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# A note on generalized $q$ -difference equations for general Al-Salam–Carlitz polynomials

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## Abstract

In this paper, we deduce the generalized  $q$ -difference equations for general Al-Salam–Carlitz polynomials and generalize Arjika's recent results (Arjika in *J. Differ. Equ. Appl.* 26:987–999, 2020). In addition, we obtain transformational identities by the method of  $q$ -difference equation. Moreover, we deduce  $U(n + 1)$  type generating functions and Ramanujan's integrals involving general Al-Salam–Carlitz polynomials by  $q$ -difference equation.

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## 1 Introduction

In this paper, we refer to the general references [2] for definitions and notations. Throughout this paper, we suppose that  $0 < q < 1$ . For complex numbers  $a$ , the  $q$ -shifted factorials are defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, 3, \dots \end{cases}$$

and  $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$ , where  $m$  is a positive integer and  $n$  is a nonnegative integer or  $\infty$ .

The  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \quad (1.1)$$

The basic (or  $q$ -) hypergeometric function of the variable  $z$  and with  $r$  numerator and  $s$  denominator parameters is defined as follows (see, for details, the monographs by Slater

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([3], Chap. 3) and by Srivastava and Karlsson ([4], p. 347, Eq. (272)); see also [5–7]):

$${}_t\Phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_\tau; \\ b_1, b_2, \dots, b_s; \end{matrix} q; z \right] := \sum_{n=0}^{\infty} [(-1)^n q^{\binom{n}{2}}]^{1+s-\tau} \frac{(a_1, a_2, \dots, a_\tau; q)_n}{(b_1, b_2, \dots, b_s; q)_n} \frac{z^n}{(q; q)_n},$$

where  $q \neq 0$  when  $\tau > s + 1$ . We also note that

$${}_{\tau+1}\Phi_\tau \left[ \begin{matrix} a_1, a_2, \dots, a_{\tau+1}; \\ b_1, b_2, \dots, b_\tau; \end{matrix} q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{\tau+1}; q)_n}{(b_1, b_2, \dots, b_\tau; q)_n} \frac{z^n}{(q; q)_n}.$$

We remark in passing that, in a recently-published survey-cum-expository review article, the so-called  $(p, q)$ -calculus was exposed to be a rather trivial and inconsequential variation of the classical  $q$ -calculus, the additional parameter  $p$  being redundant or superfluous (see, for details, ([8], p. 340)).

The basic (or  $q$ -) series and basic (or  $q$ -) polynomials, especially the basic (or  $q$ -) hypergeometric functions and basic (or  $q$ -) hypergeometric polynomials, are applicable particularly in several diverse areas [see also ([4], pp. 350–351)]. In particular, the celebrated Chu–Vandermonde summation theorem and its known  $q$ -extensions, which have already been demonstrated to be useful (see, for details, [2, 9–11]).

The usual  $q$ -differential operator, or  $q$ -derivative, is defined by [12–14]

$$D_a \{f(a)\} = \frac{f(a) - f(aq)}{a}, \quad \theta_a \{f(a)\} = \frac{f(aq^{-1}) - f(a)}{q^{-1}a}. \tag{1.2}$$

The Leibniz rule for  $D_a$  and  $\theta_a$  is the following identities [12, 13, 15]:

$$D_a^n \{f(a)g(a)\} = \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_a^k \{f(a)\} D_a^{n-k} \{g(aq^k)\}, \tag{1.3}$$

$$\theta_a^n \{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta_a^k \{f(a)\} \theta_a^{n-k} \{g(aq^{-k})\}. \tag{1.4}$$

The following property of  $D_q$  is straightforward and important [16]:

$$D_q \left\{ \frac{1}{(at; q)_\infty} \right\} = \frac{t}{(at; q)_\infty}, \quad D_q^k \left\{ \frac{1}{(at; q)_\infty} \right\} = \frac{t^k}{(at; q)_\infty},$$

$$D_q^k \{a^n\} = \begin{cases} \frac{(q; q)_n}{(q; q)_{n-k}} a^{n-k}, & 0 \leq k \leq n - 1, \\ (q; q)_n, & k = n, \\ 0, & k \geq n + 1. \end{cases} \tag{1.5}$$

The Al-Salam–Carlitz polynomials were introduced by Al-Salam and Carlitz in 1965 [17, Eqs. (1.11) and (1.15)]

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} (aq^{1-k}; q)_k x^k. \tag{1.6}$$

They play important roles in the theory of  $q$ -orthogonal polynomials. In fact, there are two families of these polynomials: one with continuous orthogonality and another with

discrete orthogonality, which are given explicitly in the book of Koekoek, Swarttouw, and Lesky [18, Eqs. (14.24.1) and (14.25.1)]. For further information about  $q$ -polynomials, see [18–23].

The generalized Al-Salam–Carlitz polynomials [24, Eq. (4.7)]

$$\begin{aligned} \phi_n^{(a,b,c)}(x,y|q) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a,b;q)_k}{(c;q)_k} x^k y^{n-k}, \\ \psi_n^{(a,b,c)}(x,y|q) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}-nk} (a,b;q)_k}{(c;q)_k} x^k y^{n-k}, \end{aligned} \tag{1.7}$$

whose generating functions are [24, Eqs. (4.10) and (4.11)]

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x,y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_{\infty}} {}_2\Phi_1 \left[ \begin{matrix} a, b; \\ c; \end{matrix} q; xt \right], \quad \max\{|yt|, |xt|\} < 1, \tag{1.8}$$

$$\sum_{n=0}^{\infty} \psi_n^{(a,b,c)}(x,y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (yt;q)_{\infty} {}_2\Phi_1 \left[ \begin{matrix} a, b; \\ c; \end{matrix} q; xt \right], \quad |xt| < 1. \tag{1.9}$$

Chen and Liu [12, 13] gave the clever way of parameter augmentation by use of the following two  $q$ -exponential operators:

$$\mathbb{T}(bD_a) = \sum_{n=0}^{\infty} \frac{(bD_a)^n}{(q;q)_n}, \quad \mathbb{E}(b\theta_a) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (b\theta_a)^n}{(q;q)_n}, \tag{1.10}$$

which is a rich and powerful tool for basic hypergeometric series, especially makes many famous results easily fall into this framework. For further information about  $q$ -exponential operators, see [12, 13, 25–28].

Recently, Srivastava, Arjika, and Sherif Kelil [29] introduced the following homogeneous  $q$ -difference operator  $\tilde{E}(a, b; D_q)$ :

$$\tilde{E}(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a;q)_n}{(q;q)_n} (bD_a)^n. \tag{1.11}$$

The operators (1.11) have turned out to be suitable for dealing with generalized Cauchy polynomials  $p_n(x, y, a)$

$$p_n(x, y, a) = \tilde{E}(a, b; D_q) \{x^n\}. \tag{1.12}$$

For more information about the relations between operators and  $q$ -polynomials, see [29].

Liu [16, 30] deduced several results involving Bailey’s  ${}_6\psi_6$ ,  $q$ -Mehler formulas for Rogers–Szegő polynomials and  $q$ -integral of Sears’ transformation by the following  $q$ -difference equations.

**Proposition 1** ([30, Theorems 1 and 2]) *Let  $f(a, b)$  be a two-variable analytic function in the neighborhood of  $(a, b) = (0, 0) \in \mathbb{C}^2$ .*

(I) If  $f(a, b)$  satisfies the difference equation

$$bf(aq, b) - af(a, bq) = (b - a)f(a, b), \tag{1.13}$$

then we have

$$f(a, b) = \mathbb{T}(bD_a)\{f(a, 0)\}. \tag{1.14}$$

(II) If  $f(a, b)$  satisfies the difference equation

$$af(aq, b) - bf(a, bq) = (a - b)f(aq, bq), \tag{1.15}$$

then we have

$$f(a, b) = \mathbb{E}(b\theta_a)\{f(a, 0)\}. \tag{1.16}$$

Arjika [1] continues to consider the following generalized  $q$ -difference equations.

**Proposition 2** ([1, Theorem 2.1]) *Let  $f(a, x, y)$  be a three-variable analytic function in the neighborhood of  $(a, x, y) = (0, 0, 0) \in \mathbb{C}^3$ . If  $f(a, x, y)$  can be expanded in terms of  $p_n(x, y, a)$  if and only if*

$$\begin{aligned} x[f(a, x, y) - f(a, x, qy)] &= y[f(a, qx, qy) - f(a, x, qy)] \\ &\quad - ay[f(a, qx, q^2y) - f(a, x, q^2y)]. \end{aligned} \tag{1.17}$$

**Proposition 3** ([1, Theorem 2.2]) *Let  $f(a, x, y)$  be a three-variable analytic function in the neighborhood of  $(a, x, y) = (0, 0, 0) \in \mathbb{C}^3$ . If  $f(a, x, y)$  satisfies the  $q$ -difference equation*

$$\begin{aligned} x[f(a, x, y) - f(a, x, qy)] &= y[f(a, qx, qy) - f(a, x, qy)] \\ &\quad - ay[f(a, qx, q^2y) - f(a, x, q^2y)], \end{aligned} \tag{1.18}$$

then we have

$$f(a, x, y) = \tilde{E}(a, b; D_q)\{f(a, x, 0)\}. \tag{1.19}$$

In this paper, our goal is to generalize the results of Arjika [1] in Sect. 2. We first construct the following  $q$ -operators:

$$\mathbb{T}(a, b, c, d, e, yD_x) = \sum_{n=0}^{\infty} \frac{(a, b, c; q)_n}{(q, d, e; q)_n} (yD_x)^n, \tag{1.20}$$

$$\mathbb{E}(a, b, c, d, e, y\theta_x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a, b, c; q)_n}{(q, d, e; q)_n} (y\theta_x)^n. \tag{1.21}$$

We remark that the  $q$ -operator (1.20) is a particular case of the homogeneous  $q$ -difference operator  $\mathbb{T}(\mathbf{a}, \mathbf{b}, cD_x)$  (see [31]) by taking

$$\mathbf{a} = (a, b, c), \quad \mathbf{b} = (d, e), \quad \text{and} \quad c = y.$$

We also built the relations between operators  $\mathbb{T}(a, b, c, d, e, yD_x)$ ,  $\mathbb{E}(a, b, c, d, e, y\theta_x)$  and the new generalized Al-Salam–Carlitz polynomials  $\phi_n^{(a,b,c)}(x, y|q)$ ,  $\psi_n^{(a,b,c)}(x, y|q)$ , respectively,

$$\phi_n^{(a,b,c)}(x, y|q) \triangleq \mathbb{T}(a, b, c, d, e, yD_x)\{x^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b, c; q)_k}{(d, e; q)_k} x^{n-k} y^k, \tag{1.22}$$

$$\psi_n^{(a,b,c)}(x, y|q) \triangleq \mathbb{E}(a, b, c, d, e, y\theta_x)\{x^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{k(k-n)} (a, b, c; q)_k}{(d, e; q)_k} x^{n-k} y^k. \tag{1.23}$$

The paper is organized as follows: In Sect. 2, we state two theorems and give the proofs. In Sect. 3, we gain generalized generating functions for new generalized Al-Salam–Carlitz polynomials by using the method of  $q$ -difference equations perspective. In Sect. 4, we obtain transformational identities involving generating functions for generalized Al-Salam–Carlitz polynomials by  $q$ -difference equations. In Sect. 5, we deduce  $U(n + 1)$  type generating functions for generalized Al-Salam–Carlitz polynomials by  $q$ -difference equation. In Sect. 6, we deduce generalizations of Ramanujan’s integrals.

## 2 Main results and proofs

In this section, we give the following two theorems.

**Theorem 1** *Let  $f(a, b, c, d, e, x, y)$  be a seven-variable analytic function in the neighborhood of  $(a, b, c, d, e, x, y) = (0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^7$ .*

(I) *If  $f(a, b, c, d, e, x, y)$  can be expanded in terms of  $\phi_n^{(a,b,c)}(x, y|q)$  if and only if*

$$\begin{aligned} &x\{f(a, b, c, d, e, x, y) - f(a, b, c, d, e, x, yq) \\ &\quad - (d + e)q^{-1}[f(a, b, c, d, e, x, yq) - f(a, b, c, d, e, x, yq^2)] \\ &\quad + deq^{-2}[f(a, b, c, d, e, x, yq^2) - f(a, b, c, d, e, x, yq^3)]\} \\ &= y\{[f(a, b, c, d, e, x, y) - f(a, b, c, d, e, xq, y)] \\ &\quad - (a + b + c)[f(a, b, c, d, e, x, yq) - f(a, b, c, d, e, xq, yq)] \\ &\quad + (ab + ac + bc)[f(a, b, c, d, e, x, yq^2) - f(a, b, c, d, e, xq, yq^2)] \\ &\quad - abc[f(a, b, c, d, e, x, yq^3) - f(a, b, c, d, e, xq, yq^3)]\}. \end{aligned} \tag{2.1}$$

(II) *If  $f(a, b, c, d, e, x, y)$  can be expanded in terms of  $\psi_n^{(a,b,c)}(x, y|q)$  if and only if*

$$\begin{aligned} &x\{f(a, b, c, d, e, xq, y) - f(a, b, c, d, e, xq, yq) \\ &\quad - (d + e)q^{-1}[f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, xq, yq^2)] \\ &\quad + deq^{-2}[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, xq, yq^3)]\} \\ &= y\{[f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, x, yq)] \\ &\quad - (a + b + c)[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, x, yq^2)] \\ &\quad + (ab + ac + bc)[f(a, b, c, d, e, xq, yq^3) - f(a, b, c, d, e, x, yq^3)] \\ &\quad - abc[f(a, b, c, d, e, xq, yq^4) - f(a, b, c, d, e, x, yq^4)]\}. \end{aligned} \tag{2.2}$$

*Remark 1* For  $c = d = e = 0$  and  $b \rightarrow \frac{1}{b}, y \rightarrow yb, b \rightarrow 0$ , then equation (2.1) reduces to (1.17).

**Theorem 2** Let  $f(a, b, c, d, e, x, y)$  be a seven-variable analytic function in the neighborhood of  $(a, b, c, d, e, x, y) = (0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^7$ .

(I) If  $f(a, b, c, d, e, x, y)$  satisfies the difference equation

$$\begin{aligned} &x\{f(a, b, c, d, e, x, y) - f(a, b, c, d, e, x, yq) \\ &\quad - (d + e)q^{-1}[f(a, b, c, d, e, x, yq) - f(a, b, c, d, e, x, yq^2)] \\ &\quad + deq^{-2}[f(a, b, c, d, e, x, yq^2) - f(a, b, c, d, e, x, yq^3)]\} \\ &= y\{[f(a, b, c, d, e, x, y) - f(a, b, c, d, e, xq, y)] \\ &\quad - (a + b + c)[f(a, b, c, d, e, x, yq) - f(a, b, c, d, e, xq, yq)] \\ &\quad + (ab + ac + bc)[f(a, b, c, d, e, x, yq^2) - f(a, b, c, d, e, xq, yq^2)] \\ &\quad - abc[f(a, b, c, d, e, x, yq^3) - f(a, b, c, d, e, xq, yq^3)]\}, \end{aligned} \tag{2.3}$$

then we have

$$f(a, b, c, d, e, x, y) = \mathbb{T}(a, b, c, d, e, yD_x)\{f(a, b, c, d, e, x, 0)\}. \tag{2.4}$$

(II) If  $f(a, b, c, d, e, x, y)$  satisfies the difference equation

$$\begin{aligned} &x\{f(a, b, c, d, e, xq, y) - f(a, b, c, d, e, xq, yq) \\ &\quad - (d + e)q^{-1}[f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, xq, yq^2)] \\ &\quad + deq^{-2}[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, xq, yq^3)]\} \\ &= y\{[f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, x, yq)] \\ &\quad - (a + b + c)[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, x, yq^2)] \\ &\quad + (ab + ac + bc)[f(a, b, c, d, e, xq, yq^3) - f(a, b, c, d, e, x, yq^3)] \\ &\quad - abc[f(a, b, c, d, e, xq, yq^4) - f(a, b, c, d, e, x, yq^4)]\}, \end{aligned} \tag{2.5}$$

then we have

$$f(a, b, c, d, e, x, y) = \mathbb{E}(a, b, c, d, e, y\theta_x)\{f(a, b, c, d, e, x, 0)\}. \tag{2.6}$$

*Remark 2* For  $c = d = e = 0$  and  $b \rightarrow \frac{1}{b}, y \rightarrow yb, b \rightarrow 0$ , then equation (2.3) reduces to (1.18).

To determine if a given function is an analytic function in several complex variables, we often use the following Hartogs theorem. For more information, please refer to [32, 33].

**Lemma 1** ([34, Hartogs theorem]) *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain  $D \in \mathbb{C}^n$ , then it is holomorphic (analytic) in  $D$ .*

In order to prove Theorem 1, we need the following fundamental property of several complex variables.

**Lemma 2** ([35, Proposition 1]) *If  $f(x_1, x_2, \dots, x_k)$  is analytic at the origin  $(0, 0, \dots, 0) \in \mathbb{C}^k$ , then  $f$  can be expanded in an absolutely convergent power series*

$$f(x_1, x_2, \dots, x_k) = \sum_{n_1, n_2, \dots, n_k=0}^{\infty} \alpha_{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

*Proof of Theorem 1* (I) From the Hartogs theorem and the theory of several complex variables, we assume that

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} A_k(a, b, c, d, e, x) y^k. \tag{2.7}$$

On one hand, substituting (2.7) into (2.1) yields

$$\begin{aligned} & x \sum_{k=0}^{\infty} [1 - q^k - (d + e)q^{k-1} + (d + e)q^{2k-1} + deq^{2k-2} - deq^{3k-2}] A_k(a, b, c, d, e, x) y^k \\ &= \sum_{k=0}^{\infty} [1 - (a + b + c)q^k + (ab + bc + ac)q^{2k} - abcq^{3k}] [A_k(a, b, c, d, e, x) \\ &\quad - A_k(a, b, c, d, e, xq)] y^{k+1}, \end{aligned} \tag{2.8}$$

which is equal to

$$\begin{aligned} & x \sum_{k=0}^{\infty} (1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1}) A_k(a, b, c, d, e, x) y^k \\ &= \sum_{k=0}^{\infty} (1 - aq^k)(1 - bq^k)(1 - cq^k) [A_k(a, b, c, d, e, x) - A_k(a, b, c, d, e, xq)] y^{k+1}. \end{aligned} \tag{2.9}$$

Equating coefficients of  $y^k$  on both sides of equation (2.9), we have

$$\begin{aligned} & x(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1}) A_k(a, b, c, d, e, x) \\ &= (1 - aq^{k-1})(1 - bq^{k-1}) \\ &\quad \times (1 - cq^{k-1}) [A_{k-1}(a, b, c, d, e, x) - A_{k-1}(a, b, c, d, e, xq)], \end{aligned} \tag{2.10}$$

which is equivalent to

$$\begin{aligned} & A_k(a, b, c, d, e, x) \\ &= \frac{(1 - aq^{k-1})(1 - bq^{k-1})(1 - cq^{k-1})}{(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1})} \cdot \frac{A_{k-1}(a, b, c, d, e, x) - A_{k-1}(a, b, c, d, e, xq)}{x} \\ &= \frac{(1 - aq^{k-1})(1 - bq^{k-1})(1 - cq^{k-1})}{(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1})} \cdot D_x \{A_{k-1}(a, b, c, d, e, x)\}. \end{aligned}$$

By iteration, we gain

$$A_k(a, b, c, d, e, x) = \frac{(a, b, c; q)_k}{(q, d, e; q)_k} \cdot D_x^k \{A_0(a, b, c, d, e, x)\}.$$

Letting  $f(a, b, c, d, e, x, 0) = A_0(a, b, c, d, e, x) = \sum_{n=0}^\infty \mu_n x^n$  yields

$$A_k(a, b, c, d, e, x) = \frac{(a, b, c; q)_k}{(q, d, e; q)_k} \cdot \sum_{n=0}^\infty \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} x^{n-k}, \tag{2.11}$$

we have

$$\begin{aligned} f(a, b, c, d, e, x, y) &= \sum_{k=0}^\infty \frac{(a, b, c; q)_k}{(q, d, e; q)_k} \sum_{n=0}^\infty \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} x^{n-k} y^k \\ &= \sum_{n=0}^\infty \mu_n \sum_{k=0}^\infty \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b, c; q)_k}{(d, e; q)_k} x^{n-k} y^k \\ &= \sum_{n=0}^\infty \mu_n \phi_n^{(a, b, c)}(x, y|q). \end{aligned}$$

On the other hand, if  $f(a, b, c, d, e, x, y)$  can be expanded in terms of  $\phi_n^{(a, b, c)}(x, y|q)$ , we verify that  $f(a, b, c, d, e, x, y)$  satisfies (2.1). Similarly, we deduce (II). The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2* From the theory of several complex variables, we begin to solve the  $q$ -difference. First we may assume that

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^\infty A_k(a, b, c, d, e, x) y^k. \tag{2.12}$$

Substituting this equation into (2.12) and comparing the coefficients of  $y^k$  ( $k \geq 1$ ), we readily find that

$$\begin{aligned} &x(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1})A_k(a, b, c, d, e, x) \\ &= (1 - aq^{k-1})(1 - bq^{k-1})(1 - cq^{k-1}) \\ &\quad \times [A_{k-1}(a, b, c, d, e, x) - A_{k-1}(a, b, c, d, e, xq)], \end{aligned} \tag{2.13}$$

which equals

$$\begin{aligned} &A_k(a, b, c, d, e, x) \\ &= \frac{(1 - aq^{k-1})(1 - bq^{k-1})(1 - cq^{k-1})}{(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1})} \cdot \frac{A_{k-1}(a, b, c, d, e, x) - A_{k-1}(a, b, c, d, e, xq)}{x} \\ &= \frac{(1 - aq^{k-1})(1 - bq^{k-1})(1 - cq^{k-1})}{(1 - q^k)(1 - dq^{k-1})(1 - eq^{k-1})} \cdot D_x \{A_{k-1}(a, b, c, d, e, x)\}. \end{aligned}$$



By iteration, we gain

$$A_k(a, b, c, d, e, x) = \frac{(a, b, c; q)_k}{(q, d, e; q)_k} \cdot D_x^k \{A_0(a, b, c, d, e, x)\}. \tag{2.14}$$

Now we return to calculate  $A_0(a, b, c, d, e, x)$ . Just taking  $y = 0$  in equation (2.12), we immediately obtain  $A_0(a, b, c, d, e, x) = f(a, b, c, d, e, x, 0)$ . Substituting (2.14) into (2.12), we achieve (2.4) directly. The proof of Theorem 2 is complete.  $\square$

### 3 Generating functions for new generalized Al-Salam–Carlitz polynomials

In this section we generalize generating functions for the new generalized Al-Salam–Carlitz polynomials by the method of  $q$ -difference equations.

We start with the following lemmas.

**Lemma 3** ([36]) *The Cauchy polynomials are given as follows:*

$$p_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y) = (y/x; q)_n x^n \tag{3.1}$$

together with the following Srivastava–Agarwal type generating function (see also [37]):

$$\sum_{n=0}^{\infty} p_n(x, y) \frac{(\lambda; q)_n t^n}{(q; q)_n} = {}_2\Phi_1 \left[ \begin{matrix} \lambda, \frac{y}{x}; \\ 0; \end{matrix} q; xt \right]. \tag{3.2}$$

**Lemma 4** ([36]) *Suppose that  $\max\{|xt|, |yt|\} < 1$ , we have*

$$\sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}. \tag{3.3}$$

Based upon the  $q$ -binomial theorem or the homogeneous version of Cauchy identity (3.3) and Heine’s transformations, Srivastava et al. [38] established a set of two presumably new theta-function identities (see, for details, [38]).

**Lemma 5** ([36, Theorem 5]) *Suppose that  $\max\{|act|, |adt|, |bct|, |bdt|\} < 1$ , we have*

$$\sum_{n=0}^{\infty} h_n(a, b|q)h_n(c, d|q) \frac{t^n}{(q; q)_n} = \frac{(abcdt^2; q)_{\infty}}{(act, adt, bct, bdt; q)_{\infty}}. \tag{3.4}$$

**Theorem 3** *Suppose that  $\max\{|xt|, |yt|\} < 1$ , we have*

$$\sum_{n=0}^{\infty} \phi_n \binom{a, b, c}{d, e} (x, y|q) \frac{t^n}{(q; q)_n} = \frac{1}{(xt; q)_{\infty}} {}_3\Phi_2 \left[ \begin{matrix} a, b, c; \\ d, e; \end{matrix} q; yt \right], \tag{3.5}$$

$$\sum_{n=0}^{\infty} \psi_n \binom{a, b, c}{d, e} (x, y|q) \frac{t^n}{(q; q)_n} = (xt; q)_{\infty} {}_3\Phi_3 \left[ \begin{matrix} a, b, c; \\ 0, d, e; \end{matrix} q; -yt \right]. \tag{3.6}$$

*Remark 3* For  $c = e = 0$  in Theorem 3, equations (3.5) and (3.6) reduce to equations (1.8) and (1.9), respectively.

*Proof of Theorem 3* By denoting the right-hand side of equation (3.5) by  $f(a, b, c, d, e, x, y)$ , we can verify that  $f(a, b, c, d, e, x, y)$  satisfies (2.1). So, we have

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{(a,b,c)}(x, y|q)$$

and

$$f(a, b, c, d, e, x, 0) = \sum_{k=0}^{\infty} \mu_n x^n = \frac{1}{(xt; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} x^n.$$

So,  $f(a, b, c, d, e, x, y)$  is equal to

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} \frac{t^n}{(q; q)_n} \phi_n^{(a,b,c)}(x, y|q),$$

equal to the right-hand side of equation (3.5).

Similarly, by denoting the right-hand side of equation (3.6) by  $f(a, b, c, d, e, x, y)$ , we can verify that  $f(a, b, c, d, e, x, y)$  satisfies (2.2). So, we can use the same way to achieve equation (3.6). The proof of Theorem 3 is complete.  $\square$

**Theorem 4** Suppose that  $\max\{|xt|, |yt|\} < 1$ , we have

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x, y|q) \frac{p_n(s, t)}{(q; q)_n} = \frac{(xs; q)_{\infty}}{(xt; q)_{\infty}} {}_4\Phi_3 \left[ \begin{matrix} a, b, c, s/t; \\ d, e, xs; \end{matrix} \middle| q; yt \right]. \tag{3.7}$$

**Corollary 1** Suppose that  $|yt| < 1$ , we have

$$\sum_{n=0}^{\infty} \phi_n^{(a,b,c)}(x, y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (xt; q)_{\infty} {}_3\Phi_3 \left[ \begin{matrix} a, b, c; \\ d, e, xt; \end{matrix} \middle| q; yt \right]. \tag{3.8}$$

*Remark 4* For  $t = 0$ , in Theorem 4, equation (3.7) reduces to (3.8). For  $s = 0$  in Theorem 4, equation (3.7) reduces to (3.5), respectively.

*Proof of Theorem 4* By denoting the right-hand side of equation (3.7) by  $f(a, b, c, d, e, x, y)$ , we can verify that  $f(a, b, c, d, e, x, y)$  satisfies (2.1). So, we have

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{(a,b,c)}(x, y|q) \tag{3.9}$$

and

$$f(a, b, c, d, e, x, 0) = \sum_{k=0}^{\infty} \mu_n x^n = \frac{(xs; q)_{\infty}}{(xt; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{p_n(t, s)}{(q; q)_n} x^n.$$

So,  $f(a, b, c, d, e, x, y)$  is equal to the right-hand side of equation (3.7). The proof of Theorem 4 is complete.  $\square$

**Theorem 5** For  $k \in \mathbb{N}$  and  $\max\{|xt|, |yt|\} < 1$ , we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{n+k}^{(a,b,c)}(x,y|q) \frac{t^n}{(q;q)_n} \\ &= \frac{x^k}{(xt;q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a,b,c;q)_n}{(q,d,e;q)_n} (yt)^n \sum_{j=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^j q^{nj - \binom{j}{2}} (q^{-k}, xt; q)_j}{(xt)^j}. \end{aligned} \tag{3.10}$$

*Remark 5* For  $k = 0$  in Theorem 5, equation (3.10) reduces to (3.5).

*Proof of Theorem 5* Denote the right-hand side of equation (3.10) equivalently by

$$\begin{aligned} & f(a,b,c,d,e,x,y) \\ &= \frac{x^k}{(xt;q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a,b,c;q)_n}{(q,d,e;q)_n} (yt)^n \sum_{j=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^j q^{nj - \binom{j}{2}} (q^{-k}, xt; q)_j}{(xt)^j}, \end{aligned} \tag{3.11}$$

and it is easy to check that (3.11) satisfies (2.1), so we have

$$f(a,b,c,d,e,x,y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{(a,b,c)}(x,y|q). \tag{3.12}$$

Setting  $y = 0$  in (3.12) leads to

$$\begin{aligned} f(a,b,c,d,e,x,0) &= \sum_{k=0}^{\infty} \mu_n x^n = \frac{x^n}{(xt;q)_{\infty}} = \sum_{n=0}^{\infty} x^k \frac{(xt)^n}{(q;q)_n} = \sum_{n=0}^{\infty} x^{n+k} \frac{t^n}{(q;q)_n} \\ &= \sum_{n=k}^{\infty} x^n \frac{t^{n-k}}{(q;q)_{n-k}}. \end{aligned}$$

Hence

$$f(a,b,c,d,e,x,y) = \sum_{n=k}^{\infty} \phi_n^{(a,b,c)}(x,y|q) \frac{t^{n-k}}{(q;q)_{n-k}} = \sum_{n=0}^{\infty} \phi_{n+k}^{(a,b,c)}(x,y|q) \frac{t^n}{(q;q)_n}.$$

The proof of Theorem 5 is complete. □

**Theorem 6** We have

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(a_1,b_1,c_1)}(x_1,y_1|q) \phi_n^{(a_2,b_2,c_2)}(x_2,y_2|q) \frac{t^n}{(q;q)_n} \\ &= \frac{1}{(x_1x_2t;q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a_2,b_2,c_2;q)_n (x_1y_2t)^n}{(q,d_2,e_2;q)_n} \\ & \quad \times \sum_{j=0}^{\infty} \frac{(q^{n-j+1}, x_1x_2t, a_1, b_1, c_1; q)_j \left(\frac{y_1}{x_1}\right)^j}{(q,d_1,e_1;q)_j} {}_3\Phi_2 \left[ \begin{matrix} a_1q^j, b_1q^j, c_1q^j; \\ d_1q^j, e_1q^j; \end{matrix} q; x_2y_1t \right]. \end{aligned} \tag{3.13}$$

*Remark 6* For  $a_1 = b_1 = c_1 = d_1 = e_1 = a_2 = b_2 = c_2 = d_2 = e_2 = 0$  in Theorem 6, equation (3.13) reduces to (3.4).

*Proof of Theorem 6* Denoting the right-hand side of equation (3.13) by  $H(a_1, b_1, c_1, d_1, e_1, x_1, y_1)$ , we have

$$\begin{aligned}
 &H(a_1, b_1, c_1, d_1, e_1, x_1, y_1) \\
 &= \frac{1}{(x_1 x_2 t; q)_\infty} \sum_{n=0}^\infty \frac{(a_2, b_2, c_2; q)_n (x_1 y_2 t)^n}{(q, d_2, e_2; q)_n} \\
 &\quad \times \sum_{j=0}^\infty \frac{(q^{n-j+1}, x_1 x_2 t, a_1, b_1, c_1; q)_j (\frac{y_1}{x_1})^j}{(q, d_1, e_1; q)_j} {}_3\Phi_2 \left[ \begin{matrix} a_1 q^j, b_1 q^j, c_1 q^j; \\ d_1 q^j, e_1 q^j; \end{matrix} q; x_2 y_1 t \right]. \tag{3.14}
 \end{aligned}$$

Because equation (3.14) satisfies (2.3), we have

$$\begin{aligned}
 &H(a_1, b_1, c_1, d_1, e_1, x_1, y_1) \\
 &= \mathbb{T}(a_1, b_1, c_1, d_1, e_1, y_1 D_{x_1}) \{ H(a_1, b_1, c_1, d_1, e_1, x_1, 0) \} \\
 &= \mathbb{T}(a_1, b_1, c_1, d_1, e_1, y_1 D_{x_1}) \left\{ \frac{1}{(x_1 x_2 t; q)_\infty} \sum_{n=0}^\infty \frac{(a_2, b_2, c_2; q)_n (x_1 y_2 t)^n}{(q, d_2, e_2; q)_n} \right\} \\
 &= \mathbb{T}(a_1, b_1, c_1, d_1, e_1, y_1 D_{x_1}) \left\{ \sum_{n=0}^\infty \phi_n^{(a_2, b_2, c_2)}(x_2, y_2 | q) \frac{(x_1 t)^n}{(q; q)_n} \right\} \\
 &= \sum_{n=0}^\infty \phi_n^{(a_2, b_2, c_2)}(x_2, y_2 | q) \frac{t^n}{(q; q)_n} \mathbb{T}(a_1, b_1, c_1, d_1, e_1, y_1 D_{x_1}) \{ x_1^n \} \\
 &= \sum_{n=0}^\infty \phi_n^{(a_1, b_1, c_1)}(x_1, y_1 | q) \phi_n^{(a_2, b_2, c_2)}(x_2, y_2 | q) \frac{t^n}{(q; q)_n}.
 \end{aligned}$$

The proof of Theorem 6 is complete. □

#### 4 Transformational identities from $q$ -difference equations

Liu [24] gave some important transformational identities by the method of  $q$ -difference operator. For more details, please refer to [18, 24, 39].

In this section we deduce the following transformational identities involving generating functions for new generalized Al-Salam–Carlitz polynomials by the method of  $q$ -difference equation.

**Theorem 7** *Let  $A(k)$  and  $B(k)$  satisfy*

$$\sum_{k=0}^\infty A(k) x^k = \sum_{k=0}^\infty B(k) \frac{(xtq^k; q)_\infty}{(xq^k; q)_\infty}, \tag{4.1}$$

and we have

$$\sum_{k=0}^\infty A(k) \phi_k^{(a, b, c)}(x, y | q) = \sum_{k=0}^\infty B(k) \frac{(xtq^k; q)_\infty}{(xq^k; q)_\infty} {}_4\Phi_3 \left[ \begin{matrix} a, b, c, 1/t; \\ d, e, xtq^k; \end{matrix} q; yq^k \right], \tag{4.2}$$

supposing that equations (4.1) and (4.2) are convergent.

**Corollary 2** Suppose that  $|r|, |x|, |xt| < 1$ , we have

$$\sum_{k=0}^{\infty} \phi_k^{(a,b,c)}(x, y|q) \frac{(t, s; q)_k}{(q, r; q)_k} = \frac{(xt, s; q)_{\infty}}{(x, r; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(r/s, x; q)_k s^k}{(q, xt; q)_k} {}_4\Phi_3 \left[ \begin{matrix} a, b, c, 1/t; \\ d, e, xtq^k; \end{matrix} q; yq^k \right]. \tag{4.3}$$

*Remark 7* Setting  $A(k)$  and  $B(k)$  in Theorem 7 by (4.6) given below, equation (4.2) reduces to (4.3). For  $y = 0$  in (4.3), equation (4.2) reduces to (4.5).

*Proof of Theorem 7* Denoting the right-hand side of equation (4.2) equivalently by  $f(a, b, c, d, e, x, y)$ , we can check that  $f(a, b, c, d, e, x, y)$  satisfies (2.1), so we have

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{(a,b,c)}(x, y|q). \tag{4.4}$$

Setting  $y = 0$  in (4.4), it becomes

$$\begin{aligned} f(a, b, c, d, e, x, 0) &= \sum_{k=0}^{\infty} \mu_n x^n = \sum_{k=0}^{\infty} B(k) \frac{(xtq^k; q)_{\infty}}{(xq^k; q)_{\infty}} by(48) \\ &= \sum_{k=0}^{\infty} A(k) x^k. \end{aligned}$$

Hence

$$f(a, b, c, d, e, x, y) = \sum_{k=0}^{\infty} A(k) \phi_k^{(a,b,c)}(x, y|q).$$

The proof of Theorem 7 is complete. □

*Proof of Corollary 2* Using Heine’s  $q$ -Euler transformations [17, Eq. (1.4.1)]

$${}_2\Phi_1 \left[ \begin{matrix} t, s; \\ r; \end{matrix} q; x \right] = \frac{(s, xt; q)_{\infty}}{(r, x; q)_{\infty}} {}_2\Phi_1 \left[ \begin{matrix} r/s, x; \\ xt; \end{matrix} q; s \right], \tag{4.5}$$

formula (4.1) is valid for the case

$$A(k) = \sum_{k=0}^{\infty} \frac{(t, s; q)_k}{(q, r)_k} \quad \text{and} \quad B(k) = \frac{(s; q)_{\infty}}{(r; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(r/s; q)_k}{(q; q)_k} s^k. \tag{4.6}$$

Using equation (4.2), we can deduce Corollary 2. □

### 5 $U(n + 1)$ type generating functions for generalized Al-Salam–Carlitz polynomials

Multiple basic hypergeometric series associated with the unitary  $U(n + 1)$  group have been investigated by various authors, see [40, 41]. In [40], Milne initiated theory and application of the  $U(n + 1)$  generalization of the classical Bailey transform and Bailey lemma, which involves the following nonterminating  $U(n + 1)$  generalizations of the  $q$ -binomial theorem.

**Proposition 4** ([16, Theorem 5.42]) *Let  $b, z$  and  $x_1, \dots, x_n$  be indeterminate, and let  $n \geq 1$ . Suppose that none of the denominators in the following identity vanishes, and that  $0 < |q| < 1$  and  $|z| < |x_1, \dots, x_n| |x_m|^{-n} |q|^{(n-1)/2}$  for  $m = 1, 2, \dots, n$ . Then*

$$\sum_{\substack{y_n \geq 0 \\ k=1,2,\dots,n}} \left\{ \prod_{1 \leq r < s \leq n} \left[ \frac{1 - (x_r/x_s)q^{y_r - y_s}}{1 - (x_r/x_s)} \right] \prod_{r,s=1}^n \left( q \frac{x_r}{x_s}; q \right)_{y_r}^{-1} \prod_{i=1}^n (x_i)^{ny_i - (y_1 + \dots + y_n)} \right. \\ \times (-1)^{(n-1)(y_1 + \dots + y_n)} q^{y_2 + 2y_3 + \dots + (n-1)y_n + (n-1)\binom{y_1}{2} + \dots + \binom{y_n}{2}}]^{-e_2(y_1, \dots, y_n)} \\ \left. \times (b; q)_{y_1 + \dots + y_n} z^{y_1 + \dots + y_n} \right\} = \frac{(bz; q)_\infty}{(z; q)_\infty}, \tag{5.1}$$

where  $e_2(y_1, \dots, y_n)$  is the second elementary symmetric function of  $\{y_1, \dots, y_n\}$ .

In this section, we deduce  $U(n + 1)$  type generating functions for generalized Al-Salam–Carlitz polynomials by the method of  $q$ -difference equation.

**Theorem 8** *Let  $b, z$  and  $x_1, \dots, x_n$  be indeterminate, and let  $n \geq 1$ . Suppose that none of the denominators in the following identity vanishes, and that  $0 < |q| < 1$  and  $|z| < |x_1, \dots, x_n| |x_m|^{-n} |q|^{(n-1)/2}$  for  $m = 1, 2, \dots, n$ . Then*

$$\sum_{\substack{y_n \geq 0 \\ k=1,2,\dots,n}} \left\{ \prod_{1 \leq r < s \leq n} \left[ \frac{1 - (x_r/x_s)q^{y_r - y_s}}{1 - (x_r/x_s)} \right] \prod_{r,s=1}^n \left( q \frac{x_r}{x_s}; q \right)_{y_r}^{-1} \prod_{i=1}^n (x_i)^{ny_i - (y_1 + \dots + y_n)} \right. \\ \times (-1)^{(n-1)(y_1 + \dots + y_n)} q^{y_2 + 2y_3 + \dots + (n-1)y_n + (n-1)\binom{y_1}{2} + \dots + \binom{y_n}{2}}]^{-e_2(y_1, \dots, y_n)} \\ \left. \times \phi_{y_1 + \dots + y_n}^{(r,s,t)}(z, y|q)(b; q)_{y_1 + \dots + y_n} \right\} = \frac{(bz; q)_\infty}{(z; q)_\infty} {}_4\Phi_3 \left[ \begin{matrix} r, s, t, b; \\ u, v, bz; \end{matrix} q; y \right], \tag{5.2}$$

where  $e_2(y_1, \dots, y_n)$  is the second elementary symmetric function of  $\{y_1, \dots, y_n\}$ .

*Remark 8* For  $y = 0$ , in Theorem 8, equation (5.2) reduces to (5.1).

*Proof of Theorem 8* Denote the right-hand side of equation (5.2) equivalently by  $f(r, s, t, u, v, z, y)$ , and we can check that  $f(r, s, t, u, v, z, y)$  satisfies (2.3), so we have

$$f(r, s, t, u, v, z, y) \\ = \mathbb{T}(r, s, t, u, v, yD_z)\{f(r, s, t, u, v, z, 0)\} \\ = \sum_{\substack{y_n \geq 0 \\ k=1,2,\dots,n}} \left\{ \prod_{1 \leq r < s \leq n} \left[ \frac{1 - (x_r/x_s)q^{y_r - y_s}}{1 - (x_r/x_s)} \right] \prod_{r,s=1}^n \left( q \frac{x_r}{x_s}; q \right)_{y_r}^{-1} \prod_{i=1}^n (x_i)^{ny_i - (y_1 + \dots + y_n)} \right. \\ \times (-1)^{(n-1)(y_1 + \dots + y_n)} q^{y_2 + 2y_3 + \dots + (n-1)y_n + (n-1)\binom{y_1}{2} + \dots + \binom{y_n}{2}}]^{-e_2(y_1, \dots, y_n)} \\ \left. \times (b; q)_{y_1 + \dots + y_n} \right\} \mathbb{T}(r, s, t, u, v, yD_z)\{z^{y_1 + \dots + y_n}\},$$

which is the left-hand side of (5.2) by (1.22). The proof of Theorem 8 is complete. □

### 6 Generalization of Ramanujan’s integrals

The following integral of Ramanujan [42] is quite famous.

**Proposition 5** ([42, Eq. (2)]) *For  $0 < q = e^{-2k^2} < 1$  and  $m \in \mathbb{R}$ . Suppose that  $|abq| < 1$ , we have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_{\infty}} dx = \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_{\infty}}{(abq; q)_{\infty}}. \tag{6.1}$$

In this section, we have the following generalization of Ramanujan’s integrals.

**Theorem 9** *For  $0 < q = e^{-2k^2} < 1$  and  $m \in \mathbb{R}$ . Suppose that  $|abq| < 1$ , we have*

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_{\infty}} {}_3\Phi_2 \left[ \begin{matrix} r, s, t; \\ u, v; \end{matrix} \begin{matrix} q; yq^{1/2}e^{2ikx} \end{matrix} \right] dx \\ &= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_{\infty}}{(abq; q)_{\infty}} {}_4\Phi_3 \left[ \begin{matrix} r, s, t, e^{2mki}/b; \\ u, v, -aqe^{2mki}; \end{matrix} \begin{matrix} q; ybq \end{matrix} \right]. \end{aligned} \tag{6.2}$$

*Remark 9* For  $y = 0$  in Theorem 9, equation (6.2) reduces to (6.1).

*Proof of Theorem 9* Denote the right-hand side of (6.2) equivalently by  $f(r, s, t, u, v, a, y)$ .  $f(r, s, t, u, v, a, y)$  is analytic near  $(r, s, t, u, v, a, y)$ , and we can check that  $f(r, s, t, u, v, a, y)$  satisfies (2.1), so we have

$$f(r, s, t, u, v, a, y) = \sum_{k=0}^{\infty} \mu_n \phi_n^{(r,s,t)}(a, y|q)$$

and

$$\begin{aligned} f(r, s, t, u, v, a, 0) &= \sum_{k=0}^{\infty} \mu_n a^n = \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_{\infty}}{(abq; q)_{\infty}} \text{ by (6.1)} \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_{\infty}} dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx}; q)_{\infty}} \left\{ \sum_{n=0}^{\infty} \frac{(q^{1/2}e^{2ikx})^n}{(q; q)_n} a^n \right\} dx. \end{aligned}$$

So we have

$$f(r, s, t, u, v, a, y) = \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx}; q)_{\infty}} \left\{ \sum_{n=0}^{\infty} \phi_n^{(r,s,t)}(a, y|q) \frac{(q^{1/2}e^{2ikx})^n}{(q; q)_n} \right\} dx,$$

which is equal to the left-hand side of equation (6.2). The proof of Theorem 9 is complete. □

### 7 Concluding remarks and observations

In our present investigation, we have introduced a set of two  $q$ -operators  $\mathbb{T}(a, b, c, d, e, yD_x)$  and  $\mathbb{E}(a, b, c, d, e, y\theta_x)$  with the aim to apply them to generalize Arjika’s recently results [1]

and derive transformational identities by means of the  $q$ -difference equations. We have also derived  $U(n+1)$ -type generating functions and Ramanujan's integrals involving general Al-Salam–Carlitz polynomials by means of the  $q$ -difference equations.

It is believed that the  $q$ -series and  $q$ -integral identities, which we have been presented in this paper, as well as the various related recent works cited here, will provide encouragement and motivation for further research on the topics that are dealt with and investigated in this paper.

In conclusion, we find it to be worthwhile to remark that some potential further applications of the methodology and findings, which we have been presented here by means of  $q$ -analysis and  $q$ -calculus, can be found in the study of zeta and  $q$ -zeta functions as well as their related functions of analytic number theory (see, for example, [43, 44]; see also [9]) and also in the study of analytic and univalent functions of geometric function theory *via* number-theoretic entities (see, for example, [45]).

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The authors declare that they have no competing interests.

#### Authors' contributions

All authors equally contributed to this manuscript and approved the final version.

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