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Explicit criteria for the oscillation of second-order differential equations with several sub-linear neutral coefficients

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Abstract
In this work, we present sufficient conditions for oscillation of all solutions of a second-order functional differential equation. We consider two special cases when $\gamma > \beta$ and $\gamma < \beta$. This new theorem complements and improves a number of results reported in the literature. Finally, we provide examples illustrating our results and state an open problem.

MSC: 34C10; 34K11

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1 Introduction

Delay differential equations are widely used in mathematical modeling to describe physical and biological systems, often inducing oscillatory behavior [1–4, 8, 13, 14, 17, 18, 24–26, 28–35].

In the literature, numerous mathematical models with different levels of complexity have been proposed for delay differential equations in order to represent the cardiovascular system (CVS).

The pioneering and remarkable paper of Ottesen [27] shows how to use delay differential equations to solve a cardiovascular model that has a discontinuous derivative. Ottesen [27] also illustrated that complex dynamic interactions between nonlinear behaviors and delays associated with the autonomic-cardiac regulation may cause instability [5].

Moreover, a model-based approach to stability analysis of autonomic-cardiac regulation was studied in [5]; specifically, it is important to underline that the autonomic-cardiac regulation operates by the interaction between autonomic nervous system (ANS) and cardiovascular system (CVS) [5].

It is clear that mathematical analysis based on physics-based models can be a versatile tool in examining delay differential equations from the point of view of biological systems.

In this article we consider the neutral differential equation

$$\big(r(t)\big(w'(t)\big)^{\gamma}\big)' + q(t)x^{\beta}\big(\vartheta(t)\big) = 0, \quad t \geq t_0, \tag{1}$$

where $w(t) = x(t) + \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\varsigma_i(t))$, α_i for $i = 1, 2, \dots, m$, γ and β are the quotients of odd positive integers. Throughout this work, we suppose that:

(A1) $\vartheta, \varsigma_i \in C([t_0,\infty),\mathbb{R}_+)$, $\varsigma_i \in C^2([t_0,\infty),\mathbb{R}_+)$, $\vartheta(t) < t$, $\varsigma_i(t) < t$, $\lim_{t\to\infty}\vartheta(t) = \infty$, $\lim_{t\to\infty}\varsigma_i(t) = \infty$ for all $i = 1, 2, \dots$.

(A2) $r \in C^1([t_0,\infty),\mathbb{R}_+)$, $q \in C([t_0,\infty),\mathbb{R}_+)$; $0 \le q(t)$ for all $t \ge 0$; $q(t)$ is not identically zero in any interval $[b,\infty)$.

(A3) $\lim_{t\to\infty} R(t) = \infty$, where $R(t) = \int_{t_1}^{t} r^{-1/\gamma}(s)\,ds$.

(A4) $p_i : [t_0,\infty) \to \mathbb{R}^+$ are continuous functions for $i = 1, 2, \dots, m$.

In 1978, Brands [11] proved that, for each bounded delay $\vartheta(t)$, the equation

$$x''(t) + q(t)x\bigl(t - \vartheta(t)\bigr) = 0$$

is oscillatory if and only if the equation

$$x''(t) + q(t)x(t) = 0$$

is oscillatory. In [12, 15] Chatzarakis *et al.* considered a more general equation

$$\bigl(r(x')^{\beta}\bigr)'(t) + q(t)x^{\beta}\bigl(\vartheta(t)\bigr) = 0, \tag{2}$$

and established new oscillation criteria for (2) when $\lim_{t\to\infty} R(t) = \infty$ and $\lim_{t\to\infty} R(t) < \infty$.

Wong [37] obtained the oscillation conditions of

$$\bigl(x(t) + px(t - \varsigma)\bigr)'' + q(t)f\bigl(x(t - \vartheta)\bigr) = 0, \quad -1 < p < 0,$$

in which the neutral coefficient and delays are constants. However, we have seen in [6, 16] that the authors Baculǐkovǎ and Džurina studied

$$\bigl(r(t)\bigl(w'(t)\bigr)^{\gamma}\bigr)' + q(t)x^{\beta}\bigl(\vartheta(t)\bigr) = 0, \qquad w(t) = x(t) + p(t)x\bigl(\varsigma(t)\bigr), \quad t \ge t_0, \tag{3}$$

and established the oscillation of solutions of (3) using comparison techniques when $\gamma = \beta = 1$, $0 \le p(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$. In the same technique, Baculikova and Džurina [7] considered (3) and obtained oscillation conditions of (3) by considering the assumptions $0 \le p(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$. In [36], Tripathy *et al.* studied (3) and established several conditions of the solutions of (3) by considering the assumptions $\lim_{t\to\infty} R(t) = \infty$ and $\lim_{t\to\infty} R(t) < \infty$ for different ranges of the neutral coefficient p. In [9], Bohner *et al.* obtained sufficient conditions for oscillation of solutions of (3) when $\gamma = \beta$, $\lim_{t\to\infty} R(t) < \infty$, and $0 \le p(t) < 1$. Grace *et al.* [19] studied the oscillation of (3) when $\gamma = \beta$ and by considering the assumptions $\lim_{t\to\infty} R(t) < \infty$, $\lim_{t\to\infty} R(t) = \infty$, and $0 \le p(t) < 1$. In [22], Li *et al.* established sufficient conditions for the oscillation of the solutions of (3) under the assumptions $\lim_{t\to\infty} R(t) < \infty$ and $p(t) \ge 0$. Karpuz and Santra [21] considered the equation

$$\bigl(r(t)\bigl(x(t) + p(t)x\bigl(\varsigma(t)\bigr)\bigr)'\bigr)' + q(t)f\bigl(x\bigl(\vartheta(t)\bigr)\bigr) = 0$$

by considering the assumptions $\lim_{t\to\infty} R(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$ for different ranges of p.

In fact, equation (1) (that is, half-linear/Emden–Fowler differential equation) arises in a variety of real world problems such as in the study of non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium [10, 23]. Neutral differential equations have several applications in the natural sciences and engineering. For example, they often appear in models biological (see, e.g., [20]). In this paper, we restrict our attention to studying oscillation and non-oscillation of (1).

2 Preliminary results

To simplify our notation, for any function $\rho : [t_0,\infty) \to \mathbb{R}^+$ which is positive, continuous decreasing to zero, we set

$$\begin{aligned}
P(t) &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i) p_i(t)\right) \ge 0;\\
Q_1(t) &= q(t)P^{\beta}\bigl(\vartheta(t)\bigr);\\
Q_2(t) &= q(t)P^{\beta}\bigl(\vartheta(t)\bigr)\rho^{\beta-1}\bigl(\vartheta(t)\bigr);\\
Q_3(t) &= q(t)P^{\beta}\bigl(\vartheta(t)\bigr)R^{\beta-1}\bigl(\vartheta(t)\bigr);\\
Q_4(t) &= q(t)P^{\beta}\bigl(\vartheta(t)\bigr)R^{\beta}\bigl(\vartheta(t)\bigr);\\
U(t) &= \int_t^{\infty} q(\zeta)x^{\beta}\bigl(\vartheta(\zeta)\bigr)\,d\zeta \ge 0.
\end{aligned}$$

We need the following lemmas for our work in the sequel.

Lemma 2.1 ([20]) *If a and b are nonnegative, then*

$$a^{\alpha} b^{1-\alpha} \le \alpha a + (1-\alpha) b \quad \text{for } 0 < \alpha \le 1,$$

where equality holds if and only if $a = b$.

Lemma 2.2 *Let* (A1)–(A4) *hold for $t \ge t_0$. If x is an eventually positive solution of* (1), *then w satisfies*

$$w(t) > 0, \qquad w'(t) > 0, \quad \text{and} \quad \bigl(r\bigl(w'\bigr)^{\gamma}\bigr)'(t) \le 0 \quad \text{for } t \ge t_1. \tag{4}$$

Proof Let x be an eventually positive solution of (1). Hence, $w(t) > 0$, and there exists $t_0 \ge 0$ such that $x(t) > 0$, $x(\varsigma_i(t)) > 0$, and $x(\vartheta(t)) > 0$ for all $t \ge t_0$ and for all $i = 1, 2, \dots$. From (1) it follows that

$$\bigl(r(t)\bigl(w'(t)\bigr)^{\gamma}\bigr)' = -q(t)x^{\beta}\bigl(\vartheta(t)\bigr) \le 0 \quad \text{for } t \ge t_0.$$

Therefore, $r(t)(w'(t))^{\gamma}$ is nonincreasing for $t \ge t_0$. Assume that $r(t)(w'(t))^{\gamma} < 0$ for $t \ge t_1 > t_0$. Hence,

$$r(t)\bigl(w'(t)\bigr)^{\gamma} \le r(t_1)\bigl(w'(t_1)\bigr)^{\gamma} < 0 \quad \text{for all } t \ge t_1,$$

that is,

$$w'(t) \leq \left(\frac{r(t_1)}{r(t)}\right)^{1/\gamma} w'(t_1) \quad \text{for } t \geq t_1.$$

Using integration from t_1 to t, we have

$$w(t) \leq w(t_1) + \big(r(t_1)\big)^{1/\gamma} w'(t_1)R(t) \to -\infty$$

as $t \to \infty$ due to (A3), which is a contradiction to $w(t) > 0$.

Therefore $r(t)(w'(t))^\gamma > 0$ for all $t \geq t_1$. From $r(t)(w'(t))^\gamma > 0$ and $r(t) > 0$, it follows that $w'(t) > 0$. This completes the proof. □

Lemma 2.3 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If* x *is an eventually positive solution of* (1), *then* w *satisfies*

$$w(t) \geq \big(r(t)\big)^{1/\gamma} w'(t)R(t) \quad \textit{for } t \geq t_1$$

and

$$\frac{w(t)}{R(t)} \textit{ is decreasing} \quad \textit{for } t \geq t_1.$$

Proof Proceeding as in the proof of Lemma 2.2, we obtain (4) for $t \geq t_1$. Since $r(t)(w'(t))^\gamma$ is decreasing, we have

$$\begin{aligned} w(t) &\geq \int_{t_1}^{t} \big(r(\eta)\big)^{1/\gamma} w'(\eta) \frac{1}{(r(\eta))^{1/\gamma}}\, d\eta \\ &\geq \big(r(t)\big)^{1/\gamma} w'(t) \int_{t_1}^{t} \frac{1}{(r(\eta))^{1/\gamma}}\, d\eta \\ &\geq \big(r(t)\big)^{1/\gamma} w'(t)R(t). \end{aligned}$$

Again, using the previous inequality, we have

$$\left(\frac{w(t)}{R(t)}\right)' = \frac{(r(t))^{1/\gamma} w'(t)R(t) - w(t)}{(r(t))^{1/\gamma} R^2(t)} \leq 0.$$

We conclude that $\frac{w(t)}{R(t)}$ is decreasing for $t \geq t_1$. This completes the proof. □

Lemma 2.4 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If* x *is an eventually positive solution of* (1), *then* w *satisfies*

$$x(t) \geq P(t)w(t) \quad \textit{for } t \geq t_1. \tag{5}$$

Proof Let x be an eventually positive solution of (1). Hence, $w(t) > 0$, and there exists $t_0 \geq 0$ such that

$$\begin{aligned} x(t) &= w(t) - \sum_{i=1}^{m} p_i(t)x^{\alpha_i}\big(\varsigma_i(t)\big) \\ &\geq w(t) - \sum_{i=1}^{m} p_i(t)w^{\alpha_i}\big(\varsigma_i(t)\big) \\ &\geq w(t) - \sum_{i=1}^{m} p_i(t)w^{\alpha_i}(t) \\ &\geq w(t) - \sum_{i=1}^{m} p_i(t)\big(\alpha_i w(t) - (1-\alpha_i)\big) \\ &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t)\right) w(t) - \sum_{i=1}^{m} (1-\alpha_i)p_i(t) \end{aligned} \tag{6}$$

using Lemma 2.1. Since $w(t)$ is positive and increasing and $\rho(t)$ is positive and decreasing to zero, there is $t_0 \geq t_1$ such that

$$w(t) \geq \rho(t) \quad \text{for } t \geq t_1. \tag{7}$$

Using (7) in (6), we obtain

$$x(t) \geq P(t)w(t).$$

This completes the proof. □

Lemma 2.5 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If* x *is an eventually positive solution of* (1), *then there exist* $t_1 > t_0$ *and* $\delta > 0$ *such that*

$$0 < w(t) \leq \delta R(t) \quad \textit{and} \tag{8}$$

$$R(t)\left[\int_t^{\infty} q(\zeta)x^{\beta}\big(\vartheta(\zeta)\big)\,d\zeta\right]^{1/\gamma} \leq w(t) \tag{9}$$

hold for all $t \geq t_1$.

Proof Let x be an eventually positive solution of (1). Then there exists $t_0 > 0$ such that $x(t) > 0$, $x(\varsigma_i(t)) > 0$, and $x(\vartheta(t)) > 0$ for all $t \geq t_0$ and for all $i = 1, 2, \ldots$. So, there exists $t_1 > t_0$ such that Lemma 2.2 holds true and w satisfies (4) for $t \geq t_1$. From $r(t)(w'(t))^{\gamma} > 0$ and being nonincreasing, we have

$$w'(t) \leq \left(\frac{r(t_1)}{r(t)}\right)^{1/\gamma} w'(t_1) \quad \text{for } t \geq t_1.$$

Integrating this inequality from t_1 to t,

$$w(t) \leq w(t_1) + \big(r(t_1)\big)^{1/\gamma} w'(t_1)R(t).$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists a positive constant δ such that (8) holds. On the other hand, $\lim_{t\to\infty} r(t)(w'(t))^\gamma$ exists, and integrating (1) from t to a, we obtain

$$r(a)\bigl(w'(a)\bigr)^\gamma - r(t)\bigl(w'(t)\bigr)^\gamma = -\int_t^a q(\eta)x^\beta\bigl(\vartheta(\eta)\bigr)\,d\eta.$$

Taking limit as $a \to \infty$,

$$r(t)\bigl(w'(t)\bigr)^\gamma \geq \int_t^\infty q(\eta)x^\beta\bigl(\vartheta(\eta)\bigr)\,d\eta, \tag{10}$$

that is,

$$w'(t) \geq \left[\frac{1}{r(t)}\int_t^\infty q(\eta)x^\beta\bigl(\vartheta(\eta)\bigr)\,d\eta\right]^{1/\gamma}.$$

Therefore,

$$\begin{aligned} w(t) &\geq \int_{t_1}^t \left[\frac{1}{r(\eta)}\int_\eta^\infty q(s)x^\beta\bigl(\vartheta(s)\bigr)\,ds\right]^{1/\gamma} d\eta \\ &\geq \int_{t_1}^t \left[\frac{1}{r(\eta)}\int_t^\infty q(s)x^\beta\bigl(\vartheta(s)\bigr)\,ds\right]^{1/\gamma} d\eta \\ &= R(t)\left[\int_t^\infty q(s)x^\beta\bigl(\vartheta(s)\bigr)\,ds\right]^{1/\gamma}. \end{aligned}$$

This completes the proof. □

3 Sufficient conditions for oscillations

Theorem 3.1 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If*

(A5) $\int_0^\infty Q_1(\eta)\,d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof Let x be an eventually positive solution of (1). Then there exists $t_0 > 0$ such that $x(t) > 0$, $x(\varsigma_i(t)) > 0$, and $x(\vartheta(t)) > 0$ for all $t \geq t_0$ and for all $i = 1, 2, \ldots$. Applying Lemmas 2.2 and 2.4 for $t \geq t_1 > t_0$, we conclude that w satisfies (4), w is increasing, and $x(t) \geq P(t)w(t)$ for all $t \geq t_1$. From (1), we have

$$\bigl(r(t)\bigl(w'(t)\bigr)^\gamma\bigr)' + q(t)P^\beta\bigl(\vartheta(t)\bigr)w^\beta\bigl(\vartheta(t)\bigr) \leq 0 \tag{11}$$

for $t \geq t_1$. Applying (4), we conclude that $\lim_{t\to\infty}(r(t)(w'(t))^\gamma)$ exists, and there exist $t_2 > t_1$ and a number $c > 0$ such that $w(t) \geq c$ for $t \geq t_2$. Integrating (11) from t_2 to t, we have

$$c^\beta \int_{t_2}^t q(s)P^\beta\bigl(\vartheta(s)\bigr)\,ds \leq -\bigl[r(s)\bigl(w'(s)\bigr)^\gamma\bigr]_{t_2}^t < \infty \quad \text{as } t \to \infty,$$

which is a contradiction to (A5).

The case where x is an eventually negative solution is similar, and we omit it here. Thus, the proof is complete. □

Remark 3.1 Theorem 3.1 holds for any β and γ.

Next, we obtain an oscillation result for equation (1) in the case $\beta > 1$.

Theorem 3.2 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If*

(A6) $\int_0^\infty Q_2(\eta)\,d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.1, we obtain (11). Applying (7) in (11), we have

$$\bigl(r(t)\bigl(w'(t)\bigr)^\gamma\bigr)' + q(t)P^\beta\bigl(\vartheta(t)\bigr)\rho^{\beta-1}\bigl(\vartheta(t)\bigr)w\bigl(\vartheta(t)\bigr) \leq 0. \tag{12}$$

The rest of the proof is similar to that of Theorem 3.1, and hence the details are omitted. □

Next, we obtain an oscillation result for equation (1) in the case $0 < \beta < 1$.

Theorem 3.3 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If*

(A7) $\int_0^\infty Q_3(\eta)\,d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.1 we obtain (11). Now (11) can be written as

$$\bigl(r(t)\bigl(w'(t)\bigr)^\gamma\bigr)' + q(t)P^\beta\bigl(\vartheta(t)\bigr)R^{\beta-1}\bigl(\vartheta(t)\bigr)\frac{w^{\beta-1}(\vartheta(t))}{R^{\beta-1}(\vartheta(t))}w\bigl(\vartheta(t)\bigr) \leq 0 \tag{13}$$

for $t \geq t_2 > t_1$. Since $\frac{w(t)}{R(t)}$ is decreasing, there is a constant k such that

$$\frac{w(t)}{R(t)} \leq k \quad \text{for } t \geq t_2. \tag{14}$$

Using (14) and $\beta < 1$ in (13), we have

$$\bigl(r(t)\bigl(w'(t)\bigr)^\gamma\bigr)' + q(t)\frac{P^\beta(\vartheta(t))R^{\beta-1}(\vartheta(t))}{k^{1-\beta}}w\bigl(\vartheta(t)\bigr) \leq 0.$$

The rest of the proof is similar to that of Theorem 3.2, and hence it is omitted. □

Next, we assume that there exists a constant β_1, the quotient of odd positive integers such that $0 < \beta < \beta_1 < \gamma$.

Theorem 3.4 *Let* (A1)–(A4) *hold for* $t \geq t_0$. *If*

(A8) $\int_0^\infty Q_4(\eta)\,d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof Let x be an eventually positive solution of (1). So, there exists $t_0 > 0$ such that $x(t) > 0$, $x(\varsigma_i(t)) > 0$, and $x(\vartheta(t)) > 0$ for all $t \geq t_0$ and for all $i = 1, 2, \ldots$. Applying Lemmas 2.2 and

2.5 for $t \geq t_1 > t_0$, we conclude that w satisfies (4), (8), and (9) for all $t \geq t_1$. We can find $t_1 > 0$ such that

$$w(t) \geq R(t)U^{1/\gamma}(t) \geq 0 \quad \text{for } t \geq t_1. \tag{15}$$

Using (5), (8), $\beta - \beta_1 < 0$, and (15), we have

$$\begin{aligned} x^\beta(t) &\geq P^\beta(t)w^{\beta-\beta_1}(t)w^{\beta_1}(t) \geq P^\beta(t)\bigl(\delta R(t)\bigr)^{\beta-\beta_1}w^{\beta_1}(t) \\ &\geq P^\beta(t)\bigl(\delta R(t)\bigr)^{\beta-\beta_1}\bigl(R(t)U^{1/\gamma}(t)\bigr)^{\beta_1} = P^\beta(t)\delta^{\beta-\beta_1}R^\beta(t)U^{\beta_1/\gamma}(t) \quad \text{for } t \geq t_2. \end{aligned}$$

Since $U'(t) = -q(t)x^\beta(\vartheta(t)) \leq 0$, $t \geq t_2$, that is, w is nonincreasing, then the last inequality becomes

$$\begin{aligned} x^\beta\bigl(\vartheta(\eta)\bigr) &\geq P^\beta\bigl(\vartheta(\eta)\bigr)\delta^{\beta-\beta_1}R^\beta\bigl(\vartheta(\eta)\bigr)U^{\beta_1/\gamma}\bigl(\vartheta(\eta)\bigr) \\ &\geq P^\beta\bigl(\vartheta(\eta)\bigr)\delta^{\beta-\beta_1}R^\beta\bigl(\vartheta(\eta)\bigr)U^{\beta_1/\gamma}(\eta). \end{aligned} \tag{16}$$

Therefore,

$$\bigl(U^{1-\beta_1/\gamma}(t)\bigr)' = \left(1 - \frac{\beta_1}{\gamma}\right)w^{-\beta_1/\gamma}(t)U'(t). \tag{17}$$

Integrating (17) from t_2 to t and then using the fact that $U > 0$, we find

$$\begin{aligned} \infty > U^{1-\beta_1/\gamma}(t_2) &\geq \left(1 - \frac{\beta_1}{\gamma}\right)\left[-\int_{t_2}^{t} U^{-\beta_1/\gamma}(\eta)U'(\eta)\,d\eta\right] \\ &= \left(1 - \frac{\beta_1}{\gamma}\right)\left[\int_{t_2}^{t} U^{-\beta_1/\gamma}(\eta)\bigl(q(\eta)x^\beta\bigl(\vartheta(\eta)\bigr)\bigr)\,d\eta\right] \\ &\geq \frac{(1 - \frac{\beta_1}{\gamma})}{\delta^{(\beta_1-\beta)}}\left[\int_{t_2}^{t} q(\eta)P^\beta\bigl(\vartheta(\eta)\bigr)R^\beta\bigl(\vartheta(\eta)\bigr)\,d\eta\right], \end{aligned}$$

which contradicts (A8) as $t \to \infty$.

This completes the proof. □

Next, we assume that there exists a constant β_2, the quotient of odd positive integers such that $\gamma < \beta_2 < \beta$.

Theorem 3.5 *Let* (A1)–(A4) *hold for* $t \geq t_0$, $\vartheta'(t) \geq \vartheta_0 > 0$ *and* $r(t)$ *is nondecreasing. If*

(A9) $\int_0^\infty [\frac{1}{r(\eta)}\int_\eta^\infty Q_1(\zeta)\,d\zeta]^{1/\gamma}\,d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof Let x be an eventually positive solution of (1). Then there exists $t_0 > 0$ such that $x(t) > 0$, $x(\varsigma_i(t)) > 0$, and $x(\vartheta(t)) > 0$ for all $t \geq t_0$ and $i = 1, 2, \ldots$. Applying Lemmas 2.2 and 2.4 for $t \geq t_1 > t_0$, we conclude that w satisfies (4), w is increasing, and $x(t) \geq P(t)w(t)$ for all $t \geq t_1$. So,

$$x^\beta(t) \geq P^\beta(t)w^\beta(t) \geq P^\beta(t)w^{\beta-\beta_2}(t)w^{\beta_2}(t) \geq P^\beta(t)w^{\beta-\beta_2}(t_1)w^{\beta_2}(t)$$

implies that

$$x^{\beta}(\vartheta(t)) \geq P^{\beta}(\vartheta(t)) w^{\beta-\beta_2}(t_1) w^{\beta_2}(\vartheta(t)) \quad \text{for } t \geq t_2 > t_1. \tag{18}$$

Using (10) and (18), we have

$$r(t)\big(w'(t)\big)^{\gamma} \geq w^{\beta-\beta_2}(t_1)\left[\int_t^{\infty} q(\eta)P^{\beta}(\vartheta(\eta))\,d\eta\right] w^{\beta_2}(\vartheta(t)) \tag{19}$$

for $t \geq t_2$. From $r(t)(w'(t))^{\gamma}$ being nonincreasing and $\vartheta(t) \leq t$, we have

$$r(\vartheta(t))\big(w'(\vartheta(t))\big)^{\gamma} \geq r(t)\big(w'(t)\big)^{\gamma}.$$

Using the last inequality in (19) and then dividing by $r(\vartheta(t))w^{\beta_2}(\vartheta(t)) > 0$, and then operating the power $1/\gamma$ on both sides, we get

$$\frac{w'(\vartheta(t))}{w^{\beta_2/\gamma}(\vartheta(t))} \geq \left[\frac{w^{\beta-\beta_2}(t_1)}{r(\vartheta_0(t))}\int_t^{\infty} q(\eta)P^{\beta}(\vartheta(\eta))\,d\eta\right]^{1/\gamma}$$

for $t \geq t_2$. Multiplying the left-hand side by $\vartheta'(t)/\vartheta_0 \geq 1$ and integrating from t_2 to t, we find

$$\begin{aligned} &\frac{1}{\vartheta_0}\int_{t_2}^{t} \frac{w'(\vartheta(\eta))\vartheta'(\eta)}{w^{\beta_2/\gamma}(\vartheta(\eta))}\,d\eta \\ &\quad \geq w^{(\beta-\beta_2)/\gamma}(t_1)\int_{t_2}^{t}\left[\frac{1}{r(\vartheta(\eta))}\int_{\eta}^{\infty} q(\zeta)P^{\beta}(\vartheta(\zeta))\,d\zeta\right]^{1/\gamma} d\eta t \geq t_2. \end{aligned} \tag{20}$$

Since $\gamma < \beta_2$, $r(\vartheta(\eta)) \leq r(\eta)$ and

$$\frac{1}{\vartheta_0(1-\beta_2/\gamma)}\big[w^{1-\beta_2/\gamma}(\vartheta(\eta))\big]_{\eta=t_2}^{t} \leq \frac{1}{\vartheta_0(\beta_2/\gamma-1)} w^{1-\beta_2/\gamma}(\vartheta(t_2)),$$

then (20) becomes

$$\int_{t_2}^{\infty}\left[\frac{1}{r(\eta)}\int_{\eta}^{\infty} q(\zeta)P^{\beta}(\vartheta(\zeta))\,d\zeta\right]^{1/\gamma} d\eta < \infty,$$

which is a contradiction to (A9). This contradiction implies that the solution x cannot be eventually positive. The case where x is eventually negative is very similar, and we omit it here. □

We finalize the paper by presenting some examples to show effectiveness and feasibility of the main results and Remark 3.3.

Example 3.1 Consider the differential equation

$$\left(t\left(\left(x(t) + \frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right) + \frac{1}{t^2}x^{\frac{1}{5}}\left(\frac{t}{3}\right)\right)'\right)^3\right)' + t^6 x^3\left(\frac{t}{2}\right) = 0 \quad \text{for } t \geq 4, \tag{21}$$

where $r(t) :\equiv y$, $q(t) :\equiv t^6$, $\vartheta(t) :\equiv \frac{t}{2}$, $\beta = \gamma = 3$, $p_i(t) :\equiv \frac{1}{t^i}$, $\alpha_i :\equiv \frac{1}{2i+1}$, and $\varsigma_i(t) :\equiv \frac{t}{i+1}$ for $i = 1, 2, \dots, m$, and $t \geq 4$. All the assumptions of Theorem 3.1 can be verified with the index $i = 1, 2$ and $\rho(t) = \frac{1}{t}$. Hence, due to Theorem 3.1 every solution of (21) is oscillatory.

Example 3.2 Consider the differential equation

$$\left(t\left(\left(x(t) + \frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{3}\right) + \frac{1}{t^2}x^{\frac{1}{5}}\left(\frac{t}{4}\right)\right)'\right)^5\right)' + t^{\frac{6}{5}}x\left(\frac{t}{2}\right) = 0 \quad \text{for } t \geq 4, \tag{22}$$

where $r(t) :\equiv t$, $q(t) :\equiv t^{\frac{6}{5}}$, $\vartheta(t) :\equiv \frac{t}{2}$, $\beta = 1 < \gamma = 5$, $p_i(t) :\equiv \frac{1}{t^i}$, $\alpha_i :\equiv \frac{1}{2i+1}$, and $\varsigma_i(t) :\equiv \frac{t}{i+2}$ for $i = 1, 2, \dots, m$ and $t \geq 2$. All the assumptions of Theorem 3.4 (or Theorem 3.1) can be verified with the index $i = 1, 2$ and $\rho(t) = \frac{1}{t}$. Hence, due to Theorem 3.4 (or Theorem 3.1) every solution of (22) is oscillatory.

Example 3.3 Consider the differential equation

$$\left((t+1)\left(x(t) + \frac{1}{t^2}x^{\frac{1}{3}}\left(\frac{t}{2}\right) + \frac{1}{t^4}x^{\frac{3}{5}}\left(\frac{t}{3}\right)\right)'\right)' + t^{12}x^3\left(\frac{t}{2}\right) = 0 \quad \text{for } t \geq 2, \tag{23}$$

where $r(t) :\equiv t+1$, $q(t) :\equiv t^{12}$, $\vartheta(t) :\equiv \frac{t}{2}$, $\vartheta'(t) > \frac{1}{3} = \vartheta_0$, $\beta = 3 > \gamma = 1$, $p_i(t) :\equiv \frac{1}{t^{2i}}$, $\alpha_i :\equiv \frac{2i-1}{2i+1}$, and $\varsigma_i(t) :\equiv \frac{t}{i+1}$ for $i = 1, 2, \dots, m$ and $t \geq 2$. All the assumptions of Theorem 3.5 (or Theorem 3.1) can be verified with the index $i = 1, 2$ and $\rho(t) = \frac{1}{t^2}$. Hence, due to Theorem 3.4 (or Theorem 3.1), every solution of (23) is oscillatory.

4 Conclusion

In this work, we have undertaken the problem by taking a second-order nonlinear neutral differential equation with sublinear neutral terms and established the sufficient conditions for oscillation of (1). However, we failed to establish the necessary and sufficient conditions for oscillation of all solutions of (1) by using the method adopted in the current paper. It seems that some other method may be required to establish the necessary and sufficient conditions for oscillation.

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