x_n \in A$, $\forall n \in \mathbb{N}$ and

$$(\beta_n - \beta) \bullet^q x_n \to 0 \quad \text{as } n \to \infty \text{ in } A.$$

The space of q-Boehmians furnished with the current convergence notions emerges to be a complete quasi-normed space. Over the set of functions

$$A = \left\{ \check{\delta}(t) : \exists M, \tau_1, \tau_2 > 0, \left|\check{\delta}(t)\right| < M E_q\left(\frac{|t|}{\tau_j}\right), t \in (-1)^j \times [0, \infty) \right\},$$

the q-analogue of the Sumudu transform of the first type was latterly defined by [8, (1.18)] as follows:

$$S_q\left\{\check{\delta}(t); s\right\} = \frac{1}{(1-q)s} \int_0^s E_q\left(\frac{q}{s} t\right) \check{\delta}(t) \, d_q t, \quad s \in (-\tau_1, \tau_2). \tag{1}$$

The properties of q-analogue S_q of the Sumudu transform including the convergence conditions and its relation with the q-Laplace integral transform have been derived by Albayrak et al. [8]. Over and above, the authors investigated certain fundamental aspects of the cited integral enfolding linearity, shifting theorems, differentiation, integration, etc. Also an attempt has been made to obtain the convolution theorem in a convergent series type. On the other hand, the authors in [6] provided some applications of the q-Sumudu transform to q-polynomials, q-functions, and q-Fox's H-functions as well.

The aimed goal of this paper is to discuss the generalized q-theory of the q-integrable functions in the space L_q^1 and to investigate fundamental properties of the q-analogue (1) in the generalized q-theory. In Sect. 2, we introduce a concept of q-Boehmians and a concept of q-delta sequences. We also establish the space B_q^1 of q-Boehmians. In Sect. 3, we establish the second space B_q^2 of q-Boehmians. In Sect. 4, we introduce the generalized q-Sumudu transform and discuss several properties. In Sect. 5, we provide a conclusion part.

2 The space B_q^1

Denote by L_q^1 the space of all q-integrable functions $\check{\delta}$ on $\mathbb{R}_+$ defined by

$$\|\check{\delta}\|_{L_q^1(\mathbb{R}_+)} = \frac{1}{1-q} \int_0^\infty \left|\check{\delta}(x)\right| d_q x < \infty, \tag{2}$$

whose comparable definition in a series expression formula is given as $\sum_{-\infty}^{\infty} q^n \check{\delta}(q^n)$, provided the series converges absolutely. Denote by D_q the q-space of all test functions of compact supports on $\mathbb{R}_+$, i.e.,

$$D_q = \left\{ \check{\delta} \in C^\infty(\mathbb{R}) : \sup_{0<x<\infty} \left|D_q \check{\delta}(x)\right| < \infty \right\}.$$

Denote by Δ_q the set of all sequences from D_q such that $\Delta_q^1 - \Delta_q^3$ satisfy

$$\Delta_q^1 : \int_0^\infty \left|x_n(x)\right| d_q x = 1, \quad \forall n \in \mathbb{N}.$$

$$\Delta_q^2: |x_n(x)| < M, \quad M > 0, M \in \mathbb{R}_+.$$
$$\Delta_q^3: \operatorname{supp}(x_n) \subseteq (0, b_n), b_n \to 0 \quad \text{as } n \to \infty, \forall n \in \mathbb{N}.$$

Between two integrable functions $\check{\delta}$ and $\hat{\delta}$ in L_q^1, we denote by $*^q$ the q-convolution product defined by

$$\left(\check{\delta} *^q \hat{\delta}\right)(x) = \int_0^x \check{\delta}(x-t)\hat{\delta}(t)\, d_q t, \tag{3}$$

provided the right-hand side integral exists for every real number $x > 0$. It is clear from the context that $\check{\delta} *^q \hat{\delta} \in L_q^1$ for all $\check{\delta}$ and $\hat{\delta}$ in L_q^1. Following Belgacem [32], the q-convolution theorem of the q-Sumudu transform of the convolution $\check{\delta} *^q \hat{\delta}$ can be easily established by using [21, (2.1)] (see, also [11]) as follows:

$$S_q\left(\check{\delta} *^q \hat{\delta}\right)(x) = x S_q\check{\delta}(x) S_q\hat{\delta}(x). \tag{4}$$

An imperative result for categorizing the q-delta sequences may be introduced as follows.

Lemma 1 *Let (x_n) and (y_n) be arbitrary in Δ_q. Then the sequence $(x_n *^q y_n)$ is a q-delta sequence in Δ_q.*

Proof To establish this lemma, we examine that Δ_q^1, Δ_q^2, and Δ_q^3 satisfy for $(x_n *^q y_n)$. To examine the correctness of Δ_q^1, we use integral Eq. (3) to write

$$\int_0^\infty \left(x_n *^q y_n\right)(x)\, d_q x = \int_0^\infty \left(\int_0^x x_n(x-t) y_n(t)\, d_q t\right) d_q x.$$

Now we change the order of integration to have

$$\int_0^\infty \left(x_n *^q y_n\right)(x)\, d_q x = \int_0^\infty \int_t^\infty x_n(x-t) y_n(t)\, d_q t\, d_q x.$$

By changing the variables in the inner integral, i.e., substituting the change of variables $x - t = y$, hence $d_q x = d_q y$, the above equation reveals

$$\int_0^\infty \left(x_n *^q y_n\right)(x)\, d_q x = \int_0^\infty y_n(t) x_n(y)\, d_q t\, d_q y = \int_0^\infty y_n(t)\, d_q t \int_0^\infty x_n(y)\, d_q y = 1.$$

This proves Δ_q^1. The proof of Δ_q^2 follows from the fact that $|x_n *^q y_n| \le |x_n||y_n|$. The proof of Δ_q^3 follows from the fact that $\operatorname{supp}(x_n *^q y_n) \subset \operatorname{supp}(x_n) + \operatorname{supp}(y_n)$ for $(x_n), (y_n) \in \Delta_q$. This completes the proof of the lemma. □

Hence, Lemma 1 shows that every sequence in Δ_q forms an appropriate q-delta sequence.

Lemma 2 *Let $\check{\delta}$ and $\hat{\delta}$ be in L_q^1. Then, for every $\gamma, \hat{\gamma} \in D_q$ and $\alpha \in \mathbb{C}$, the following assertions are valid*:

(i) $\gamma *^q \hat{\gamma} = \hat{\gamma} *^q \gamma$, (ii) $(\check{\delta} + \hat{\delta}) *^q \gamma = \check{\delta} *^q \gamma + \hat{\delta} * \gamma$,

(iii) $(\alpha\check{\delta}) *^q \gamma = \alpha\left(\check{\delta} *^q \gamma\right)$, (iv) $\check{\delta} *^q \left(\gamma *^q \hat{\gamma}\right) = \left(\check{\delta} *^q \gamma\right) *^q \hat{\gamma}$.

Proof (i) By using the definition of the convolution product $*^q$ given by (3) and applying the change of variables $y = x - t$, hence $d_q y = -d_q t$, we get

$$\begin{aligned}\left(\gamma *^q \hat{\psi}\right)(x) &= \int_0^x \gamma(x-t)\hat{\gamma}(t)\, d_q t \\ &= \int_x^0 \gamma(y)\hat{\gamma}(x-y)(-1)\, d_q y \\ &= \int_0^x \hat{\gamma}(x-y)\gamma(y)\, d_q y.\end{aligned}$$

That is,

$$\left(\gamma *^q \hat{\psi}\right)(x) = \left(\hat{\gamma} *^q \gamma\right)(x). \tag{5}$$

Proof of (ii) and (iii) follows from simple integral calculus. The proof of (iv) follows from a similar argument to that of (i). This completes the proof of the lemma. □

Lemma 3 *Let $\check{\delta}$ and $(\check{\delta}_n)$ be sequences of integrable functions in the space L_q^1 such that $\check{\delta}_n \to \check{\delta}$ as $n \to \infty$. Then we have*

$$\check{\delta}_n *^q \gamma \to \check{\delta} *^q \gamma \quad \text{as } n \to \infty \tag{6}$$

for every $\gamma \in D_q$.

The proof of this lemma follows from simple integration. Hence, we delete the details.

Finally, we have to establish the following lemma.

Lemma 4 *Let $\check{\delta}$ and $\hat{\delta}$ be arbitrary functions in L_q^1 and (x_n) be in Δ_q such that $\check{\delta} *^q x_n = \hat{\delta} *^q x_n$. Then we have $\check{\delta} = \hat{\delta}$ in L_q^1 for every $n \in \mathbb{N}$.*

Proof The proof of this lemma follows from Eq. (2) and Lemma 3. Thus, we omit the details. □

Lemma 5 *Let $\check{\delta}$ be an integrable function in L_q^1, then, for every (x_n) in Δ_q, we have $\check{\delta} *^q x_n \to \check{\delta}$ as $n \to \infty$.*

Proof By the assumption that Δ_q^1 and Δ_q^3 hold for (x_n) and allowing the support of x_n to be included in the interval $(0, b_n)$, for some real numbers b_n, $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\left\|\check{\delta} *^q x_n - \check{\delta}\right\|_{L_q^1(\mathbb{R}_+)} &= \frac{1}{1-q}\int_0^\infty \left|\left(\check{\delta} *^q x_n\right)(x) - \check{\delta}(x)\right| d_q x \\ &\le \frac{1}{1-q}\int_0^\infty \int_0^x \left|\check{\delta}(x-t) - \check{\delta}(x)\right|\left|x_n(t)\right| d_q t\, d_q x \\ &= \frac{1}{1-q}\int_0^\infty \int_0^{b_n} \left|\check{\delta}(x-t) - \check{\delta}(x)\right|\left|x_n(t)\right| d_q t\, d_q x.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left\|\check{\delta} *^q x_n - \check{\delta}\right\|_{L_q^1(\mathbb{R}_+)} \leq{}& \frac{1}{1-q}\int_0^\infty \int_0^{b_n} \left|\check{\delta}(x-t)\right|\left|x_n(t)\right| d_q t\, d_q x \\ &+ \frac{1}{1-q}\int_0^\infty \int_0^{b_n} \left|\check{\delta}(x)\right|\left|x_n(t)\right| d_q t\, d_q x.\end{aligned}$$

Since $\check{\delta} \in L_q^1$, the above equation can be developed to give

$$\left\|\check{\delta} *^q x_n - \check{\delta}\right\|_{L_q^1(\mathbb{R}_+)} \leq A\frac{1}{1-q}\int_0^{b_n} \left|x_n(t)\right| d_q t + A\frac{1}{1-q}\int_0^{b_n} \left|x_n(t)\right| d_q t \tag{7}$$

for some constant A. Hence, by applying Δ_q^2 and integrating, from Eq. (7) we get

$$\left\|\check{\delta} *^q x_n - \check{\delta}\right\|_{L_q^1(\mathbb{R}_+)} \leq AMb_n + BMb_n \to 0$$

as $n \to \infty$, where M is a certain positive constant. Hence the proof of this lemma is completed. □

Therefore, the space B_q^1 is defined with the sets $(L_q^1, *^q)$, $(D_q, *^q)$, and Δ_q. The canonical q-embedding of the space L_q^1 in the space B_q^1 is given as

$$\check{\delta} \to \frac{\check{\delta} *^q x_n}{x_n}. \tag{8}$$

Therefore, every $\check{\delta}$ in L_q^1 can be identified in B_q^1 as $\frac{\check{\delta} *^q x_n}{x_n}$. In addition, scalar multiplication, differentiation, Δ_q convergence, and δ_q convergence are defined in a natural way. If $(\check{\delta}_n) \in L_q^1$ and $(x_n) \in \Delta_q$, then the pair $(\check{\delta}_n, x_n)$ (or $\frac{\check{\delta}_n}{x_n}$) is said to be a quotient of sequences if $\check{\delta}_n *^q x_m = \check{\delta}_m *^q x_n$, $\forall n, m \in \mathbb{N}$. Therefore, if $\frac{\check{\delta}_n}{x_n}$ and $\frac{\hat{\delta}_n}{y_n}$ are quotients of sequences and $\check{\delta} \in L_q^1$, then it is easy to see that

$$\frac{\check{\delta} *^q x_n}{x_n} \quad \text{and} \quad \frac{\check{\delta}_n *^q x_n + \hat{\delta}_n *^q x_n}{x_n *^q y_n}$$

are also quotients of sequences. Further, the following equivalence relations can be easily checked:

$$\frac{\check{\delta}_n}{x_n *^q \check{\delta}} \sim \frac{\check{\delta}_n *^q \check{\delta}}{x_n} \quad \text{and} \quad \frac{\check{\delta}_n}{x_n *^q \hat{\delta}_n} \sim \frac{\check{\delta}_n *^q \hat{\delta}_n}{x_n}.$$

Two quotients of sequences $\frac{\check{\delta}_n}{x_n}$ and $\frac{\hat{\delta}_n}{y_n}$ are said to be equivalent if $\check{\delta}_n *^q y_m = \hat{\delta}_m *^q x_n$, $\forall n, m \in \mathbb{N}$. The equivalent class $\breve{w}_B = (\frac{\check{\delta}_n}{x_n})$ of quotients of sequences containing $\frac{\check{\delta}_n}{x_n}$ is said to be a q-Boehmian. The space of such q-Boehmians is denoted by B_q^1.

For two q-Boehmians $\breve{w}_B = \frac{\check{\delta}_n}{x_n}$ and $\check{z}_B = \frac{\hat{\delta}_n}{y_n}$ in B_q^1, the following are well defined on B_q^1:

$$\text{(i)}\quad \breve{w}_B + \check{z}_B = \frac{\check{\delta}_n *^q x_n + \hat{\delta}_n *^q \delta_n}{x_n *^q y_n}, \qquad \text{(ii)}\quad \beta \breve{w}_B = \frac{\beta\check{\delta}_n}{x_n},$$

$$\text{(iii)}\quad \breve{w}_B *^q \breve{z}_B = \frac{\breve{\delta}_n *^q \hat{\delta}_n}{x_n *^q y_n}, \qquad \text{(iv)}\quad D^k \breve{w}_B = \frac{D^k \breve{\delta}_n}{x_n}, \quad \text{and} \quad \text{(v)}\quad \breve{w}_B *^q \breve{\delta} = \frac{\breve{\delta}_n *^q \breve{\delta}}{x_n},$$

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$, $D^k \breve{w}_B$ is the kth derivative of $\breve{w}_B$ and $\breve{\delta} \in L_q^1$.

Definition 6 For $n = 1, 2, 3, \ldots$ and $\breve{w}_{B,n}, \breve{w}_B \in B_q^1$, the sequence $(\breve{w}_{B,n})$ is said to be δ_q-convergent to $\breve{w}_B$, denoted by $\delta_q - \lim_{n\to\infty} \breve{w}_{B,n} = \breve{w}_B$, provided there can be found a delta sequence (x_n) such that

(a) $(\breve{w}_{B,n} *^q x_k)$, $(\breve{w}_B *^q x_k)$ in L_q^1 for all $n, k \in \mathbb{N}$;
(b) $\lim_{n\to\infty} \breve{w}_{B,n} *^q x_k = \breve{w}_B *^q x_k$ in L_q^1 for every $k \in \mathbb{N}$.

Definition 7 For $n = 1, 2, 3, \ldots$ and $\breve{w}_{B,n}, \breve{w}_B \in B_q^1$, the sequence $(\breve{w}_{B,n})$ is said to be Δ_q-convergent to $\breve{w}_B$, denoted by $\Delta_q\text{-}\lim_{n\to\infty} \breve{w}_{B,n} = \breve{w}_B$, provided there can be found a delta sequence (x_n) such that

(a) $(\breve{w}_{B,n} - \breve{w}_B) *^q x_n \in L_q^1 (\forall n \in \mathbb{N})$;
(b) $\lim_{n\to\infty} (\breve{w}_{B,n} - \breve{w}_B) *^q x_n = 0$ in L_q^1.

Remark 8 Let $\breve{\delta} \in L_q^1$ and $(x_n) \in \Delta_q$ be fixed. Then the mapping

$$\breve{\delta} \to \breve{w}_B,$$

where $\breve{w}_B = \frac{\breve{\delta} *^q x_n}{x_n}$ is an injective mapping from L_q^1 into B_q^1.

Therefore, it can be easily checked that L_q^1 may be identified as a subspace of B_q^1.

Remark 9 Let $(x_n) \in \Delta_q$. Then, if $\breve{\delta}_n \to \breve{\delta}$ in L_q^1 as $n \to \infty$, then for all $k \in \mathbb{N}$,

$$\breve{\delta}_n *^q x_k \to \breve{\delta} *^q x_k$$

as $n \to \infty$. That is, $\breve{w}_{B,n} \to \breve{w}_B$ in B_q^1 as $n \to \infty$.

The above remark states the following.

Theorem 10 *The mapping* $\breve{\delta} \to \breve{w}_B$, $\breve{w}_B = \frac{\breve{\delta} *^q x_n}{x_n}$, *is a continuous q-embedding of the space* L_q^1 *into the space* B_q^1.

3 The space B_q^2

In this section we provide the basic essentials that we need in defining the space B_q^2 of q-Boehmians, with the set $(L_q^1, \bullet^q)$, the subset $(D_q^{S_q}, *^q)$, the set of delta sequences $(\Delta_q^{S_q}, *^q)$, the convolution products $*^q$, and the operation $\bullet^q$; see Eq. (9). To establish the space B_q^2, let us define the following convolution product.

Definition 11 Let $\breve{\delta}$ and $\hat{\delta}$ be integrable functions in L_q^1. Between $\breve{\delta}$ and $\hat{\delta}$ we define the q-product $\bullet^q$ as follows:

$$\left(\breve{\delta} \bullet^q \hat{\delta}\right)(x) = x S_q \breve{\delta}(x) \hat{\delta}(x), \tag{9}$$

where $S_q \breve{\delta}$ is the q-Sumudu transform of $\breve{\delta}$.

We have the following assertion.

Theorem 12 *Let $\check{\delta}$ be an integrable function in L_q^1, then $\check{\delta} \bullet^q \psi \in L_q^1$ for all $\psi \in D_q^{S_q}$.*

Proof Let $\check{\delta} \in L_q^1$ and $\psi = S_q \gamma$ for some $\gamma \in D_q$. Then, by the definition of the space L_q^1 and the product $\bullet^q$, we write

$$\|\check{\delta} \bullet^q \psi\|_{L_q^1(\mathbb{R}_+)} = \frac{1}{1-q} \int_0^\infty \left|\left(\check{\delta} \bullet^q \psi\right)(x)\right| d_q x = \frac{1}{1-q} \int_0^\infty x S_q \check{\delta}(x) \psi(x)\, d_q x.$$

Hence, we have

$$\|\check{\delta} \bullet^q \psi\|_{L_q^1(\mathbb{R}_+)} \leq \frac{1}{1-q} \int_0^\infty x S_q \check{\delta}(x) S_q(\gamma)(x)\, d_q x. \tag{10}$$

Therefore, if $[a, b]$ denotes a closed interval containing the support of ψ, then the hypothesis that $\check{\delta} \in L_q^1$ yields

$$\begin{aligned} \|\check{\delta} \bullet^q \psi\|_{L_q^1(\mathbb{R}_+)} &= \frac{1}{1-q} \int_0^\infty \left|\left(\check{\delta} \bullet^q \psi\right)(x)\right| d_q x \\ &\leq M \frac{1}{1-q} \int_a^b \left|x S_q \gamma(x)\right| d_q x \\ &= M \frac{1}{1-q} \int_a^b \left|x \psi(x)\right| d_q x \\ &< \infty \end{aligned}$$

for every $\psi \in D_q^{S_q}$, M being a positive constant.

This establishes the theorem. □

Now, we reciprocate the product $\bullet^q$ as follows.

Theorem 13 *Let $\check{\delta}$ be an integrable function in L_q^1, then $\check{\delta} \bullet^q (\psi \bullet^q \hat{\psi}) = (\check{\delta} \bullet^q \psi) \bullet^q \hat{\psi}$ for all $\psi, \hat{\psi} \in D_q^{S_q}$.*

Proof Let $\psi = S_q \gamma$, $\hat{\psi} = S_q \hat{\gamma}$, $\gamma, \hat{\gamma} \in D_q$. Then, by the concept of q-convolution $\bullet^q$ and Eq. (3), we get

$$\left(\check{\delta} \bullet^q \left(\psi \bullet^q \hat{\psi}\right)\right)(x) = x(S_q \check{\delta})(x)\left(S_q \gamma \bullet^q S_q \hat{\gamma}\right)(x) = x S_q \check{\delta}(x) x S_q \gamma(x) S_q \hat{\gamma}(x). \tag{11}$$

Rearranging Eq. (11) and again the concept of q-convolution $\bullet^q$ gives

$$\begin{aligned} \left(\check{\delta} \bullet^q \left(\psi \bullet^q \hat{\psi}\right)\right)(x) &= x\left(x S_q \check{\delta}(x) x S_q \gamma(x)\right)\left(S_q \hat{\gamma}(x)\right) \\ &= x\left(\check{\delta} \bullet^q \psi\right) S_q(\hat{\gamma})(x) \\ &= \left(\left(\check{\delta} \bullet^q \psi\right) \bullet^q \hat{\psi}\right)(x). \end{aligned}$$

Hence the proof of the theorem follows. □

Theorem 14 (i) *Let $\check{\delta}$ and $\hat{\delta}$ be integrable functions in L_q^1, then $(\check{\delta}+\hat{\delta}) \bullet^q \psi = \check{\delta} \bullet^q \psi + \hat{\delta} \bullet^q \psi$ for all $\psi \in D_q^{S_q}$.*

(ii) *Let $\check{\delta}$ be an integrable function in L_q^1, then $(\alpha\check{\delta} \bullet^q \psi) = \alpha(\check{\delta} \bullet^q \psi)$ for all $\psi \in D_q^{S_q}$, $\alpha \in \mathbb{C}$.*

Proof (i) Let $\psi \in D_q^{S_q}$, $\psi \in D_q^{S_q}$, $\psi = S_q\gamma$, $\gamma \in D_q$. Then, by the definition of $\bullet^q$ and Eq. (11), we write

$$\begin{aligned}\bigl((\check{\delta}+\hat{\delta}) \bullet^q \psi\bigr)(x) &= xS_q(\check{\delta}+\hat{\delta})(x)\psi(x)\\ &= xS_q\check{\delta}(x)S_q\gamma(x) + xS_q\hat{\delta}(x)S_q\gamma(x)\\ &= \bigl(\check{\delta} \bullet^q \psi\bigr)(x) + \bigl(\hat{\delta} \bullet^q \psi\bigr)(x).\end{aligned}$$

The proof of the second part is similar. The proof of this theorem is therefore completed. □

Theorem 15 *Let $\check{\delta}$, $\hat{\delta}$ and $(\hat{\delta}_n)$ be integrable in L_q^1. Then the following hold.*

(i) *If $\hat{\delta}_n \to \check{\delta}$ in L_q^1 as $n \to \infty$, then $\hat{\delta}_n \bullet^q \psi \to \check{\delta} \bullet^q \psi$ for all $\psi \in D_q^{S_q}$, $\psi = S_q\gamma$, $\gamma \in D_q$ as $n \to \infty$.*

(ii) *If $\check{\delta} \bullet^q x_n = \hat{\delta} \bullet^q z_n$, then $\check{\delta} = \hat{\delta}$ in L_q^1 for all $(z_n) \in \Delta_q^{S_q}$, $z_n = S_q x_n$, $n \in \mathbb{N}$.*

(iii) *$\check{\delta} \bullet^q x_n \to 0$ in L_q^1 for all $(x_n) \in \Delta_q^{S_q}$ as $n \to \infty$.*

Proof (i) Let $\check{\delta}$, $\hat{\delta}$ and $(\hat{\delta}_n)$ be integrable in L_q^1 and $\psi \in D_q^{S_q}$, $\psi = S_q\gamma$, $\gamma \in D_q$, such that $\hat{\delta}_n \to \hat{\delta}$ in L_q^1 as $n \to \infty$. Then we have

$$\bigl(\hat{\delta}_n \bullet^q \psi\bigr)(x) = xS_q\hat{\delta}_n(x)S_q\gamma(x) \to xS_q\hat{\delta}(x)S_q\gamma(x) = xS_q\hat{\delta}(x)\psi(x) = \check{\delta} \bullet^q \psi \text{ as } n \to \infty.$$

Proof (ii) Let $\check{\delta}$ and $\hat{\delta}$ be integrable functions in L_q^1 and $(z_n) \in \Delta_q^{S_q}$, $z_n = S_q x_n$, $n \in \mathbb{N}$ such that $\check{\delta} \bullet^q z_n = \hat{\delta} \bullet^q z_n$. Then $xS_q\check{\delta}(x)S_q x_n(x) - xS_q\hat{\delta}(x)S_q(x_n)(x) = 0$ for all $x \in L_q^1$. Hence, we have

$$xS_q x_n(x)S_q(\check{\delta}-\hat{\delta})(x) = 0$$

for all $x \in L_q^1$. Therefore, e infers that $\check{\delta}(x) = \hat{\delta}(x)$ for all $x \in L_q^1$. The proof of (iii) is analogous. Hence the proof is completed. □

If $(\check{\delta}_n) \in L_q^1$ and $(z_n) \in \Delta_q^{S_q}$, $z_n = S_q x_n$, $n \in \mathbb{N}$, then the pair $(\check{\delta}_n, z_n)$ (or $\frac{\check{\delta}_n}{z_n}$) is said to be a quotient of sequences if $\check{\delta}_n \bullet^q z_m = \check{\delta}_m \bullet^q z_n$, $\forall n, m \in \mathbb{N}$. Therefore, if $\frac{\check{\delta}_n}{z_n}$ and $\frac{\hat{\delta}_n}{y_n}$ are quotients of sequences and $\check{\delta} \in L_q^1$, then it is easy to see that

$$\frac{\check{\delta} \bullet^q z_n}{z_n} \quad \text{and} \quad \frac{\check{\delta}_n \bullet^q z_n + \hat{\delta}_n \bullet^q z_n}{z_n \bullet^q y_n}$$

are quotients of sequences. Further, we can easily check the following equivalence relations:

$$\frac{\check{\delta}_n}{z_n \bullet^q \check{\delta}} \sim \frac{\check{\delta}_n \bullet^q \check{\delta}}{z_n} \quad \text{and} \quad \frac{\check{\delta}_n}{z_n \bullet^q \hat{\delta}_n} \sim \frac{\check{\delta}_n \bullet^q \hat{\delta}_n}{z_n}.$$

Two quotients of sequences $\frac{\check{\delta}_n}{z_n}$ and $\frac{\hat{\delta}_n}{y_n}$ are said to be equivalent if $\check{\delta}_n \bullet^q y_m = \hat{\delta}_m \bullet^q z_n$, $\forall n, m \in \mathbb{N}$. The equivalent class $\check{w}_B = \frac{\check{\delta}_n}{z_n}$ of quotients of sequences containing $\frac{\check{\delta}_n}{z_n}$ is said to be a q-Boehmian. The space of such q-Boehmians is denoted by B_q^2. For two q-Boehmians $\check{w}_B = \frac{\check{\delta}_n}{z_n}$ and $\check{z}_B = \frac{\hat{\delta}_n}{y_n}$ in B_q^2, the following are well defined on B_q^2:

(i) $\check{w}_B + \check{z}_B = \dfrac{\check{\delta}_n \bullet^q z_n + \hat{\delta}_n \bullet^q z_n}{z_n \bullet^q y_n}$, (ii) $\beta \check{w}_B = \frac{\beta \check{\delta}_n}{z_n}$,

(iii) $\check{w}_B \bullet^q \check{z}_B = \dfrac{\check{\delta}_n \bullet^q \hat{\delta}_n}{z_n \bullet^q y_n}$, (iv) $D^k \check{w}_B = \frac{D^k \check{\delta}_n}{z_n}$, and

(v) $\check{w}_B \bullet^q \check{\delta} = \dfrac{\check{\delta}_n \bullet^q \check{\delta}}{z_n}$,

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$, and $D^k \check{w}_B$ is the kth derivative of $\check{w}_B$ and $\check{\delta} \in L_q^1$.

Definition 16 For $n = 1, 2, 3, \ldots$ and $\check{w}_{B,n}, \check{w}_B \in B_q^2$, the sequence $(\check{w}_{B,n})$ is said to be δ_q-convergent to $\check{w}_B$, denoted by $\delta_q - \lim_{n\to\infty} \check{w}_{B,n} = \check{w}_B$, provided there can be found a delta sequence (z_n), $z_n = S_q x_n$, $n \in \mathbb{N}$, such that

(a) $(\check{w}_{B,n} \bullet^q z_k)$, $(\check{w}_B \bullet^q z_k)$ in L_q^1 for all $n, k \in \mathbb{N}$;
(b) $\lim_{n\to\infty} \check{w}_{B,n} \bullet^q z_k = \check{w}_B \bullet^q z_k$ in L_q^1 for every $k \in \mathbb{N}$.

Definition 17 For $n = 1, 2, 3, \ldots$ and $\check{w}_{B,n}, \check{w}_B \in B_q^2$, the sequence $(\check{w}_{B,n})$ is said to be $\Delta_q^{S_q}$-convergent to $\check{w}_B$, denoted by Δ_q-$\lim_{n\to\infty} \check{w}_{B,n} = \check{w}_B$, provided there can be found a delta sequence (z_n) such that

(a) $(\check{w}_{B,n} - \check{w}_B) \bullet^q z_n \in L_q^1$ ($\forall n \in \mathbb{N}$);
(b) $\lim_{n\to\infty} (\check{w}_{B,n} - \check{w}_B) \bullet^q z_n = 0$ in L_q^1.

Remark 18 Let $\check{\delta} \in L_q^1$ and $(z_n) \in \Delta_q^{S_q}$, $z_n = S_q x_n$, $n \in \mathbb{N}$, be fixed. Then the mapping

$$\check{\delta} \to \check{w}_B,$$

where $\check{w}_B = \frac{\check{\delta} \bullet^q z_n}{z_n}$ is an injective mapping from L_q^1 into B_q^2.

Therefore, it can be easily checked that L_q^1 may be identified as a subspace of B_q^2.

Remark 19 Let $(z_n) \in \Delta_q^{S_q}$, $z_n = S_q x_n$, $n \in \mathbb{N}$. Then if $\check{\delta}_n \to \check{\delta}$ in L_q^1 as $n \to \infty$, then, for all $k \in \mathbb{N}$,

$$\check{\delta}_n \bullet^q z_k \to \check{\delta} \bullet^q z_k$$

as $n \to \infty$. That is, $\check{w}_{B,n} \to \check{w}_B$ in B_q^2 as $n \to \infty$.

The above remark states the following.

Theorem 20 *The mapping $\psi \to \check{w}_B$, $\check{w}_B = \frac{\check{\delta} \bullet^q z_n}{z_n}$, is a continuous q-embedding of the space L_q^1 into the space B_q^2.*

4 The q-Sumudu transform of generalized q-theory

Definition 21 Let $\frac{\check{\delta}_n}{x_n}$ be a q-Boehmian in the space B_q^1. Then we define the q-Sumudu transform of $\frac{\check{\delta}_n}{x_n}$ as follows:

$$S_q^1 \frac{\check{\delta}_n}{x_n} = \frac{S_q \check{\delta}_n}{S_q x_n} \quad \text{for all } (\check{\delta}_n) \in L_q^1 \text{ and } (x_n) \in \Delta_q.$$

It is clear that $S_q^1 \frac{\check{\delta}_n}{x_n}$ belongs to B_q^2 as $(S_q \check{\delta}_n)$ and $(S_q x_n)$ are elements of the spaces L_q^1 and $\Delta_q^{S_q}$, respectively. The linearity of S_q^1 follows by easy techniques.

Theorem 22 *The operator $S_q^1 : B_q^1 \to B_q^2$ is q-sequentially continuous, i.e., if $\Delta_q - \lim_{n\to\infty} \breve{w}_{B,n} = \breve{w}_B$ in B_q^1, then*

$$\Delta_q - \lim_{n\to\infty} S_q^1 \breve{w}_{B,n} = S_q^1 \breve{w}_B \quad \textit{in } B_q^2.$$

Proof Let $\Delta_q - \lim_{n\to\infty} \breve{w}_{B,n} = \breve{w}_B$ in B_q^1, then there is (x_n) in Δ_q such that $\Delta_q - \lim_{n\to\infty} (\breve{w}_{B,n} - \breve{w}_B) *^q x_n = 0$ in B_q^1. The continuity of the integral operator gives

$$\Delta_q^{S_q} - \lim_{n\to\infty} S_q^1 \bigl((\breve{w}_{B,n} - \breve{w}_B) *^q x_n\bigr) = \Delta_q^{S_q} - \lim_{n\to\infty} \bigl((S_q^1 \breve{w}_{B,n} - S_q^1 \breve{w}_B) \bullet^q S_q x_n\bigr) = 0.$$

Thus, we have $\Delta_q^{S_q} - \lim_{n\to\infty} S_q^1 \breve{w}_{B,n} = S_q^1 \breve{w}_B$ in B_q^2.

This finishes the proof of the theorem. □

Theorem 23 *$S_q^1 : B_q^1 \to B_q^2$ is one-one, onto, continuous with respect to δ_q and Δ_q-convergence and consists with the classical operator S_q.*

Proof Proofs of the parts that S_q^1 is one-one, onto, continuous with respect to δ_q and Δ_q-convergence are analogous to those given in the literature. To prove that S_q^1 consists with the classical operator S_q^1, let $\hat{\delta} \in L_q^1$ and let $\frac{\hat{\delta} *^q z_n}{z_n}$ be its representative in B_q^2 for all $z_n = S_q x_n$, $(x_n) \in \Delta_q$. Clearly, for all $n \in \mathbb{N}$, (z_n) is independent of the representative. Hence, by the convolution theorem we get

$$S_q^1 \frac{\hat{\delta} *^q z_n}{z_n} = \frac{S_q \hat{\delta} \bullet^q S_q x_n}{S_q x_n} = S_q \hat{\delta} \bullet^q S_q^1 \frac{x_n}{x_n}.$$

That is, $\frac{S_q \hat{\delta} \bullet^q S_q x_n}{S_q x_n} = \frac{S_q \hat{\delta} \bullet^q z_n}{z_n}$ is the representative of $S_q \hat{\delta}$ in the space L_q^1. The proof is, therefore, finished.

We introduce the transform inversion formula as follows. □

Definition 24 We define the inverse integral operator of S_q^1 of a q-Boehmian $\frac{S_q \check{\delta}_n}{z_n}$ in B_q^2 as a q-Boehmian in B_q^1 defined by

$$S_q^{-1} \frac{S_q \check{\delta}_n}{z_n} = \frac{\check{\delta}_n}{x_n},$$

where $z_n = S_q x_n$, (x_n) is a delta sequence in Δ_q, and $(\check{\delta}_n)$ is a sequence of integrable functions in L_q^1.

Theorem 25 *Let $\frac{S_q\check{\delta}_n}{z_n}$ be a q-Boehmian in B_q^2, $z_n = S_q x_n$, $(x_n) \in \Delta_q$, and $\hat{\delta} \in L_q^1$. Then we have*

$$S_q^1\left(\frac{\check{\delta}_n}{x_n} *^q \hat{\delta}\right) = \frac{S_q\check{\delta}_n}{z_n} \bullet^q \hat{\delta} \quad \textit{and} \quad S_q^{-1}\left(\frac{S_q\check{\delta}_n}{z_n} \bullet^q \hat{\delta}\right) = \frac{\check{\delta}_n}{x_n} *^q \hat{\delta}.$$

Proof Assume that $\frac{S_q\check{\delta}_n}{z_n}$ is a q-Boehmian in the space $B_q^2, z_n = S_q x_n$, $(x_n) \in \Delta_q$, and $\hat{\delta} \in L_q^1$. Then, by using the convolution theorem, Definition 21, and Eq. (9), we have

$$\begin{aligned} S_q^1\left(\frac{\check{\delta}_n}{x_n} *^q \hat{\delta}\right) &= S_q^1\left(\frac{\check{\delta}_n *^q \hat{\delta}}{x_n}\right) \\ &= \frac{S_q(\check{\delta}_n *^q \hat{\delta})}{S_q x_n} \\ &= \frac{x S_q\check{\delta}_n(x) S_q\hat{\delta}(x)}{S_q x_n} \\ &= \frac{S_q\check{\delta}_n \bullet^q \hat{\delta}}{S_q x_n} \\ &= \frac{S_q\check{\delta}_n}{S_q x_n} \bullet^q \hat{\delta} \\ &= \frac{S_q\check{\delta}_n}{z_n} \bullet^q \hat{\delta}. \end{aligned}$$

Similarly, by using the convolution theorem, Definition 24, and Eq. (9), we obtain

$$\begin{aligned} S_q^{-1}\left(\frac{S_q\check{\delta}_n}{z_n} \bullet^q \hat{\delta}\right) &= S_q^{-1}\frac{S_q\check{\delta}_n \bullet^q \hat{\delta}}{z_n} \\ &= S_q^{-1}\frac{x S_q\check{\delta}_n(x) S_q\hat{\delta}(x)}{z_n} \\ &= S_q^{-1}\frac{S_q(\check{\delta}_n *^q \hat{\delta})}{S_q x_n} \\ &= \frac{\check{\delta}_n *^q \hat{\delta}}{x_n} \\ &= \frac{\check{\delta}_n}{x_n} *^q \hat{\delta}. \end{aligned}$$

This completely finishes the proof of the theorem. □

5 Conclusion

This paper could be an evolution of idea. It gives an extension to a set of q-integrable functions to a set of q-integrable q-generalized functions. It verifies that the q-analysis of this paper generalizes the q-analysis followed by Albayrak et al. 2013. Moreover, this paper has also shown that the generalized q-Sumudu transform and its q-inversion formula are well-defined mappings possessing properties alike to the classical properties of their corresponding classical versions.

Acknowledgements
The author would like to thank his parents and friends for their support.

Funding
No funding sources to be declared.

Availability of data and materials
Please contact the author for data requests.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 August 2020 Accepted: 26 November 2020 Published online: 07 January 2021

References

1. Kritika, A.R., Purohit, S.D.: Solution of fractional kinetic equations associated with the (p,q)-Mathieu-type series. Discrete Dyn. Nat. Soc. **2020**, 1–7 (2020)
2. Nisar, K.S.: Generalized Mittag-Leffler type function: fractional integrations and application to fractional kinetic equations. Front. Phys. **8**, 1–7 (2020)
3. Jackson, F.H.: On q-definite integrals. Q. J. Pure Appl. Math. **41**, 193–203 (1910)
4. Ucar, F.: q-Sumudu transforms of q-analogues of Bessel functions. Sci. World J. **2014**, 1–7 (2014)
5. Abdi, W.H.: On q-Laplace transforms. Proc. Natl. Acad. Sci., India **29**, 389–408 (1961)
6. Albayrak, D., Purohit, S.D., Ucar, F.: On q-Sumudu transforms of certain q-polynomials. Filomat **27**(2), 413–429 (2013)
7. Hahn, W.: Beitrage Zur Theorie Der Heineschen Reihen, die 24 Integrale der hypergeometrischen q-diferenzengleichung, das q-Analog on der Laplace transformation. Math. Nachr. **2**, 340–379 (1949)
8. Al-Omari, S.K.: On q-analogues of the natural transform of certain q-Bessel functions and some application. Filomat **31**(9), 2587–2598 (2017)
9. Fitouhi, A., Bettaibi, N.: Applications of the Mellin transform in quantum calculus. J. Math. Anal. Appl. **328**, 518–534 (2007)
10. Exton, H.: A basic analogue of the Bessel–Clifford equation. Jñānābha **8**, 49–56 (1978)
11. Shanoja, R.N., Haubold, H.J.: On the q-Laplace transform and related special functions. Axioms **5**, 24 (2016)
12. Al-Omari, S.K.Q.: On q-analogues of the Mangontarum transform for certain q-Bessel functions and some application. J. King Saud Univ., Sci. **28**(4), 375–379 (2016)
13. Albayrak, D., Purohit, S.D., Ucar, F.: On q-analogues of Sumudu transform. An. Ştiinţ. Univ. 'Ovidius' Constanţa **21**(1), 239–260 (2013)
14. Al-Omari, S.K.Q.: On q-analogues of Mangontarum transform of some polynomials and certain class of H-functions. Nonlinear Stud. **23**(1), 51–61 (2016)
15. Meena, S., Bhatter, S., Jangid, K., Purohit, S.D.: Certain integral transforms concerning the product of family of polynomials and generalized incomplete functions. Moroccan J. Pure Appl. Anal. **6**(2), 243–254 (2020)
16. Albayrak, D., Purohit, S.D., Ucar, F.: On q-Laplace and q-Sumudu transforms of a product of generalized q-Bessel functions. Math. Eng. Sci. Aerosp. **11**(2), 355–369 (2020)
17. Albayrak, D., Purohit, S.D., Ucar, F.: Certain inversion and representation formulas for q-Sumudu transforms. Hacet. J. Math. Stat. **43**(5), 699–713 (2014)
18. Nisar, K.S., Shaikh, A., Rahman, G., Kumar, D.: Solution of fractional kinetic equations involving class of functions and Sumudu transform. Adv. Differ. Equ. **2020**, 39 (2020)
19. Purohit, S.D., Ucar, F.: An application of q-Sumudu transform for fractional q-kinetic equation. Turk. J. Math. **42**, 726–734 (2018)
20. Boehme, T.K.: The support of Mikusinski operators. Trans. Am. Math. Soc. **176**, 319–334 (1973)
21. Belgacem, F.B.M., Karaballi, A.A., Kalla, S.: Analytical investigations of the Sumudu transform and applications to integral production equations. Math. Probl. Eng. **2003**(3), 103–118 (2003)
22. Al-Omari, S.K.Q.: Hartley transforms on certain space of generalized functions. Georgian Math. J. **20**(3), 415–426 (2013)
23. Al-Omari, S.K.Q., Baleanu, D.: On generalized space of quaternions and its application to a class of Mellin transforms. J. Nonlinear Sci. Appl. **9**, 3898–3908 (2016)
24. Al-Omari, S.K.Q.: A fractional Fourier integral operator and its extension to classes of function spaces. Adv. Differ. Equ. **195**, 1–9 (2018)
25. Al-Omari, S.K.Q.: Some estimate of a generalized Bessel–Struve transform on certain space of generalized functions. Ukr. Math. J. **69**(9), 1155–1165 (2017)
26. Al-Omari, S.K.Q., Baleanu, D.: On the generalized Stieltjes transform of Fox's kernel function and its properties in the space of generalized functions. J. Comput. Anal. Appl. **23**, 108–118 (2017)
27. Al-Omari, S.K.Q., Agarwal, P., Choi, J.: Real covering of the generalized Hankel–Clifford transform of fox kernel type of a class of Boehmians. Bull. Korean Math. Soc. **52**(5), 1607–1619 (2015)
28. Al-Omari, S.K.Q., Baleanu, D.: Quaternion Fourier integral operators for spaces of generalized quaternions. Math. Methods Appl. Sci. **41**, 9477–9484 (2018)

29. Al-Omari, S.K.Q.: On some Whittaker transform of a special function kernel for a class of generalized functions. Nonlinear Stud. **26**(2), 435–443 (2019)
30. Al-Omari, S.K.Q.: On Some Variant of a Whittaker integral operator and its representative in a class of square integrable Boehmians. Boletim da Sociedade Paranadade de Matematics **38**(1) (2020)
31. Al-Omari, S.K.Q.: Some remarks on short-time Fourier integral operators and classes of rapidly decaying functions. Math. Methods Appl. Sci. **41**, 1–8 (2018)
32. Belgacem, B.M., Karaballi, A.: Sumudu transform fundamental properties investigations and applications. J. Appl. Math. Stoch. Anal. **2006**, Article ID 91083 (2006)