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Coupled fixed point analysis in fuzzy cone metric spaces with an application to nonlinear integral equations

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Abstract

In this paper, we introduce the concept of coupled type and cyclic coupled type fuzzy cone contraction mappings in fuzzy cone metric spaces. We establish some coupled fixed point results without the mixed monotone property, and also present some coupled fixed results using the partial order metric in the said space. We present some strong coupled fixed point theorems using cyclic coupled type fuzzy cone contraction mappings in fuzzy cone metric spaces. Moreover, we present an application of nonlinear integral equations for the existence of a unique solution to support our work.

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Keywords: Coupled fixed point; Strong coupled fixed point; Contraction conditions; Fuzzy cone metric space

1 Introduction

Initially, Kirk et al. [1] introduced a cyclic contractive type mapping which ensures the existence of the best proximity points in complete metric spaces. Several cyclic type mapping results can be found (see, e.g., [2–4]). Later on, Lakshmikantham and Ćirić [5] presented the concept of coupled fixed point in partially ordered metric spaces which has a wide range of applications in partial differential equations and boundary value problems. In 2014, Choudhury and Maity [6] proved the result on cyclic coupled Kannan type contraction for a strong coupled fixed point. For more coupled fixed point results, see [7, 8]. More related works and references are in [9–12].

Huang and Zhang [13] presented an idea of a cone metric space by using an ordered Banach space instead of real numbers. They proved some nonlinear contractive type fixed point results in cone metric spaces. After this article, several authors have contributed their ideas in the field of cone metric spaces. They established different types of contractive results for fixed point, coincidence point, and a common fixed point in cone metric spaces (see [14–23] and the references therein).

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The fuzzy set theory was initiated by Zadeh [24], while Kramosil et al. [25] introduced fuzzy metric spaces and some more notions. They compared the notion of fuzzy metric with the statistical metric spaces and proved that both the conceptions are equivalent in some cases. Later, George et al. [26] presented the stronger form of the metric fuzziness. Some fixed point and common fixed point results in fuzzy metric spaces can be found in, e.g., [27–30].

Oner et al. [31] introduced the fuzzy cone metric space or shortly (*FCM*-space) and proved a fuzzy cone Banach contraction theorem for a fixed point in *FCM*-spaces with the assumption of Cauchy sequences. Some more topological properties, fixed point and common fixed point results can be found in, e.g., [32–36].

In this paper, we present a new concept of coupled type and cyclic coupled type fuzzy cone contraction mappings in *FCM*-spaces. The rest of the paper is organized as follows. Section 2 consists of preliminary concepts. In Sect. 3, we prove some coupled fixed point results without the mixed monotone property in the sense of Sintunavarat et al. [8], and we prove some coupled fixed point theorems via partial ordered metric *FCM*-spaces. In Sect. 4, we establish some strong coupled fixed point results for the generalized cyclic type fuzzy cone contraction mapping in *FCM*-space in the sense of Choudhury et al. [6]. In Sect. 5, we present an application of nonlinear integral equations for the existence of a unique solution to support our work. Finally, the conclusion is discussed in Sect. 6, and some illustrative examples are presented in the paper to support our work.

2 Preliminaries

Definition 2.1 ([37]) An operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is known as a continuous t -norm if it satisfies the following:

- (i) $*$ is associative, commutative, and continuous.
- (ii) $1 * \alpha_0 = \alpha_0$ and $\alpha_0 * \beta_0 \leq \alpha_1 * \beta_1$, whenever $\alpha_0 \leq \alpha_1$ and $\beta_0 \leq \beta_1$ for each $\alpha_0, \alpha_1, \beta_0, \beta_1 \in [0, 1]$.

The basic continuous t -norms (see [37]), the minimum, the product, and the Lukasiewicz t -norms, are defined respectively as follows:

$$\alpha_1 * \beta_1 = \min\{\alpha_1, \beta_1\}, \quad \alpha_1 * \beta_1 = \alpha_1 \beta_1, \quad \text{and} \quad \alpha_1 * \beta_1 = \max\{\alpha_1 + \beta_1 - 1, 0\}.$$

Throughout this paper \mathbb{E} represents the real Banach space and θ is the zero of \mathbb{E} , while \mathbb{N} represents the set of natural numbers.

Definition 2.2 ([13]) A subset $P \subset \mathbb{E}$ is known as a cone if:

- (i) $P \neq \emptyset$, closed and $P \neq \{\theta\}$.
- (ii) $\alpha_1, \beta_1 \in [0, \infty)$ and $\mu, \nu \in P$, then $\alpha_1 \mu + \beta_1 \nu \in P$.
- (iii) both $\mu - \mu \in P$, then $\mu = \theta$.

A partial ordering on a given cone $P \subset \mathbb{E}$ is defined by $\mu \leq \nu \Leftrightarrow \nu - \mu \in P$. $\mu < \nu$ stands for $\mu \leq \nu$ and $\mu \neq \nu$, while $\mu \ll \nu$ stands for $\nu - \mu \in \text{int}(P)$. In this paper, all cones have nonempty interior.

Definition 2.3 ([31]) A three-tuple $(U, F_m, *)$ is said to be a *FCM*-space if a cone $P \subset \mathbb{E}$, U is an arbitrary set, $*$ is a continuous t -norm, and F_m is a fuzzy set on $U^2 \times \text{int}(P)$ satisfying the following:

- (i) $F_m(\mu, \nu, t) > 0$ and $F_m(\mu, \nu, t) = 1 \Leftrightarrow \mu = \nu$;
 - (ii) $F_m(\mu, \nu, t) = F_m(\nu, \mu, t)$;
 - (iii) $F_m(\mu, \nu, t) * F_m(\nu, \omega, s) \leq F_m(\mu, \omega, t + s)$;
 - (iv) $F_m(\mu, \nu, \cdot) : \text{int}(P) \rightarrow [0, 1]$ is continuous;
- for all $\mu, \nu, \omega \in U$ and $t, s \in \text{int}(P)$.

Definition 2.4 ([31]) Let $(U, F_m, *)$ be a FCM-space, $\mu \in U$, and (μ_i) be a sequence in U . Then

- (i) (μ_i) is said to converge to μ if, for $t \gg \theta$ and $0 < r < 1$, $\exists i_1 \in \mathbb{N}$ such that $F_m(\mu_i, \mu, t) > 1 - r, \forall i \geq i_1$. We denote this by $\lim_{i \rightarrow \infty} \mu_i = \mu$ or $\mu_i \rightarrow \mu$ as $i \rightarrow \infty$.
- (ii) (μ_i) is said to be a Cauchy sequence if, for $t \gg \theta, 0 < r < 1, \exists i_1 \in \mathbb{N}$ such that $F_m(\mu_k, \mu_i, t) > 1 - r, \forall k, i \geq i_1$.
- (iii) $(U, F_m, *)$ is complete if every Cauchy sequence is convergent in U .
- (iv) (μ_i) is known as a fuzzy cone contraction if $\exists 0 < \beta < 1$ such that

$$\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \leq \beta \left(\frac{1}{F_m(\mu_{i-1}, \mu_i, t)} - 1 \right), \quad \forall t \gg \theta, i \geq 1.$$

Definition 2.5 Let $(U, F_m, *)$ be an FCM-space. The fuzzy cone metric F_m is triangular if

$$\frac{1}{F_m(\mu, \omega, t)} - 1 \leq \left(\frac{1}{F_m(\mu, \nu, t)} - 1 \right) + \left(\frac{1}{F_m(\nu, \omega, t)} - 1 \right), \quad \forall \mu, \nu, \omega \in U, t \gg \theta.$$

Lemma 2.6 ([31]) Let $(U, F_m, *)$ be an FCM-space. Let $\mu \in X$ and (μ_i) be a sequence in U . Then $\mu_i \rightarrow \mu \Leftrightarrow \lim_{i \rightarrow \infty} F_m(\mu_i, \mu, t) = 1$, for $t \gg \theta$.

Definition 2.7 ([31]) Let $(U, F_m, *)$ be an FCM-space and $T : U \rightarrow U$. Then T is known as a fuzzy cone contraction if $\exists 0 < h < 1$ such that

$$\frac{1}{F_m(T\mu, T\nu, t)} - 1 \leq h \left(\frac{1}{F_m(\mu, \nu, t)} - 1 \right), \quad \forall \mu, \nu \in U, t \gg \theta. \tag{2.1}$$

3 Coupled fixed point results in FCM-spaces

Definition 3.1 ([38]) Let U be a nonempty set, and an ordered pair $(\mu, \nu) \in U \times U$ is called a coupled fixed point of the mapping $T : U \times U \rightarrow U$ if $\mu = T(\mu, \nu)$ and $\nu = T(\nu, \mu)$.

Example 3.2 Let $U = [0, \infty)$ and a mapping $T : U \times U \rightarrow U$ be defined as

$$T(\mu, \nu) = \frac{\mu + \nu}{2}, \quad \forall \mu, \nu \in U.$$

Then T has a unique coupled fixed point in U for all $\mu = \nu$.

Definition 3.3 ([39]) Let $T : U \times U \rightarrow U$ be a mapping in a metric space (U, m) and A be a nonempty subset of U^4 . Then we say that A is a T -invariant subset of U^4 if and only if $\forall \mu, \nu, g, h \in U$ satisfies:

- (i) $(\mu, \nu, g, h) \in A \Leftrightarrow (h, g, \nu, \mu) \in A$.
- (ii) $(\mu, \nu, g, h) \in A \Leftrightarrow (T(\mu, \nu), T(\nu, \mu), T(g, h), T(h, g)) \in A$.

Definition 3.4 ([8]) Let (U, m) be a metric space and $A \subseteq U^4$ which satisfies the transitive property if and only if $\forall \mu, \nu, g, h, x, y \in U$ such that

$$(\mu, \nu, g, h) \in A \quad \text{and} \quad (g, h, x, y) \in A \quad \Rightarrow \quad (\mu, \nu, x, y) \in A.$$

Remark 3.5 ([8]) Easily one can verify that the set $A = U^4$ is trivially T -invariant, which satisfies the transitive property.

Theorem 3.6 Let A be a nonempty subset of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular. Let a function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $0 = \psi(0) < \psi(\tau) < \tau$, and $\lim_{r \rightarrow \tau^+} \psi(r) < \tau$ for each $\tau > 0$. Suppose that $T : U^2 \rightarrow U$ is a mapping such that

$$\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right) \right), \tag{3.1}$$

for all $(\mu, \nu, x, y) \in A$. Assume that either:

- (1) T is continuous, or
- (2) If for any two sequences (μ_i) and (ν_i) with $(\mu_{i+1}, \nu_{i+1}, \mu_i, \nu_i) \in A$, where $(\mu_i) \rightarrow \mu$, and $(\nu_i) \rightarrow \nu$ for all $i \geq 1$, then $(\mu, \nu, \mu_i, \nu_i) \in A$ for all $i \geq 1$. If $\exists (\mu_o, \nu_o) \in U \times U$ such that $(T(\mu_o, \nu_o), T(\nu_o, \mu_o), \mu_o, \nu_o) \in A$, and A is a T -invariant set which satisfies the transitive property, then T has a coupled fixed point such that $\mu = T(\mu, \nu)$ and $\nu = T(\nu, \mu)$.

Proof Let (μ_i) and (ν_i) be two sequences in U and $T(U^2) \subseteq U$ such that

$$\mu_i = T(\mu_{i-1}, \nu_{i-1}) \quad \text{and} \quad \nu_i = T(\nu_{i-1}, \mu_{i-1}), \quad \forall i \in \mathbb{N}. \tag{3.2}$$

If $\exists i^* \in \mathbb{N}$ such that $\mu_{i^*-1} = \mu_{i^*}$ and $\nu_{i^*-1} = \nu_{i^*}$, then we have

$$\mu_{i^*-1} = T(\mu_{i^*-1}, \nu_{i^*-1}) \quad \text{and} \quad \nu_{i^*-1} = T(\nu_{i^*-1}, \mu_{i^*-1}).$$

Thus, $(\nu_{i^*-1}, \mu_{i^*-1})$ is a coupled fixed point of T and the proof is complete. Otherwise, we may assume that

$$\mu_{i-1} \neq \mu_i \quad \text{or} \quad \nu_{i-1} \neq \nu_i, \quad \forall i \in \mathbb{N}.$$

Since $(T(\mu_o, \nu_o), T(\nu_o, \mu_o), \mu_o, \nu_o) = (\mu_1, \nu_1, \mu_o, \nu_o) \in A$ and A is a T -invariant set, we have $(T(\mu_1, \nu_1), T(\nu_1, \mu_1), T(\mu_o, \nu_o), T(\nu_o, \mu_o)) = (\mu_2, \nu_2, \mu_1, \nu_1) \in A$. Similarly, by the fact that A is a T -invariant set, we have

$$(T(\mu_2, \nu_2), T(\nu_2, \mu_2), T(\mu_1, \nu_1), T(\nu_1, \mu_1)) = (\mu_3, \nu_3, \mu_2, \nu_2) \in A.$$

Repeating the same argument, we get

$$(T(\mu_{i-1}, \nu_{i-1}), T(\nu_{i-1}, \mu_{i-1}), \mu_{i-1}, \nu_{i-1}) = (\mu_i, \nu_i, \mu_{i-1}, \nu_{i-1}) \in A.$$

Let us denote $(\frac{1}{F_m(\mu_i, \mu_{i-1}, t)} - 1 + \frac{1}{F_m(v_i, v_{i-1}, t)} - 1)$ by δ_{i-1} , for $t \gg \theta$, i.e.,

$$\delta_{i-1} = \left(\frac{1}{F_m(\mu_i, \mu_{i-1}, t)} - 1 + \frac{1}{F_m(v_i, v_{i-1}, t)} - 1 \right) > 0, \quad \text{for all } i \in \mathbb{N}.$$

Now we have to show that

$$\delta_i \leq 2\psi\left(\frac{\delta_{i-1}}{2}\right), \quad \text{for all } i \in \mathbb{N}.$$

Since $(\mu_i, v_i, \mu_{i-1}, v_{i-1}) \in A, \forall i \in \mathbb{N}$, and by (3.1), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 &= \frac{1}{F_m(T(\mu_{i-1}, v_{i-1}), T(\mu_i, v_i), t)} - 1 \\ &\leq \psi\left(\frac{1}{2}\left(\frac{1}{F_m(\mu_{i-1}, \mu_i, t)} - 1 + \frac{1}{F_m(v_{i-1}, v_i, t)} - 1\right)\right) \\ &= \psi\left(\frac{\delta_{i-1}}{2}\right). \end{aligned} \tag{3.3}$$

Since A is T -invariant set and $(\mu_i, v_i, \mu_{i-1}, v_{i-1}) \in A, \forall i \in \mathbb{N}$, we get that $(v_{i-1}, \mu_{i-1}, v_i, \mu_i) \in A, \forall i \in \mathbb{N}$. Now, again from (3.1), for $t \gg \theta$, and $(v_{i-1}, \mu_{i-1}, v_i, \mu_i) \in A, \forall i \in \mathbb{N}$, we may get

$$\begin{aligned} \frac{1}{F_m(v_i, v_{i+1}, t)} - 1 &= \frac{1}{F_m(T(v_{i-1}, \mu_{i-1}), T(v_i, \mu_i), t)} - 1 \\ &\leq \psi\left(\frac{1}{2}\left(\frac{1}{F_m(v_{i-1}, v_i, t)} - 1 + \frac{1}{F_m(\mu_{i-1}, \mu_i, t)} - 1\right)\right) \\ &= \psi\left(\frac{\delta_{i-1}}{2}\right). \end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\delta_i = 2\psi\left(\frac{\delta_{i-1}}{2}\right), \quad \text{for all } i \in \mathbb{N}. \tag{3.5}$$

Since $\forall \tau > 0, \psi(\tau) < \tau$, then from (3.5) we have

$$\delta_i = 2\psi\left(\frac{\delta_{i-1}}{2}\right) < \delta_{i-1}, \quad \text{for all } i \in \mathbb{N}.$$

Hence, (δ_i) is a monotone decreasing sequence, therefore $\lim_{i \rightarrow \infty} \delta_i = \delta$ for $\delta \geq 0$.

Next, we have to show that $\delta = 0$. By the contrary case, let $\delta > 0$, taking the limit $i \rightarrow \infty$ on both sides of (3.5), i.e., $\lim_{r \rightarrow \tau^+} \psi(r) < \tau, \forall \tau > 0$, then

$$\delta = \lim_{i \rightarrow \infty} \delta_i \leq 2 \lim_{i \rightarrow \infty} \psi\left(\frac{\delta_{i-1}}{2}\right) = 2 \lim_{\delta_{i-1} \rightarrow \delta^+} \psi\left(\frac{\delta_{i-1}}{2}\right) < 2\left(\frac{\delta}{2}\right) = \delta,$$

which is a contradiction to the fact that $\delta > 0$. Hence $\delta = 0$, therefore for $t \gg \theta$ we have

$$\delta = \lim_{i \rightarrow \infty} \delta_i = \lim_{i \rightarrow \infty} \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 + \frac{1}{F_m(v_i, v_{i+1}, t)} - 1 \right) = 0. \tag{3.6}$$

Next, we have to show that (μ_i) and (v_i) are Cauchy sequences in $(U, F_m, *)$. By supposition, let at least one, (μ_i) or (v_i) , be not a Cauchy sequence. Then $\exists \varepsilon > 0$ and the two subsequences of integers i_k and j_k with $i_k > j_k \geq k$ such that

$$r_k = \frac{1}{F_m(\mu_{j_k}, \mu_{i_k}, t)} - 1 + \frac{1}{F_m(v_{j_k}, v_{i_k}, t)} - 1 \geq \varepsilon, \quad \text{for all } k \in \{1, 2, 3, \dots\}. \tag{3.7}$$

Further, we choose i_k is the smallest integer such that $i_k > j_k \geq k$ and (3.6) holds, for $t \gg \theta$, we have

$$\frac{1}{F_m(\mu_{j_k}, \mu_{i_{k-1}}, t)} - 1 + \frac{1}{F_m(v_{j_k}, v_{i_{k-1}}, t)} - 1 < \varepsilon. \tag{3.8}$$

By using the F_m triangle inequality and from (3.7) and (3.8), for $t \gg \theta$, we have

$$\begin{aligned} \varepsilon \leq r_k &= \frac{1}{F_m(\mu_{j_k}, \mu_{i_k}, t)} - 1 + \frac{1}{F_m(v_{j_k}, v_{i_k}, t)} - 1 \\ &\leq \left(\frac{1}{F_m(\mu_{j_k}, \mu_{i_{k-1}}, t)} - 1 + \frac{1}{F_m(\mu_{i_{k-1}}, \mu_{i_k}, t)} - 1 \right) \\ &\quad + \left(\frac{1}{F_m(v_{j_k}, v_{i_{k-1}}, t)} - 1 + \frac{1}{F_m(v_{i_{k-1}}, v_{i_k}, t)} - 1 \right) \\ &= \left(\frac{1}{F_m(\mu_{j_k}, \mu_{i_{k-1}}, t)} - 1 + \frac{1}{F_m(v_{j_k}, v_{i_{k-1}}, t)} - 1 \right) \\ &\quad + \left(\frac{1}{F_m(\mu_{i_k}, \mu_{i_{k-1}}, t)} - 1 + \frac{1}{F_m(v_{i_k}, v_{i_{k-1}}, t)} - 1 \right) \\ &< \varepsilon + \delta_{i_{k-1}}, \\ \Rightarrow \quad \varepsilon \leq r_k &< \varepsilon + \delta_{i_{k-1}}. \end{aligned} \tag{3.9}$$

Now taking limit $k \rightarrow \infty$, and from (3.6), we have $\lim_{k \rightarrow \infty} r_k = \varepsilon > 0$. Since $i_k > j_k$ and A satisfies the transitive property, we get

$$(\mu_{i_k}, v_{i_k}, \mu_{j_k}, v_{j_k}) \in A \quad \text{and} \quad (v_{j_k}, \mu_{j_k}, v_{i_k}, \mu_{i_k}) \in A. \tag{3.10}$$

Now, in the view of (3.1) and (3.10), for $t \gg \theta$, we get

$$\begin{aligned} \frac{1}{F_m(\mu_{j_{k+1}}, \mu_{i_{k+1}})} - 1 &= \frac{1}{F_m(T(\mu_{j_k}, v_{j_k}), T(\mu_{i_k}, v_{i_k}), t)} - 1 \\ &\leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu_{j_k}, \mu_{i_k}, t)} - 1 + \frac{1}{F_m(v_{j_k}, v_{i_k}, t)} - 1 \right) \right) \\ &= \psi \left(\frac{r_k}{2} \right). \end{aligned} \tag{3.11}$$

Similarly, for $t \gg \theta$,

$$\frac{1}{F_m(v_{j_{k+1}}, v_{i_{k+1}})} - 1 = \frac{1}{F_m(T(v_{j_k}, \mu_{j_k}), T(v_{i_k}, \mu_{i_k}), t)} - 1$$

$$\begin{aligned} &\leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(v_{j_k}, v_{i_k}, t)} - 1 + \frac{1}{F_m(\mu_{j_k}, \mu_{i_k}, t)} - 1 \right) \right) \\ &= \psi \left(\frac{r_k}{2} \right). \end{aligned} \tag{3.12}$$

Adding (3.11) and (3.12), we have

$$r_{k+1} \leq 2\psi \left(\frac{r_k}{2} \right), \quad \text{for all } k \in \{1, 2, 3, \dots\}. \tag{3.13}$$

Now, using the limit $k \rightarrow \infty$ on both sides of (3.13), i.e., $\lim_{r \rightarrow \tau^+} \psi(r) < \tau, \forall \tau > 0$, we have

$$\varepsilon = \lim_{k \rightarrow \infty} r_{k+1} \leq 2 \lim_{k \rightarrow \infty} \psi \left(\frac{r_k}{2} \right) = 2 \lim_{r_k \rightarrow \varepsilon^+} \psi \left(\frac{r_k}{2} \right) < 2 \left(\frac{\varepsilon}{2} \right) = \varepsilon,$$

which is contradiction. Hence (μ_i) and (v_i) are Cauchy sequences. Since $(U, F_m, *)$ is complete, $\exists \mu, v \in U$ such that

$$\mu_i \rightarrow \mu \quad \text{and} \quad v_i \rightarrow v, \quad \text{as } i \rightarrow \infty. \tag{3.14}$$

Now, finally we have to show that $T(\mu, v) = \mu$ and $T(v, \mu) = v$. If assertion (1) holds, then we have

$$\mu = \lim_{i \rightarrow \infty} \mu_{i+1} = \lim_{i \rightarrow \infty} T(\mu_i, v_i) = T \left(\lim_{i \rightarrow \infty} \mu_i, \lim_{i \rightarrow \infty} v_i \right) = T(\mu, v),$$

and

$$v = \lim_{i \rightarrow \infty} v_{i+1} = \lim_{i \rightarrow \infty} T(v_i, \mu_i) = T \left(\lim_{i \rightarrow \infty} v_i, \lim_{i \rightarrow \infty} \mu_i \right) = T(v, \mu).$$

Hence, $\mu = T(\mu, v)$ and $v = T(v, \mu)$, i.e., T has a coupled fixed point in U . Suppose that, if assertion (2) holds. We obtain two sequences (μ_i) and (v_i) converging to μ and v respectively for some $\mu, v \in U$. Then, by supposition, we have $(\mu, v, \mu_i, v_i) \in A, \forall i \in \mathbb{N}$. Since F_m is triangular and by (3.1), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{F_m(T(\mu, v), \mu, t)} - 1 &\leq \frac{1}{F_m(T(\mu, v), \mu_{i+1}, t)} - 1 + \frac{1}{F_m(\mu_{i+1}, \mu, t)} - 1 \\ &= \frac{1}{F_m(T(\mu, v), T(\mu_i, v_i), t)} - 1 + \frac{1}{F_m(\mu_{i+1}, \mu, t)} - 1 \\ &\leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, \mu_i, t)} - 1 + \frac{1}{F_m(v, v_i, t)} - 1 \right) \right) + \frac{1}{F_m(\mu_{i+1}, \mu, t)} - 1 \\ &\rightarrow 0, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Hence we get that $F_m(T(\mu, v), \mu, t) = 1$, this implies that $\mu = T(\mu, v)$. Similarly, we can prove that $v = T(v, \mu)$. Thus, T has a coupled fixed point in U . □

Example 3.7 Let $U = [0, \infty)$, $*$ be a continuous t -norm, and $F_m : U \times U \times (0, \infty) \rightarrow [0, 1]$ be defined as

$$F_m(\mu, v, t) = \frac{t}{t + m(\mu, v)},$$

where $m(\mu, \nu) = |\mu - \nu|$ is a usual metric $\forall \mu, \nu \in U$, and $t > 0$. The partial order is usually defined as $\mu \leq \nu \Leftrightarrow \nu - \mu \in [0, \infty)$. Let a continuous mapping $T : U \times U \rightarrow U$ be defined as

$$T(\mu, \nu) = \frac{2\mu + 2\nu + 4}{5}, \quad \forall \mu, \nu \in U^2.$$

The mixed monotone property is not satisfied. If we choose $\nu_1 = 4$ and $\nu_2 = 5$, then $\nu_1 \leq \nu_2$, implies that $T(\mu, \nu_1) \leq T(\mu, \nu_2)$. Further, we define a mapping $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(\tau) = \frac{4}{5}\tau, \forall \tau > 0$. By a direct calculation, $\forall \mu, \nu, x, y \in U$, we have that

$$\begin{aligned} \frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 &= \frac{1}{t} (m(T(\mu, \nu), T(x, y))) \\ &= \frac{1}{t} \left| \frac{2\mu + 2\nu + 4}{5} - \frac{2x + 2y + 4}{5} \right| \\ &\leq \frac{2}{5t} (m(\mu, x) + m(\nu, y)) \\ &= \frac{4}{5t} \left(\frac{1}{2} (m(\mu, x) + m(\nu, y)) \right) \\ &= \frac{4}{5} \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right) \right) \\ &= \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right) \right). \end{aligned}$$

Hence all the conditions of Theorem 3.6 are satisfied with $A = U^4$ and T has a unique coupled fixed point, i.e., $T(4, 4) = 4$.

If we define a mapping $\psi(\tau) = \lambda\tau$ for any $\lambda \in [0, 1)$ in Theorem 3.6, then we get the following.

Corollary 3.8 *Let A be a nonempty subset of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular. Let $T : U \times U \rightarrow U$ be a mapping and $\exists \lambda \in [0, 1)$ such that*

$$\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \leq \lambda \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right), \tag{3.15}$$

for all $(\mu, \nu, x, y) \in A$. Assume that either:

- (1) T is continuous, or
- (2) If for any two sequences (μ_i) and (ν_i) with $(\mu_{i+1}, \nu_{i+1}, \mu_i, \nu_i) \in A$, where $(\mu_i) \rightarrow \mu$, and $(\nu_i) \rightarrow \nu$, for all $i \geq 1, \exists \mu, \nu \in U$ such that $\mu = T(\mu, \nu)$ and $\nu = T(\nu, \mu)$, then A is a T -invariant set which satisfies the transitive property. Then T has a coupled fixed point.

In the following theorem we prove the uniqueness of a coupled fixed point of a mapping T on U .

Theorem 3.9 *By addition to the hypotheses of Theorem 3.6, assume that $\forall (\mu, \nu), (g, h) \in X^2, \exists (x, y) \in U^2$ such that $(\mu, \nu, \mu, \nu) \in A$ and $(g, h, x, y) \in A$. Then T has a unique coupled fixed point in U .*

Proof From the proof of Theorem 3.6, the mapping T has a coupled fixed point in U . Assume that (μ, ν) and (g, h) are coupled fixed points of T , i.e., $T(\mu, \nu) = \mu$, $T(\nu, \mu) = \nu$, $T(g, h) = g$, and $T(h, g) = h$. We have to show that $\mu = g$ and $\nu = h$.

By hypotheses, $\exists (x, y) \in X^2$ such that $(\mu, \nu, g, h) \in A$ and $(g, h, x, y) \in A$.

Let $x = x_0$ and $y = y_0$. We define two sequences (x_i) and (y_i) such that $x_i = T(x_{i-1}, y_{i-1})$, $y_i = T(y_{i-1}, x_{i-1})$ for all $i \in \mathbb{N}$. Since A is T -invariant and $(\mu, \nu, x_0, y_0) = (\mu, \nu, x, y) \in A \Rightarrow (T(\mu, \nu), T(\nu, \mu), T(x_0, y_0), T(y_0, x_0)) \in A$, i.e., $(\mu, \nu, x_1, y_1) \in A$.

Again, by the property of T -invariance, we have

$$(T(\mu, \nu), T(\nu, \mu), T(x_1, y_1), T(y_1, x_1)) \in A, \quad \text{i.e. } (\mu, \nu, x_2, y_2) \in A.$$

Repeating this argument, we get $(\mu, \nu, x_i, y_i) \in A, \forall i \in \mathbb{N}$. Now, in view of (3.1), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{F_m(\mu, x_{i+1}, t)} - 1 &= \frac{1}{F_m(T(\mu, \nu), T(x_i, y_i), t)} - 1 \\ &\leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x_i, t)} - 1 + \frac{1}{F_m(\nu, y_i, t)} - 1 \right) \right). \end{aligned} \tag{3.16}$$

Since A is T -invariant and $(\mu, \nu, x_i, y_i) \in A, \forall i \in \mathbb{N}$, we have $(y_i, x_i, \nu, \mu) \in A, \forall i \in \mathbb{N}$. Then, again by (3.1), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{F_m(y_{i+1}, \nu, t)} - 1 &= \frac{1}{F_m(T(y_i, x_i), T(\nu, \mu), t)} - 1 \\ &\leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(y_i, \nu, t)} - 1 + \frac{1}{F_m(x_i, \mu, t)} - 1 \right) \right). \end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(y_{i+1}, \nu, t)} - 1 \right) \\ \leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x_i, t)} - 1 + \frac{1}{F_m(\nu, y_i, t)} - 1 \right) \right). \end{aligned} \tag{3.18}$$

Repeating the same argument, for $t \gg \theta$ and $\forall i \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(y_{i+1}, \nu, t)} - 1 \right) \\ \leq \psi^i \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x_1, t)} - 1 + \frac{1}{F_m(\nu, y_1, t)} - 1 \right) \right). \end{aligned} \tag{3.19}$$

Now from mapping ψ , $\psi(\tau) < \tau$ and $\lim_{r \rightarrow \tau^+} \psi(r) < \tau$, it follows that $\lim_{i \rightarrow \infty} \psi^i(\tau) = 0, \forall \tau > 0$. Hence, from (3.19), for $t \gg \theta$, we get

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(y_{i+1}, \nu, t)} - 1 \right) \\ = \lim_{i \rightarrow \infty} \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(\nu, y_{i+1}, t)} - 1 \right) = 0. \end{aligned} \tag{3.20}$$

Similarly, for $t \gg \theta$, we can prove that

$$\begin{aligned} & \lim_{i \rightarrow \infty} \left(\frac{1}{F_m(g, x_{i+1}, t)} - 1 + \frac{1}{F_m(y_{i+1}, h, t)} - 1 \right) \\ &= \lim_{i \rightarrow \infty} \left(\frac{1}{F_m(g, x_{i+1}, t)} - 1 + \frac{1}{F_m(h, y_{i+1}, t)} - 1 \right) = 0. \end{aligned} \tag{3.21}$$

Since F_m is triangular and from (3.20) and (3.21) $\forall i \in \mathbb{N}$, and $t \gg \theta$, we have

$$\begin{aligned} & \frac{1}{F_m(\mu, g, t)} - 1 + \frac{1}{F_m(v, h, t)} - 1 \\ & \leq \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(x_{i+1}, g, t)} - 1 \right) + \left(\frac{1}{F_m(v, y_{i+1}, t)} - 1 + \frac{1}{F_m(y_{i+1}, h, t)} - 1 \right) \\ &= \left(\frac{1}{F_m(\mu, x_{i+1}, t)} - 1 + \frac{1}{F_m(v, y_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(x_{i+1}, g, t)} - 1 + \frac{1}{F_m(y_{i+1}, h, t)} - 1 \right) \\ & \rightarrow 0, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

We get that $\left(\frac{1}{F_m(\mu, g, t)} - 1 + \frac{1}{F_m(v, h, t)} - 1 \right) = 0 \Rightarrow F_m(\mu, g, t) = 1$, i.e., $\mu = g$ and $F_m(v, h, t) = 1$, i.e., $v = h$. Hence T has a unique coupled fixed point. This completes the proof.

In the following we prove some coupled fixed point results by partial ordered metric space in *FCM*-spaces.

Let $C(U)$ denote the collection of all subsets of a set U , and a pair (U, \leq) denotes the partially ordered set with partially ordered \leq . A mapping $F : U \rightarrow U$ is known as nondecreasing (resp. nonincreasing) if $\forall a, b \in U$ such that $a \leq b \Rightarrow F(a) \leq F(b)$ (resp. $F(b) \leq F(a)$). \square

Definition 3.10 ([38]) Let a pair (U, \leq) be a partially ordered set, and a mapping $T : U \times U \rightarrow U$ is known to have a mixed monotone property if $T : U \times U \rightarrow U$ is monotone nondecreasing in the first argument, and T is monotone nonincreasing in the second argument $\forall \mu, v \in U$:

- (i) $\mu, \mu^* \in U, \mu \leq \mu^* \Rightarrow T(\mu, v) \leq T(\mu^*, v)$.
- (ii) $v, v^* \in U, v \leq v^* \Rightarrow T(\mu, v) \geq T(\mu, v^*)$.

Example 3.11 Let (U, m) be a metric space with partial ordered “ \leq ”, let $(U, F_m, *)$ be an *FCM*-space defined $\forall \mu, v \in U$ and $t \gg \theta$ as follows: $F_m(\mu, v, t) = \frac{t}{t+m(\mu, v)}$ with $m(\mu, v) = |\mu - v|$. A mapping $T : U^2 \rightarrow U$ satisfies the mixed monotone property $\forall \mu, v \in U$ such that

$$\mu, \mu^* \in U, \quad \mu \leq \mu^* \quad \Rightarrow \quad T(\mu, v) \leq T(\mu^*, v),$$

and

$$v, v^* \in U, \quad v \leq v^* \quad \Rightarrow \quad T(\mu, v^*) \leq T(\mu, v).$$

We define a subset $A \subseteq U^4$ by $A = \{(\mu, \mu^*, v, v^*) \in U^4 : \mu \geq v, \mu^* \leq v^*\}$. Then A is a T -invariant subset of U^4 which satisfies the transitive property.

Theorem 3.12 *Assume that (U, \preceq) is a partial ordered set, and let $(U, F_m, *)$ be a complete FCM-space in which F_m is triangular. Let there be a mapping $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0 < \psi(\tau) < \tau$, and $\lim_{r \rightarrow \tau^+} \psi(r) < \tau$ for $\tau > 0$, and suppose that a mapping $T : U \times U \rightarrow U$ has the mixed monotone property and satisfies*

$$\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \leq \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right) \right), \tag{3.22}$$

for all $(\mu, \nu, x, y) \in U$ for which $\mu \succeq x$ and $\nu \preceq y$. Assume that either:

- (1) T is continuous, or
- (2) U has the following properties:
 - (i) If a nondecreasing sequence (μ_i) in U with $\mu_i \rightarrow \mu$, then $\mu_i \preceq \mu, \forall i \in \mathbb{N}$;
 - (ii) If a nonincreasing sequence (ν_i) in U with $\nu_i \rightarrow \nu$, then $\nu \succeq \nu_i, \forall i \in \mathbb{N}$.
 If there exist $\mu_o, \nu_o \in U$ such that $\mu_o \preceq T(\mu_o, \nu_o)$ and $\nu_o \succeq T(\nu_o, \mu_o)$ such that $\mu = T(\mu, \nu)$ and $\nu = T(\nu, \mu)$, then T has a coupled fixed point.

Proof Let a subset $A \subseteq U^4$ be defined as $A = \{(\mu, \mu^*, \nu, \nu^*) \in U^4 : \mu \succeq \nu \text{ and } \mu^* \preceq \nu^*\}$. Now from Example 3.11 we conclude that A is a T -invariant set which satisfies the transitive property. In view of (3.22), $\forall \mu, \nu, x, y \in U$ with $(\mu, \nu, x, y) \in A$. Since $\mu_o, \nu_o \in U$ such that $\mu_o \preceq T(\mu_o, \nu_o)$ and $\nu_o \succeq T(\nu_o, \mu_o)$, we get that $(T(\mu_o, \nu_o), T(\nu_o, \mu_o), \mu_o, \nu_o) \in A$.

Now for assertion (2), (μ_i) and (ν_i) are any two sequences in U such that (μ_i) is a non-decreasing sequence with $\mu_i \rightarrow \mu$ and (ν_i) is a nonincreasing sequence with $\nu_i \rightarrow \nu$. Then we have

$$\mu_1 \preceq \mu_2 \preceq \dots \preceq \mu_i \preceq \mu \quad \text{and} \quad \nu_1 \succeq \nu_2 \succeq \dots \nu_i \succeq \dots \succeq \nu, \quad \forall i \in \mathbb{N}.$$

Therefore, $(\mu, \nu, \mu_i, \nu_i) \in A, i \in \mathbb{N}$, hence assertion (2) of Theorem 3.6 is satisfied. Since all the hypotheses of Theorem 3.6 are satisfied, F has a coupled fixed point. □

Corollary 3.13 *In addition to the hypotheses of Theorem 3.12, assume that $\forall (\mu, \nu), (g, h) \in U^2, \exists (x, y) \in X^2$ such that $\mu \succeq x, \nu \preceq y$ and $g \succeq x, h \preceq y$. Then T has a unique coupled fixed point.*

Proof Let a subset $A \subseteq X^4$ be defined as $A = \{(\mu, \mu^*, \nu, \nu^*) \in X^4 : \mu \succeq \nu \text{ and } \mu^* \preceq \nu^*\}$. Now, from Example 3.11, we conclude that A is a T -invariant set which satisfies the transitive property. Then, through the proof of Theorem 3.12, easily from simple calculation we can get the existence of a coupled fixed point.

Uniqueness: Now we have to show the unique coupled fixed point of the mapping T . Since $\forall (\mu, \nu), (g, h) \in U^2, \exists (x, y) \in U^2$ such that $\mu \succeq x, \nu \preceq y$ and $g \succeq x, h \preceq y$, we conclude that $(\mu, \nu, x, y) \in A$ and $(g, h, x, y) \in A$. Thus all the hypotheses of Theorem 3.9 hold and T has a unique coupled fixed point. □

4 Strong coupled fixed point results in FCM-spaces

Definition 4.1 ([6]) Let A and B be two nonempty subsets of a given set U . A mapping $T : U \times U \rightarrow U$ such that $T(\mu, \nu) \in A$ if $\mu \in B$ and $\nu \in A$, and $T(\mu, \nu) \in B$ if $\mu \in A$ and $\nu \in B$ is called a cyclic map w.r.t A and B .

Definition 4.2 ([6, 38]) Let U be a nonempty set, and an element $(\mu, \nu) \in U^2$ is called a coupled fixed point of the mapping $T : U \times U \rightarrow U$ if $T(\mu, \nu) = \mu$ and $T(\nu, \mu) = \nu$, and it is called a strong coupled fixed point if $\mu = \nu$, that is, $T(\mu, \mu) = \mu$.

Definition 4.3 ([6]) Let A and B be two nonempty subsets of a metric space (U, m) . Then a mapping $T : U \times U \rightarrow U$ is called a cyclic coupled Kannan type contraction w.r.t A and B if T is cyclic w.r.t A and B satisfying

$$m(T(\mu, \nu), T(x, y)) \leq a(m(u, T(\mu, \nu)) + m(x, T(x, y))), \tag{4.1}$$

where $\mu, y \in A, \nu, x \in B$, and $a \in (0, \frac{1}{2})$.

Definition 4.4 Let A and B be two nonempty closed subsets of an FCM -space $(U, F_m, *)$. A mapping $T : U \times U \rightarrow U$ is called a cyclic coupled Kannan type fuzzy cone contraction w.r.t. A and B if T is cyclic w.r.t. A and B satisfying the inequality

$$\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \leq a \left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 + \frac{1}{F_m(x, T(x, y), t)} - 1 \right), \tag{4.2}$$

where $\mu, y \in A, \nu, x \in B, t \gg \theta$, and $a \in (0, \frac{1}{2})$.

In the following, we shall study a more generalized cyclic coupled type fuzzy cone contraction condition in $(U, F_m, *)$ and prove some strong coupled fixed point results in FCM -spaces. A mapping $T : U \times U \rightarrow U$ is known as a generalized cyclic coupled type fuzzy cone contraction condition in FCM -spaces if T satisfies the inequality

$$\begin{aligned} & \frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \\ & \leq a \left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \right) + b \left(\frac{1}{F_m(x, T(x, y), t)} - 1 \right) \\ & \quad + c \left(\frac{1}{F_m(x, T(\mu, \nu), t)} - 1 \right) + d \left(\frac{1}{F_m(\mu, T(x, y), t)} - 1 \right), \end{aligned} \tag{4.3}$$

where $\mu, y \in A, \nu, x \in B, t \gg \theta$, and $a, b, c, d \in [0, \infty)$. We note that (4.3) is the same as (4.2) if $a = b \in (0, \frac{1}{2})$ and $c = d = 0$. Also, we illustrate some examples to support our results.

Theorem 4.5 Assume that A and B are two nonempty closed subsets of a complete FCM -space $(U, F_m, *)$ in which F_m is triangular and $T : U \times U \rightarrow U$ is a generalized cyclic coupled type fuzzy cone contraction w.r.t A and B . Suppose that T satisfies (4.3) with $(a + b + 2c + 2d) < 1$. Then $A \cap B \neq \emptyset$ and T has a strong coupled fixed in $A \cap B$.

Proof Fix $\mu_0 \in A$ and $\nu_0 \in B$. Let (μ_i) and (ν_i) be two sequences defined as

$$\mu_{i+1} = T(\nu_i, \mu_i) \quad \text{and} \quad \nu_{i+1} = T(\mu_i, \nu_i), \quad \text{for all } i \geq 0. \tag{4.4}$$

Then $(\mu_i) \subset A$ and $(\nu_i) \subset B$ since T is a cyclic mapping w.r.t A and B . We denote the following:

$$h = \frac{a + d}{1 - (b + d)}.$$

Then $h \in (0, 1)$ for $a + b + 2c + 2d < 1$. We claim that, for $t \gg \theta$ and $i \geq 0$,

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) \\ & \leq h^i \left(\frac{1}{F_m(\mu_0, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_0, \mu_1, t)} - 1 \right). \end{aligned} \tag{4.5}$$

It is clear that (4.5) holds for $i = 0$. Suppose that (4.5) holds for $i = k$ for $t \gg \theta$, then by (4.3) we have

$$\begin{aligned} & \frac{1}{F_m(\mu_{k+1}, \nu_{k+2}, t)} - 1 \\ & = \frac{1}{F_m(T(\nu_k, \mu_k), T(\mu_{k+1}, \nu_{k+1}), t)} - 1 \\ & \leq a \left(\frac{1}{F_m(\nu_k, T(\nu_k, \mu_k), t)} - 1 \right) + b \left(\frac{1}{F_m(\mu_{k+1}, T(\mu_{k+1}, \nu_{k+1}), t)} - 1 \right) \\ & \quad + c \left(\frac{1}{F_m(\mu_{k+1}, T(\nu_k, \mu_k), t)} - 1 \right) + d \left(\frac{1}{F_m(\nu_k, T(\mu_{k+1}, \nu_{k+1}), t)} - 1 \right) \\ & \leq a \left(\frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 \right) + b \left(\frac{1}{F_m(\mu_{k+1}, \nu_{k+2}, t)} - 1 \right) \\ & \quad + d \left(\frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 + \frac{1}{F_m(\mu_{k+1}, \nu_{k+2}, t)} - 1 \right), \end{aligned}$$

which implies that

$$\frac{1}{F_m(\mu_{k+1}, \nu_{k+2}, t)} - 1 \leq h \left(\frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 \right), \quad \text{for } t \gg \theta.$$

Similarly, in view of (4.3),

$$\begin{aligned} & \frac{1}{F_m(\nu_{k+1}, \mu_{k+2}, t)} - 1 \\ & = \frac{1}{F_m(T(\mu_k, \nu_k), T(\nu_{k+1}, \mu_{k+1}), t)} - 1 \\ & \leq a \left(\frac{1}{F_m(\mu_k, T(\mu_k, \nu_k), t)} - 1 \right) + b \left(\frac{1}{F_m(\nu_{k+1}, T(\nu_{k+1}, \mu_{k+1}), t)} - 1 \right) \\ & \quad + c \left(\frac{1}{F_m(\nu_{k+1}, T(\mu_k, \nu_k), t)} - 1 \right) + d \left(\frac{1}{F_m(\mu_k, T(\nu_{k+1}, \mu_{k+1}), t)} - 1 \right) \\ & \leq a \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 \right) + b \left(\frac{1}{F_m(\nu_{k+1}, \mu_{k+2}, t)} - 1 \right) \\ & \quad + d \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 + \frac{1}{F_m(\nu_{k+1}, \mu_{k+2}, t)} - 1 \right), \end{aligned}$$

which implies that

$$\frac{1}{F_m(\nu_{k+1}, \mu_{k+2}, t)} - 1 \leq h \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 \right), \quad \text{for } t \gg \theta.$$

Thus, by the induction hypothesis, i.e., (4.5) with $i = k$ for $t \gg \theta$, we have

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_{k+1}, \nu_{k+2}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_{k+1}, \mu_{k+2}, t)} - 1 \right) \\ & \leq h \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 + \frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 \right) \\ & \leq \dots \leq h^{k+1} \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{M(\nu_o, \mu_1, t)} - 1 \right). \end{aligned}$$

That is, (4.5) holds for $i = k + 1$. Therefore, we have proved that (4.5) holds for all $i \geq 0$ by induction. Meanwhile, by (4.3), for $i \geq 0$,

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \nu_{i+1}, t)} - 1 \right) \\ & \leq \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 + \frac{1}{F_m(\nu_{i+1}, \mu_{i+1}, t)} - 1 \right) \\ & \quad + \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 + \frac{1}{F_m(\mu_{i+1}, \nu_{i+1}, t)} - 1 \right) \\ & = \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) + 2 \left(\frac{1}{F_m(\mu_{i+1}, \nu_{i+1}, t)} - 1 \right) \\ & = \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) + 2 \left(\frac{1}{F_m(F(\nu_i, \mu_i), F(\mu_i, \nu_i), t)} - 1 \right) \\ & \leq \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) + 2\alpha \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) \\ & \quad + 2b \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right) + 2c \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + 2d \left(\frac{1}{F_m(\nu_i, \nu_{i+1}, t)} - 1 \right). \end{aligned}$$

Here, we suppose that $\alpha = \max\{a, b\}$ and $\beta = \max\{c, d\}$, then we have

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \nu_{i+1}, t)} - 1 \right) \\ & \leq (1 + 2\alpha) \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 + \frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) \\ & \quad + 2\beta \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 + \frac{1}{F_m(\nu_i, \nu_{i+1}, t)} - 1 \right). \end{aligned}$$

This together with (4.5) implies that

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_i, \nu_{i+1}, t)} - 1 \right) \\ & \leq \frac{1 + 2\alpha}{1 - 2\beta} h^i \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right), \quad \text{for } t \gg \theta. \end{aligned} \tag{4.6}$$

Then, for $i, j \geq 0$, without loss of generality we assume that $i \leq j$,

$$\frac{1}{F_m(\mu_i, \mu_j, t)} - 1 \leq \sum_{k=i}^{j-1} \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right)$$

$$\begin{aligned} &\leq \sum_{k=i}^{j-1} \frac{1+2\alpha}{1-2\beta} h^i \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right) \\ &= \frac{1+2\alpha}{(1-2\beta)(1-h)} h^i \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right) \\ &\rightarrow \theta, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

This implies that (μ_i) is a Cauchy sequence and hence convergent in X . Since A is a nonempty closed subset of U , therefore

$$\mu_i \rightarrow \mu \in A, \quad \text{as } i \rightarrow \infty. \tag{4.7}$$

Similarly,

$$\nu_i \rightarrow \nu \in B, \quad \text{as } i \rightarrow \infty. \tag{4.8}$$

So, from (4.7) and (4.8), we have

$$\lim_{i \rightarrow \infty} F_m(\mu_i, \nu_i, t) = F_m(\mu, \nu, t), \quad \text{for } t \gg \theta. \tag{4.9}$$

Since F_m is triangular, by (4.5) and (4.6),

$$\begin{aligned} \frac{1}{F_m(\mu_i, \nu_i, t)} - 1 &\leq \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\mu_{i+1}, \nu_i, t)} - 1 \right) \\ &\leq \left(\frac{1+2\alpha}{1-2\beta} + 1 \right) h^i \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right) \\ &\rightarrow \theta, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Therefore, $F_m(\mu, \nu, t) = 1$, for $t \gg \theta$ and hence $\mu = \nu \in A \cap B$.

Now we have to prove that μ is a strong coupled fixed point of T by using the F_m triangularity condition, we have

$$\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \leq \left(\frac{1}{F_m(\mu, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\mu_{i+1}, T(\mu, \nu), t)} - 1 \right), \tag{4.10}$$

for $t \gg \theta$. In view of (4.3), (4.7), and (4.8),

$$\begin{aligned} &\frac{1}{F_m(\mu_{i+1}, T(\mu, \nu), t)} - 1 \\ &= \frac{1}{F_m(T(\nu_i, \mu_i), T(\mu, \nu), t)} - 1 \\ &\leq a \left(\frac{1}{F_m(\nu_i, T(\nu_i, \mu_i), t)} - 1 \right) + b \left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \right) \\ &\quad + c \left(\frac{1}{F_m(\mu, T(\nu_i, \mu_i), t)} - 1 \right) + d \left(\frac{1}{F_m(\nu_i, T(\mu, \nu), t)} - 1 \right) \\ &\leq a \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right) + b \left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \right) \end{aligned}$$

$$\begin{aligned}
 &+ c \left(\frac{1}{F_m(\mu, \mu_{i+1}, t)} - 1 \right) + d \left(\frac{1}{F_m(v_i, \mu, t)} - 1 + \frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right) \\
 &\rightarrow (b + d) \left(\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right), \quad \text{as } i \rightarrow \infty.
 \end{aligned}$$

Then

$$\limsup_{i \rightarrow \infty} \left(\frac{1}{F_m(\mu_{i+1}, T(\mu, v), t)} - 1 \right) \leq (b + d) \left(\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right), \quad \text{for } t \gg \theta.$$

Hence this together with (4.10) implies that

$$\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \leq (b + d) \left(\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right), \quad \text{for } t \gg \theta.$$

Since $b + d < 1$, which yields that $F_m(\mu, T(\mu, v), t) = 1$. This implies that $T(\mu, v) = \mu = v$ is the strong coupled fixed point of T . □

Corollary 4.6 *Assume that A and B are two nonempty subsets of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular and $T : U \times U \rightarrow U$ is a cyclic coupled type fuzzy cone contraction w.r.t. A and B . Suppose that T satisfies the inequality*

$$\begin{aligned}
 &\frac{1}{F_m(T(\mu, v), T(x, y), t)} - 1 \\
 &\leq a \left(\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right) \\
 &\quad + b \left(\frac{1}{F_m(x, T(x, y), t)} - 1 \right) + c \left(\frac{1}{F_m(x, T(\mu, v), t)} - 1 \right),
 \end{aligned} \tag{4.11}$$

where $\mu, y \in A, v, x \in B$, and $t \gg \theta$ for $a, b, c \in [0, \infty)$ with $(a + b + 2c) < 1$. Then $A \cap B \neq \emptyset$ and T has a strong coupled fixed point in $A \cap B$.

Corollary 4.7 *Assume that A and B are two nonempty closed subsets of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular and $T : U \times U \rightarrow U$ is a cyclic coupled type fuzzy cone contraction w.r.t. A and B . Suppose that T satisfies the inequality*

$$\begin{aligned}
 &\frac{1}{F_m(T(\mu, v), T(x, y), t)} - 1 \\
 &\leq a \left(\frac{1}{F_m(\mu, T(\mu, v), t)} - 1 \right) \\
 &\quad + b \left(\frac{1}{F_m(x, T(x, y), t)} - 1 \right) + d \left(\frac{1}{F_m(\mu, T(x, y), t)} - 1 \right),
 \end{aligned} \tag{4.12}$$

where $\mu, y \in A, v, x \in B$, and $t \gg \theta$ for some $a, b, d \in [0, \infty)$ with $(a + b + 2d) < 1$. Then $A \cap B \neq \emptyset$, and T has a strong coupled fixed point in $A \cap B$.

If $a = b$ and $c = d = 0$ in (4.3), then we may get the following corollary of Kannan type for a cyclic coupled fixed point in FCM-spaces.

Corollary 4.8 Assume that A and B are two nonempty closed subsets of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular and $T : U \times U \rightarrow U$ is a cyclic coupled Kannan type fuzzy cone contraction w.r.t A and B satisfying (4.2) for some $a \in (0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$, and T has a strong coupled fixed in $A \cap B$.

Remark 4.9 In a special case, Theorem 4.5, Corollary 4.6, Corollary 4.7, Corollary 4.8, and the result in reference (see [6, Theorem 5]) contains the same results if $a = b \in (0, 1/2)$ and $c = d = 0$.

Example 4.10 Let $U = [0, \infty)$, $*$ be a continuous t -norm, and $F_m : U \times U \times (0, \infty) \rightarrow [0, 1]$ be defined as follows:

$$F_m(\mu, \nu, t) = \frac{t}{t + m(\mu, \nu)},$$

where $m(\mu, \nu) = |\mu - \nu|, \forall \mu, \nu \in U$, and $t \gg \theta$. Then easily one can prove that F_m is triangular and $(U, F_m, *)$ is a complete FCM-space. Let $A = [0, 1]$ and $B = [0, \frac{1}{3}]$ be two nonempty closed subsets of U and $m(A, B) = 0$. Let a mapping $T : U \times U \rightarrow U$ be defined as follows:

$$T(\mu, \nu) = \begin{cases} \frac{\mu + 4\nu}{20}, & \text{if } \mu, \nu \in [0, 1], \\ \frac{2\mu + \nu + 10}{5}, & \text{if } \mu, \nu \in (1, \infty). \end{cases}$$

Then easily one can verify that T is a cyclic mapping w.r.t A and B for any $\mu, y \in A$ and $\nu, x \in B$. Now, for $t \gg \theta$, we have

$$\begin{aligned} & \frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \\ &= \frac{1}{t} m(T(\mu, \nu), T(x, y)) \\ &= \frac{1}{t} \left| \frac{\mu - x}{20} + \frac{4(v - y)}{20} \right| \\ &\leq \frac{12}{35t} \left| \frac{19(\mu + x)}{20} - \frac{4(v + y)}{20} \right| \\ &\leq \frac{1}{5t} \left| \frac{19(\mu + x)}{20} - \frac{4(v + y)}{20} \right| + \frac{1}{7t} \left| \frac{19(\mu + x)}{20} - \frac{4(v + y)}{20} \right| \\ &= \frac{1}{5t} \left(\left| \mu - \frac{\mu + 4\nu}{20} + x - \frac{x + 4y}{20} \right| \right) + \frac{1}{7t} \left(\left| x - \frac{\mu + 4\nu}{20} + \mu - \frac{x + 4y}{20} \right| \right) \\ &\leq \frac{1}{5t} \left(\left| \mu - \frac{\mu + 4\nu}{20} \right| + \left| x - \frac{x + 4y}{20} \right| \right) + \frac{1}{7t} \left(\left| x - \frac{\mu + 4\nu}{20} \right| + \left| \mu - \frac{x + 4y}{20} \right| \right) \\ &= \frac{1}{5t} m(\mu, T(\mu, \nu)) + \frac{1}{5t} m(x, T(x, y)) + \frac{1}{7t} m(x, T(\mu, \nu)) + \frac{1}{7t} m(\mu, T(x, y)) \\ &= \frac{1}{5} \left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \right) + \frac{1}{5} \left(\frac{1}{F_m(x, T(x, y), t)} - 1 \right) \\ &\quad + \frac{1}{7} \left(\frac{1}{F_m(x, T(\mu, \nu), t)} - 1 \right) + \frac{1}{7} \left(\frac{1}{F_m(\mu, T(x, y), t)} - 1 \right). \end{aligned}$$

Hence, inequality (4.3) is satisfied with $a = b = \frac{1}{5}$ and $c = d = \frac{1}{7}$ for $t \gg \theta$. Thus all the conditions of Theorem 4.5 are satisfied and T has a strong coupled fixed point, that is, $T(5, 5) = 5 \in (1, \infty)$.

Theorem 4.11 *Assume that A and B are two nonempty closed subsets of a complete FCM-space $(U, F_m, *)$ in which F_m is triangular and $T : U \times U \rightarrow U$ is a cyclic coupled contractive type mapping w.r.t. A and B for some $a \in [0, 1)$ satisfying*

$$\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \leq a \left(\frac{1}{\min\{F_m(\mu, T(\mu, \nu), t), F_m(x, T(x, y), t)\}} - 1 \right), \tag{4.13}$$

where $\mu, y \in A, \nu, x \in B$, and $t \gg \theta$. Then $A \cap B \neq \emptyset$ and T has a strong coupled fixed point in $A \cap B$.

Proof Let $\mu_o \in A$ and $\nu_o \in B$ be two fixed elements, and let (μ_i) and (ν_i) be any two sequences in A and B , respectively, which are defined as follows:

$$\mu_{i+1} = T(\nu_i, \mu_i) \quad \text{and} \quad \nu_{i+1} = T(\mu_i, \nu_i), \quad \forall i \geq 0. \tag{4.14}$$

Now, we have to show that (μ_i) is a Cauchy sequence. Then from (4.13) we have

$$\begin{aligned} \frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1 &= \frac{1}{F_m(T(\nu_i, \mu_i), T(\mu_{i+1}, \nu_{i+1}), t)} - 1 \\ &\leq a \left(\frac{1}{\min\{F_m(\nu_i, T(\nu_i, \mu_i), t), F_m(\mu_{i+1}, T(\mu_{i+1}, \nu_{i+1}), t)\}} - 1 \right) \\ &= a \left(\frac{1}{\min\{F_m(\nu_i, \mu_{i+1}, t), F_m(\mu_{i+1}, \nu_{i+2}, t)\}} - 1 \right). \end{aligned} \tag{4.15}$$

If $F_m(\mu_{i+1}, \nu_{i+2}, t)$ is minimum, then $(\frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1)$ is the maximum in (4.15), which is not possible. Therefore,

$$\frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1 \leq a \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \right), \quad \text{for } t \gg \theta. \tag{4.16}$$

Similarly,

$$\frac{1}{F_m(\nu_{i+1}, \mu_{i+2}, t)} - 1 \leq a \left(\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right), \quad \text{for } t \gg \theta. \tag{4.17}$$

Adding (4.16) and (4.17), for $t \gg \theta$, we have

$$\begin{aligned} &\left(\frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_{i+1}, \mu_{i+2}, t)} - 1 \right) \\ &\leq a \left(\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 + \frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \right). \end{aligned} \tag{4.18}$$

Now, again by (4.13) and similar as above, we get

$$\frac{1}{F_m(\nu_i, \mu_{i+1}, t)} - 1 \leq a \left(\frac{1}{F_m(\mu_{i-1}, \nu_i, t)} - 1 \right), \quad \text{for } t \gg \theta \tag{4.19}$$

and

$$\frac{1}{F_m(\mu_i, \nu_{i+1}, t)} - 1 \leq a \left(\frac{1}{F_m(\nu_{i-1}, \mu_i, t)} - 1 \right), \quad \text{for } t \gg \theta. \tag{4.20}$$

Adding (4.19) and (4.20), and substituting in (4.18) for $t \gg \theta$, we get

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_{i+1}, \mu_{i+2}, t)} - 1 \right) \\ & \leq a^2 \left(\frac{1}{F_m(\mu_{i-1}, \nu_i, t)} - 1 + \frac{1}{F_m(\nu_{i-1}, \mu_i, t)} - 1 \right). \end{aligned}$$

Continuing this process, for $t \gg \theta$, we have

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_{i+1}, \nu_{i+2}, t)} - 1 \right) + \left(\frac{1}{F_m(\nu_{i+1}, \mu_{i+2}, t)} - 1 \right) \\ & \leq a^{i+1} \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right). \end{aligned} \tag{4.21}$$

Thus (4.21) is true $\forall i \geq 0$. Now, for an integer k ,

$$\begin{aligned} \frac{1}{F_m(\mu_{k+1}, \nu_{k+1}, t)} - 1 &= \frac{1}{F_m(T(\nu_k, \mu_k), T(\mu_k, \nu_k), t)} - 1 \\ &\leq a \left(\frac{1}{\min\{F_m(\nu_k, T(\nu_k, \mu_k), t), F_m(\mu_k, T(\mu_k, \nu_k), t)\}} - 1 \right) \\ &\leq a \left(\frac{1}{\min\{F_m(\nu_k, \mu_{k+1}, t), F_m(\mu_k, \nu_{k+1}, t)\}} - 1 \right). \end{aligned} \tag{4.22}$$

Then, again we may have the following two cases:

- (i) If $F_m(\nu_k, \mu_{k+1}, t)$ is minimum, then $(\frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1)$ is the maximum in (4.22) such that

$$\frac{1}{F_m(\mu_{k+1}, \nu_{k+1}, t)} - 1 \leq a \left(\frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 \right), \quad \text{for } t \gg \theta. \tag{4.23}$$

- (ii) If $F_m(\mu_k, \nu_{k+1}, t)$ is minimum, then $(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1)$ will be maximum in (4.22) such that

$$\frac{1}{F_m(\mu_{k+1}, \nu_{k+1}, t)} - 1 \leq a \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 \right), \quad \text{for } t \gg \theta. \tag{4.24}$$

Adding (4.23) and (4.24),

$$\frac{1}{F_m(\mu_{k+1}, \nu_{k+1}, t)} - 1 \leq a^* \left(\frac{1}{F_m(\mu_k, \nu_{k+1}, t)} - 1 + \frac{1}{F_m(\nu_k, \mu_{k+1}, t)} - 1 \right),$$

where $a^* = \frac{a}{2}$, and by (4.21) for $t \gg \theta$, we have

$$\frac{1}{F_m(\mu_{k+1}, \nu_{k+1}, t)} - 1 \leq a^* a^k \left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1 \right), \quad \text{for } k \geq 0. \tag{4.25}$$

Since F_m is triangular, and by (4.21) and (4.25),

$$\begin{aligned} & \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(v_i, v_{i+1}, t)} - 1 \right) \\ & \leq \left(\frac{1}{F_m(\mu_i, v_i, t)} - 1 + \frac{1}{F_m(v_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(v_i, \mu_i, t)} - 1 + \frac{1}{F_m(\mu_i, v_{i+1}, t)} - 1 \right) \\ & = \left(\frac{1}{F_m(\mu_i, v_i, t)} - 1 + \frac{1}{F_m(v_i, \mu_i, t)} - 1 \right) + \left(\frac{1}{F_m(v_i, \mu_{i+1}, t)} - 1 + \frac{1}{F_m(\mu_i, v_{i+1}, t)} - 1 \right) \\ & \leq 2a^* a^{i-1} \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right) \\ & \quad + a^i \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right) \\ & = \left(1 + \frac{2a^*}{a} \right) a^i \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right), \quad \text{for } i \geq 0. \end{aligned}$$

Now, for $m, i \geq 0$, without loss of generality we may assume that $m > i$,

$$\begin{aligned} \frac{1}{F_m(\mu_i, \mu_m, t)} - 1 & \leq \sum_{n=i}^{m-1} \left(\frac{1}{F_m(\mu_n, \mu_{n+1}, t)} - 1 \right) \\ & \leq \sum_{n=i}^{m-1} \left(1 + \frac{2a^*}{a} \right) a^n \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right) \\ & \leq \left(1 + \frac{2a^*}{a} \right) \frac{a^i}{1-a} \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right) \\ & \rightarrow 0, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

This shows that (μ_i) is a Cauchy sequence and hence convergent in X . Since $A \neq \emptyset$ a closed subset of U , therefore

$$\mu_i \rightarrow \mu \in A, \quad \text{as } i \rightarrow \infty. \tag{4.26}$$

Similarly,

$$v_i \rightarrow v \in B, \quad \text{as } i \rightarrow \infty. \tag{4.27}$$

Hence, from (4.26) and (4.27), we have

$$\lim_{i \rightarrow \infty} F_m(\mu_i, v_i, t) = F_m(\mu, v, t), \quad \text{for } t \gg \theta.$$

Since F_m is triangular, by (4.21) and (4.26), we get

$$\begin{aligned} \frac{1}{F_m(\mu_i, v_i, t)} - 1 & \leq \left(\frac{1}{F_m(\mu_i, \mu_{i+1}, t)} - 1 \right) + \left(\frac{1}{F_m(\mu_{i+1}, v_i, t)} - 1 \right) \\ & \leq \left(\frac{a + 2a^*}{a} + 1 \right) a^i \left(\frac{1}{F_m(\mu_0, v_1, t)} - 1 + \frac{1}{F_m(v_0, \mu_1, t)} - 1 \right) \end{aligned}$$

$$\begin{aligned}
 &= 2\left(1 + \frac{a^*}{a}\right)a^i\left(\frac{1}{F_m(\mu_o, \nu_1, t)} - 1 + \frac{1}{F_m(\nu_o, \mu_1, t)} - 1\right) \\
 &\rightarrow 0, \quad \text{as } i \rightarrow \infty.
 \end{aligned}$$

Therefore, $F_m(\mu, \nu, t) = 1 \Rightarrow (\mu = \nu) \in A \cap B$.

Now we show that μ is a strong coupled fixed point of T . Since F_m is triangular, for $t \gg \theta$, we have

$$\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 \leq \left(\frac{1}{F_m(\mu, \mu_{i+1}, t)} - 1\right) + \left(\frac{1}{F_m(\mu_{i+1}, T(\mu, \nu), t)} - 1\right). \tag{4.28}$$

Then, in view of (4.13), (4.26), and (4.27),

$$\begin{aligned}
 \frac{1}{F_m(\mu_{i+1}, T(\mu, \nu), t)} - 1 &= \frac{1}{F_m(T(\nu_i, \mu_i), T(\mu, \nu), t)} - 1 \\
 &\leq a\left(\frac{1}{\min\{F_m(\nu_i, T(\nu_i, \mu_i), t), F_m(\mu, T(\mu, \nu), t)\}} - 1\right) \\
 &= a\left(\frac{1}{\min\{F_m(\nu_i, \mu_{i+1}, t), F_m(\mu, T(\mu, \nu), t)\}} - 1\right) \\
 &\rightarrow a\left(\frac{1}{\min\{1, F_m(\mu, T(\mu, \nu), t)\}} - 1\right), \quad \text{as } i \rightarrow \infty.
 \end{aligned}$$

If 1 is the minimum of $\{1, F_m(\mu, T(\mu, \nu), t)\}$, then directly from (4.28) we may get that $F_m(\mu, T(\mu, \nu), t) = 1$ as $i \rightarrow \infty$, which implies that $T(\mu, \nu) = \mu = \nu$. Secondly, if $F_m(\mu, T(\mu, \nu), t)$ is the minimum of $\{1, F_m(\mu, T(\mu, \nu), t)\}$, then we have

$$\limsup_{i \rightarrow \infty} \left(\frac{1}{F_m(\mu_{i+1}, T(\mu, \nu), t)} - 1\right) \leq a\left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1\right), \quad \text{for } t \gg \theta.$$

Now, from (4.28),

$$\begin{aligned}
 \frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1 &\leq a\left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1\right), \\
 (1 - a)\left(\frac{1}{F_m(\mu, T(\mu, \nu), t)} - 1\right) &\leq 0, \quad \text{for } t \gg \theta,
 \end{aligned}$$

which is a contradiction. Hence $F_m(\mu, T(\mu, \nu), t) = 1 \Rightarrow T(\mu, \nu) = \mu = \nu$ is a strong coupled fixed point of T . □

Example 4.12 As from Example 4.10, and in view of (4.13), we have that

$$\begin{aligned}
 \frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 &= \frac{1}{t}m(T(\mu, \nu), T(x, y)) \\
 &= \frac{1}{t}\left|\frac{\mu - x}{20} + \frac{4(\nu - y)}{20}\right| \\
 &\leq \frac{6}{140t}(\max\{|19\mu - 4\nu|, |19x - 4y|\}) \\
 &= \frac{6}{7t}\left(\max\left\{\left|\frac{19\mu - 4\nu}{20}\right|, \left|\frac{19x - 4y}{20}\right|\right\}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{6}{7t} \left(\max \left\{ \left| \mu - \frac{\mu + 4\nu}{20} \right|, \left| x - \frac{x + 4y}{20} \right| \right\} \right) \\
 &= \frac{6}{7t} (\max \{m(\mu, T(\mu, \nu)), m(x, T(x, y))\}) \\
 &\leq \frac{6}{7} \left(\frac{1}{\min \{F_m(\mu, T(\mu, \nu), t), F_m(x, T(x, y), t)\}} - 1 \right).
 \end{aligned}$$

Hence all the conditions of Theorem 4.11 are satisfied with $a = \frac{6}{7}$ for $t \gg \theta$, and 5 is the strong coupled fixed point of T , that is, $T(5, 5) = 5$.

5 An application of nonlinear integral equations

In this section, we present an application of nonlinear integral equations to support our results. Let $U = C([0, \ell], \mathbb{R})$ be the space of all \mathbb{R} -valued continuous functions on the interval $[0, \ell]$, where $0 < \ell \in \mathbb{R}$. The two nonlinear integral equations are:

$$\mu(\tau) = \int_0^\ell \Gamma(\tau, \mu(r), \nu(r)) \, dr \quad \text{and} \quad \nu(\tau) = \int_0^\ell \Gamma(\tau, \mu(r), \nu(r)) \, dr, \tag{5.1}$$

where $\tau \in [0, \ell]$ and Γ is a mapping, i.e., $\Gamma : [0, \ell] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The induced metric $m : U \times U \rightarrow \mathbb{R}$ is defined as follows:

$$m(\mu, \nu) = \sup_{\tau \in [0, \ell]} |\mu(\tau) - \nu(\tau)|, \quad \text{where } \mu, \nu \in C([0, \ell], \mathbb{R}).$$

The binary operation $*$ is defined by $a * b = ab, \forall a, b \in [0, \ell]$. A standard fuzzy metric $F_m : U \times U \times (0, \infty) \rightarrow [0, 1]$ is defined as follows:

$$F_m(\mu, \nu, t) = \frac{t}{t + m(\mu, \nu)}, \quad \text{for } t > 0, \text{ and } \mu, \nu \in ([0, \ell], \mathbb{R}). \tag{5.2}$$

Then easily one can show that F_m is triangular and $(U, F_m, *)$ is a complete FCM-space. An element $(\mu, \nu) \in C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R})$ is called a lower and upper coupled solution of integral equation (5.1) if $\mu(\tau) \leq \nu(\tau)$, and

$$\mu(\tau) \leq \int_0^\ell \Gamma(\tau, \mu(r), \nu(r)) \, dr \quad \text{and} \quad \nu(\tau) \geq \int_0^\ell \Gamma(\tau, \nu(r), \mu(r)) \, dr, \quad \forall \tau \in [0, \ell].$$

- (O₁) $\Gamma : [0, \ell] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
- (O₂) $\forall \tau \in [0, \ell]$ and $\forall \mu, \nu, x, y \in \mathbb{R}$ for which $\mu \geq x$ and $\nu \leq y$, we have

$$0 \leq \Gamma(\tau, \mu, \nu) - \Gamma(\tau, x, y) \leq \frac{1}{\ell} \psi \left(\frac{1}{2} (\mu - x + y - \nu) \right),$$

$\psi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, continuous and satisfies $0 = \psi(0) < \psi(\tau) < \tau$ and $\lim_{r \rightarrow \tau^+} \psi(\tau) < \tau$ for each $\tau > 0$.

Now we are in the position to present a result of an integral equation.

Theorem 5.1 *Assume that assertions (O₁) and (O₂) hold. Then equations (5.1) have a unique solution, i.e., $(\mu^*, \nu^*) \in C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R})$ if \exists a lower and upper coupled solution for (5.1).*

Proof Consider the mapping $T : C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R}) \rightarrow C([0, \ell], \mathbb{R})$

$$T(\mu, \nu)(\tau) \leq \int_0^\ell \Gamma(\tau, \mu(r), \nu(r)) \, dr, \quad \forall \mu, \nu \in C([0, \ell], \mathbb{R}) \text{ and } \tau \in [0, \ell].$$

Let $A = \{(\mu, \nu, x, y) \in U^2 \times U^2 : \mu(\tau) \geq x(\tau) \text{ and } \nu(\tau) \leq y(\tau) \forall \tau \in [0, \ell]\}$. It is obvious that a subset A of U^4 is T -invariant which satisfies the transitive property. Easily one can verify that assertion (b) given in Theorem 3.6 is satisfied.

Now we shall show that an element $(\mu^*, \nu^*) \in C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R})$ has a coupled fixed point of a mapping T .

Let $(\mu, \nu, x, y) \in A$, by using assertion $(O_1) \forall \tau \in [0, \ell]$, then we have

$$\begin{aligned} |T(\mu, \nu)(\tau) - T(x, y)(\tau)| &= \int_0^\ell (\Gamma(\tau, \mu(r), \nu(r)) - \Gamma(\tau, x(r), y(r))) \, dr \\ &\leq \frac{1}{\ell} \int_0^\ell \psi \left(\frac{1}{2} (\mu(r) - x(r) + y(r) - \nu(r)) \right) \, dr \\ &\leq \frac{1}{\ell} \int_0^\ell \psi \left(\frac{1}{2} \left(\sup_{s \in [0, \ell]} |\mu(s) - x(s)| + \sup_{s \in [0, \ell]} |y(s) - \nu(s)| \right) \right) \, dr \\ &= \psi \left(\frac{1}{2} \left(\sup_{s \in [0, \ell]} |\mu(s) - x(s)| + \sup_{s \in [0, \ell]} |y(s) - \nu(s)| \right) \right). \end{aligned}$$

This implies that

$$\sup_{\tau \in [0, \ell]} |T(\mu, \nu)(\tau) - T(x, y)(\tau)| \leq \psi \left(\frac{1}{2} \left(\sup_{s \in [0, \ell]} |\mu(s) - x(s)| + \sup_{s \in [0, \ell]} |y(s) - \nu(s)| \right) \right). \tag{5.3}$$

Thus, we get

$$\begin{aligned} &\frac{1}{F_m(T(\mu, \nu), T(x, y), t)} - 1 \\ &= \frac{1}{t} (m(T(\mu, \nu), T(x, y))) \\ &= \frac{1}{t} \left(\sup_{\tau \in [0, \ell]} |T(\mu, \nu)(\tau) - T(x, y)(\tau)| \right) \\ &\leq \frac{1}{t} \psi \left(\frac{1}{2} \left(\sup_{s \in [0, \ell]} |\mu(s) - x(s)| + \sup_{s \in [0, \ell]} |y(s) - \nu(s)| \right) \right) \\ &\leq \psi \left(\frac{1}{2t} \left(\sup_{s \in [0, \ell]} |\mu(s) - x(s)| + \sup_{s \in [0, \ell]} |y(s) - \nu(s)| \right) \right) \\ &= \psi \left(\frac{1}{2} \left(\frac{m(\mu, x)}{t} + \frac{m(\nu, y)}{t} \right) \right) \\ &= \psi \left(\frac{1}{2} \left(\frac{1}{F_m(\mu, x, t)} - 1 + \frac{1}{F_m(\nu, y, t)} - 1 \right) \right), \quad \forall (\mu, \nu, x, y) \in A. \end{aligned}$$

Thus (3.6) is satisfied. Moreover, easily one can verify that $\exists (\mu_0, \nu_0) \in C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R})$ such that $(T(\mu_0, \nu_0), T(\nu_0, \mu_0), \mu_0, \nu_0) \in A$, and all the conditions of Theorem 3.6 are satisfied. Therefore, from Theorem 3.6 we get the solution $(\mu^*, \nu^*) \in C([0, \ell], \mathbb{R}) \times C([0, \ell], \mathbb{R})$. □

6 Conclusion

In this paper, we have introduced the concept of coupled type and cyclic coupled type fuzzy cone contraction mappings in fuzzy cone metric spaces. We have established some coupled fixed point results without the mixed monotone property and also we have presented some more coupled fixed results via partial order metric in fuzzy cone metric spaces. We have proved some strong coupled fixed point theorems for cyclic type fuzzy cone contraction mappings. As a consequence, the main results of this paper extend and unify several results given in the literature of coupled fixed points. Moreover, we presented an integral type application for the existence of unique solution in fuzzy cone metric spaces to support our work.

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Abbreviations

FM, Fuzzy metric; *FCM*, Fuzzy cone metric; F_m , Fuzzy set; \mathbb{E} , Real Banach space; \mathbb{N} , Set of natural numbers; θ , The zero element of \mathbb{E} ; $C(U)$, The collection of all subsets of a set U ; (U, \preceq) , Partially ordered set.

Availability of data and materials

No dataset were generated or analysed during this current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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