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Different dimensional fractional-order discrete chaotic systems based on the Caputo h -difference discrete operator: dynamics, control, and synchronization

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Abstract

The paper investigates control and synchronization of fractional-order maps described by the Caputo h -difference operator. At first, two new fractional maps are introduced, i.e., the Two-Dimensional Fractional-order Lorenz Discrete System (2D-FoLDS) and Three-Dimensional Fractional-order Wang Discrete System (3D-FoWDS). Then, some novel theorems based on the Lyapunov approach are proved, with the aim of controlling and synchronizing the map dynamics. In particular, a new hybrid scheme is proposed, which enables synchronization to be achieved between a master system based on a 2D-FoLDS and a slave system based on a 3D-FoWDS. Simulation results are reported to highlight the effectiveness of the conceived approach.

Keywords: Discrete fractional calculus; Control; Synchronization; Discrete Lorenz system; Discrete Wang system; Lyapunov approach

1 Introduction

The primary interpretation of the discrete fractional calculus (DFC), which is being deemed one of the great modern branches in the field of calculus, was first proposed by Diaz and Olserin in 1974 [1]. Over the previous decade, several results in theories and applications of such a branch have been carried out, with the aim to provide mathematical models for a number of natural phenomena. Examples include the use of DFC for modeling the movement of a bead on a wire [2], as well as for describing optimal control systems [3].

As a matter of fact, several types of difference discrete operators (DDOs), which are frequently proposed in the DFC field, could be derived and formulated through discretizing their corresponding continuous operators as in [1]. These operators are related to various fractional-order derivatives (FoDs), such as the Riemann–Liouville (RL) derivative [4, 5], the Caputo derivative [6], and the Grünwald–Letnikov (GL) derivative [7]. In particular, these three types of operators fall under the name of fractional h -DDOs [8]. Further de-

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velopments of another operator, called h -sum operator, have been recently accomplished in [8]. Moreover, some discrete chaotic systems have been explored in accordance with the so-called ν -Caputo delta DDO (see [9–11]). Additionally, in [12] different fractional difference operators have been studied and formulated on the isolated time scale with step h . In particular, difference operators with different discrete kernels have been studied in [12], including the power law, the exponential law, and the Mittag-Leffler law. In [13] fractional difference operators with discrete generalized Mittag-Leffler kernels have been illustrated for both the Atangana–Baleanu–Riemann type and the Atangana–Baleanu–Caputo type. In [14] two types of dual identities for Caputo fractional differences are investigated. Namely, the first type relates nabla and delta type fractional sums and differences, whereas the second type (represented by the Q -operator) relates left and right fractional sums and differences [14]. In [15] two types of dual identities for Riemann fractional sums and differences have been investigated, along with the solution for a higher-order Riemann fractional difference equation. In [16] the stability of discrete nonautonomous systems (in the sense of the Caputo fractional difference) has been studied via the Lyapunov direct method. In particular, the conditions for uniform stability, uniform asymptotic stability, and global uniform asymptotic stability for such systems have been deeply analyzed [16, 17]. Despite all these studies, so far an accurate definition of the fractional-order DDO (FoDDO) has not been agreed upon. Namely, what has been agreed upon is the fact that such an operator has infinite memory (unlike the integer operator [18]) and represents an extension of the binomial formulation via the Gamma function [19].

Over the last few years, several fractional-order difference models have been discretized based on efficient tools introduced by the DFC. The so-called fractional-order chaotic discrete systems (FoCDSs) are the most significant among those models [19]. The first examples of chaotic maps have been presented in [20, 21]. In particular, in [20] a fractional logistic map is proposed and its chaotic behavior is numerically illustrated. In [21] a fractional Lorenz map is introduced and its chaotic synchronization is studied based on the stability results. References [20] and [21] have received significant consideration by many researchers, also due to the wide range of applications of chaotic maps in applied science and engineering areas. In particular, the infinite memory characteristic of fractional maps allows flexible modeling, along with the capability to achieve higher degree of chaotic behaviors. These features actually increase the usefulness of chaotic maps in several applications, including data encryption [22, 23], secure communications [24–27], and control theory [28]. Typical examples of such chaotic discrete-time systems include the Lozi map [29], the Hénon map [30], the generalized Hénon map [31], the Stefanski map [32], the Baier–Klein system [33], the Rossler map [34], the Lorenz system [35], and the Wang map [36]. It has become apparent that these maps have richer dynamics compared to the classical ones [19]. In other words, their trajectories highlight very complex dynamic behaviors, depending on both the initial conditions and the fractional-order values [18, 37]. Referring to the very recent results in the literature regarding fractional discrete-time systems, a new fractional logistic map with two parameters is proposed in [38], along with an image encryption application. In [39] a novel short-memory fractional modeling approach is applied to memristors, neural networks, and material's relaxation property. In [40] new variable-order fractional chaotic systems are proposed. In particular, the fractional order is defined by the use of a piecewise constant function, which leads to rich chaotic dynamics, for both continuous- and discrete-time systems [40].

In general, when dealing with FoCDSs, two main aspects should be explored, i.e., control and synchronization of their chaotic modes [37]. Controlling these systems consists in proposing a suitable adaptive controller for their chaotic modes, so that their states are forced to be asymptotically stable, or are stabilized at zero [18, 41, 42]. Control issues are, for instance, of great importance in several industrial processes, like in robotics where chaotic motions of a rigid body need to be controlled [43–45]. On the other hand, synchronization, which has been considered a key concept in chaos theory over the last three decades, targets to compel the states of a slave system to tend towards the exact trajectories that are determined by a master system, assuming that both systems start from different initial points in phase space [19]. Different synchronization and control techniques have been suggested and implemented on some FoCDSs [18, 19, 35, 46]. One could observe that all the aforementioned works, that have discussed both issues of synchronization and control, have employed some linearization methods or some nonlinear laws to implement their strategies [47]. As far as we know, the topic of controlling and synchronizing FoCDSs based on h -DDOs remains, to date, a new and almost unexplored field.

Based on these considerations, this paper makes a contribution to the topic of FoCDSs by presenting novel versions of two- and three-dimensional Lorenz and Wang fractional chaotic maps, respectively, as well as by providing efficient improvements in the schemes for controlling and synchronizing their dynamics. This objective is achieved by introducing novel theorems that exploit Lyapunov-based approaches [48, 49]. The paper is organized as follows. Section 2 introduces the definition of fractional h -DDOs and useful results related to the Lyapunov stability. In Sect. 3 some new versions of FoCDSs are introduced via the Caputo h -DDO, and their chaotic dynamics are analyzed in detail. In Sect. 4 linear control laws are proposed to stabilize the dynamics of the considered FoCDSs at the origin. In Sect. 5 a new hybrid scheme is proposed, which enables synchronization to be achieved between a master system based on the two-dimensional fractional Lorenz map and a slave system based on the three-dimensional fractional Wang map. Finally, simulation results are reported to highlight the effectiveness of the conceived approach.

2 Fractional h -DDOs and Lyapunov stability

As already mentioned, the DFC is considered a relatively new topic that has not been settled, yet. From this perspective, this section presents some preliminaries and notations related to such topic for completeness.

Definition 1 ([5]) Let $X : (h\mathbb{N})_a \rightarrow \mathbb{R}$. For a given $\nu > 0$, the ν th-order h -sum is given by

$${}_h\Delta_a^{-\nu} X(t) = \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu} (t - \sigma(sh))^{\nu-1} x(sh), \quad \sigma(sh) = (s+1)h, t \in (h\mathbb{N})_{a+\nu h}, \quad (1)$$

where $a \in \mathbb{R}$ is a starting point and the h -falling factorial function is defined as

$$t_h^{(\nu)} = h^\nu \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \nu)}, \quad t, \nu \in \mathbb{R},$$

while $(h\mathbb{N})_{a+(1-\nu)h} = \{a + (1-\nu)h, a + (2-\nu)h, \dots\}$.

Definition 2 ([6, 12]) For a function $x(t)$ defined on $(h\mathbb{N})_a$, and for a given $\nu > 0$ such that $\nu \notin \mathbb{N}$, the Caputo h -DDO is defined by

$${}^C_h \Delta_a^\nu X(t) = \Delta_a^{-(n-\nu)} \Delta^n X(t), \quad t \in (h\mathbb{N})_{a+(n-\nu)h}, \tag{2}$$

where $\Delta X(t) = \frac{X(t+h)-X(t)}{h}$ and $n = \lceil \nu \rceil + 1$.

Remark 1 Using the Caputo h -difference operator is useful when dealing with applications of control theory. Namely, controllability (i.e., the possibility to transfer the considered system from a given initial state to a final state using controls from some set) and observability (i.e., the possibility of reconstruction of an initial state using control inputs and output sequences) are both readily achievable when a fractional discrete system is described via the Caputo h -difference operator [50, 51]. Examples of the usefulness in adopting the Caputo h -difference operator are illustrated in [52–54], regarding the controllability and observability of fractional control systems.

From the point of view of obtaining a significant result and a useful inequality for Lyapunov functions reported in [48], which are briefly illustrated below, some stability conditions of the zero equilibrium point for a nonlinear fractional-order difference discrete system will be identified later on. Such a nonlinear system has the form:

$${}^C_h \Delta_a^\nu X(t) = f(t + \nu h, X(t + \nu h)), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{3}$$

Theorem 1 ([48]) *Let $x = 0$ be an equilibrium point of system (3). If there exists a positive definite and decreasing scalar function $V(t, X(t))$ such that ${}^C_h \Delta_a^\nu V(t, X(t)) \leq 0$, then the equilibrium point is asymptotically stable.*

Lemma 2 ([48]) *For any discrete time $t \in (h\mathbb{N})_{a+(1-\nu)h}$, the following inequality holds:*

$${}^C_h \Delta_a^\nu X^2(t) \leq 2X(t + \nu h) {}^C_h \Delta_a^\nu X(t), \tag{4}$$

where $0 < \nu \leq 1$.

3 Some new forms of FoDCSs

In this part, two new forms of the FoDCSs are introduced using fractional h -DDOs. The first is associated with the Two-Dimensional Fractional-order Lorenz Discrete System (2D-FoLDS), while the second one is associated with the Three-Dimensional Fractional-order Wang Discrete System (3D-FoWDS).

3.1 2D-FoLDS

The earlier release of the FoLDS was established in [55] using the ν -Caputo delta DDO. It turned out that this map, which possesses two nonlinear terms, is actually chaotic for some proper values of its parameters (α, β) and fractional-orders ν , where $\nu \in (0, 1]$. Herein, a new version of 2D-FoLDS is derived using, this time, the Caputo h -DDO. In particular, the following equations are proposed:

$$\begin{cases} {}^C_h \Delta_a^\nu x_m(t) = \alpha \beta x(t + \nu h) - \beta y(t + \nu h)x(t + \nu h), \\ {}^C_h \Delta_a^\nu y_m(t) = \beta(x(t + \nu h))^2 - y(t + \nu h), \end{cases} \tag{5}$$

where ${}^C_h \Delta_a^\nu$ denotes the Caputo h -DDO, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, $a \in \mathbb{R}$ is the starting point, and α and β are the system's parameters. Map (5), however, can be regarded as a generalized form of the FoLDS constructed in [55]. Its solution, moreover, can be obtained via employing the fractional h -sum operator. That is, the two corresponding implicit discrete formulae of the two equations given in (5) are reported in [8] as follows:

$$\begin{cases} x(n) = x(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha\beta x(j) - \beta y(j)x(j)), \\ y(n) = y(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\beta(x^2(j) - y(j))), \end{cases} \tag{6}$$

subject to the given initial conditions $x(0)$ and $y(0)$.

In the light of the predictor–corrector scheme (see [56]), the two equations given in (6) could be converted into another two explicit forms which might then be utilized for examining the dynamic behavior of system (5). Anyhow, taking the initial conditions $x(0) = 0.2$ and $y(0) = 0.3$, the fractional-order value $\nu = 0.9$, and the system's parameters $\alpha = 1$ and $\beta = 0.73$ yields the attractor of map (5) exhibited in Fig. 1(a). Furthermore, Fig. 1(b)–(c) shows the resultant calculations of both the bifurcation diagram and the largest Lyapunov exponent (LLE) as a function of α . Obviously, the chaotic behavior of system (5) has been demonstrated in those figures according when $\alpha = 0.95$, $\beta = 1$, and $\nu = 0.9$.

3.2 3D-FoDWS

The first form of the FoDWS, with its classical case due to Wang, was addressed and explored well by considering also the ν -Caputo DDO in [37]. Like the previously proposed map, the Caputo h -DDO is employed to propose the following new 3D-FoDWS:

$$\begin{cases} {}^C_h \Delta_a^\nu x(t) = \alpha_3 y(t + \nu h) + \alpha_4 x(t + \nu h), \\ {}^C_h \Delta_a^\nu y(t) = \alpha_1 x(t + \nu h) + \alpha_2 z(t + \nu h), \\ {}^C_h \Delta_a^\nu z(t) = \alpha_7 z(t + \nu h) + \alpha_6 y(t + \nu h)z(t + \nu h) + \alpha_5, \end{cases} \tag{7}$$

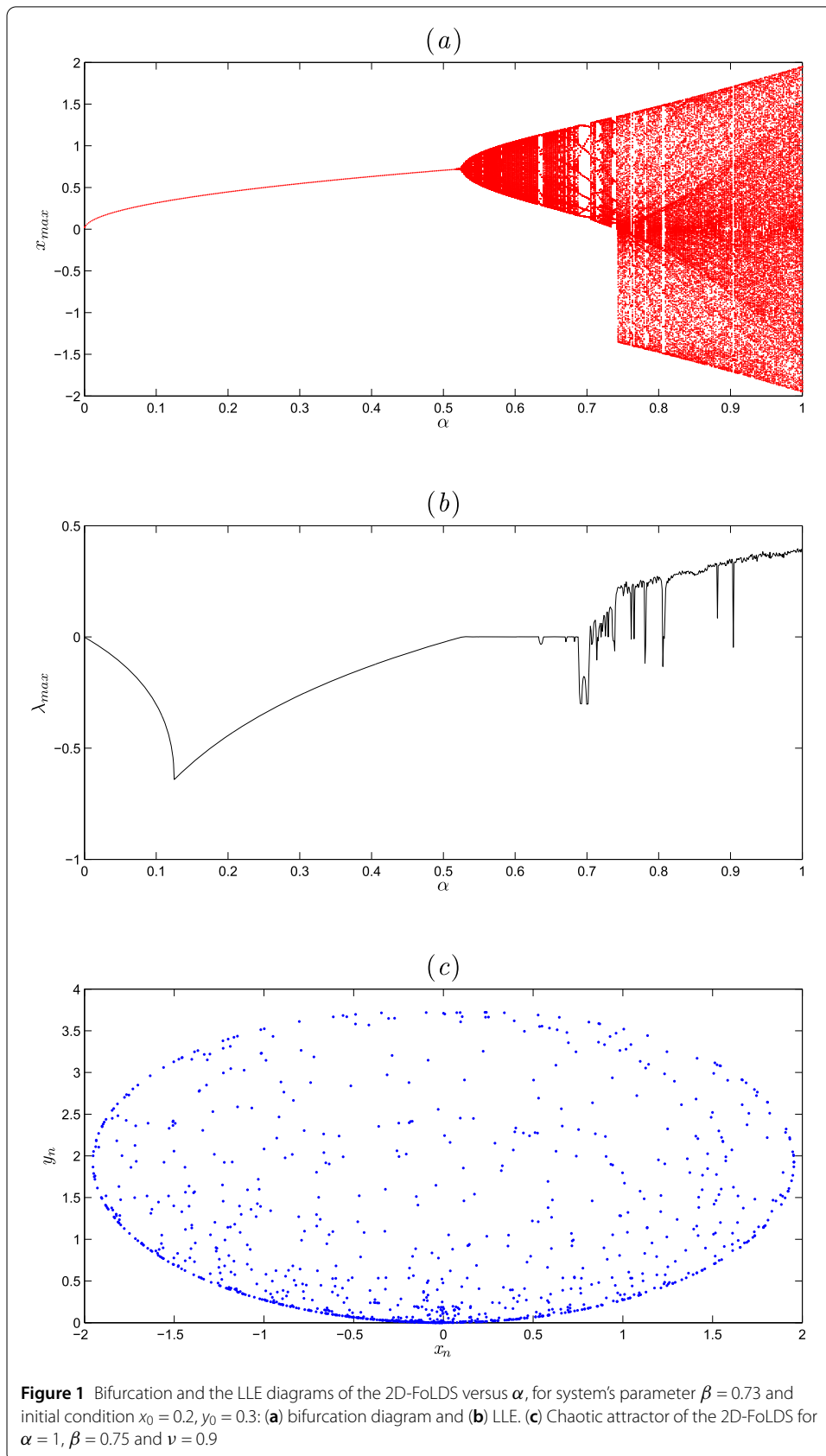
where $t \in (h\mathbb{N})_{a+(1-\nu)h}$ and α_i 's are the system's parameters, $i = 1, 2, \dots, 7$. Accordingly, the three equations given in (7) have the following equivalent numerical formulae:

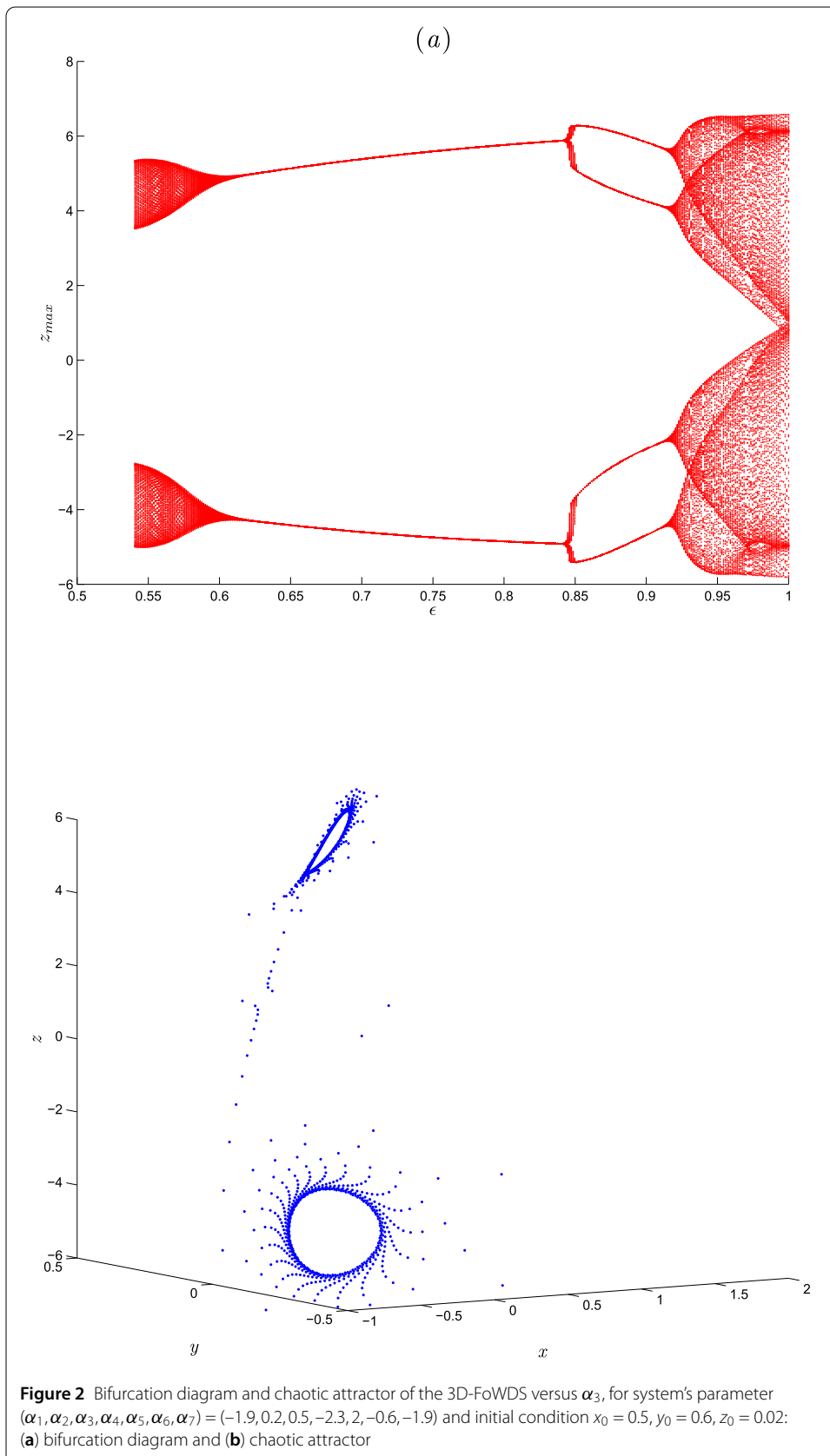
$$\begin{cases} x(n) = x(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_3 y(j) - \alpha_4 x(j)), \\ y(n) = y(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_1 x(j) + \alpha_2 z(j)), \\ z(n) = z(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_7 z(j) + \alpha_6 y(j)z(j) + \alpha_5). \end{cases} \tag{8}$$

In Fig. 2(b), the attractor of map (7) is displayed by considering the initial conditions $x(0) = 0.5$, $y(0) = 0.6$, $z(0) = 0.02$, and assuming $\nu = 0.9$, whereas the system's parameters are set to be $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$. Furthermore, the resultant calculation of the bifurcation diagram as a function of α_3 is exhibited in Fig. 2(a). Thence, it has been clearly shown that the chaotic behavior of system (7) will occur, e.g., when $h = 0.1$, $\nu = 0.9$, and when the same values of α_i 's are taken as above, where $i = 1, 2, \dots, 7$.

4 Linear control laws

This section proposes two control laws related to the 2D-FoLDS and 3D-FoWDS. Besides, the Lyapunov approach is employed to establish the asymptotic convergence of the two controllers.





Theorem 3 *The 2D-FoLDS given in (5) can be controlled under the following one-dimensional control law:*

$$C(t) = -(1 + \alpha\beta)x(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{9}$$

Proof Considering (5) yields its controlled version, of course, under the controller given in (9). This version has the form

$$\begin{cases} {}^C_h \Delta_a^\nu x(t) = \alpha\beta x(t + \nu h) - \beta y(t + \nu h)x(t + \nu h) + C(t + \nu h), \\ {}^C_h \Delta_a^\nu y(t) = \beta(x(t + \nu h))^2 - y(t + \nu h). \end{cases} \tag{10}$$

Consequently, (10) takes the form

$$\begin{cases} {}^C_h \Delta_a^\nu x(t) = -x(t + \nu h) - \beta y(t + \nu h)x(t + \nu h), \\ {}^C_h \Delta_a^\nu y(t) = \beta(x(t + \nu h))^2 - y(t + \nu h). \end{cases} \tag{11}$$

One might employ the Lyapunov approach by first considering the Lyapunov function, $V(t)$, in the form:

$$V = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t). \tag{12}$$

The adoption of the Caputo h -DDO yields

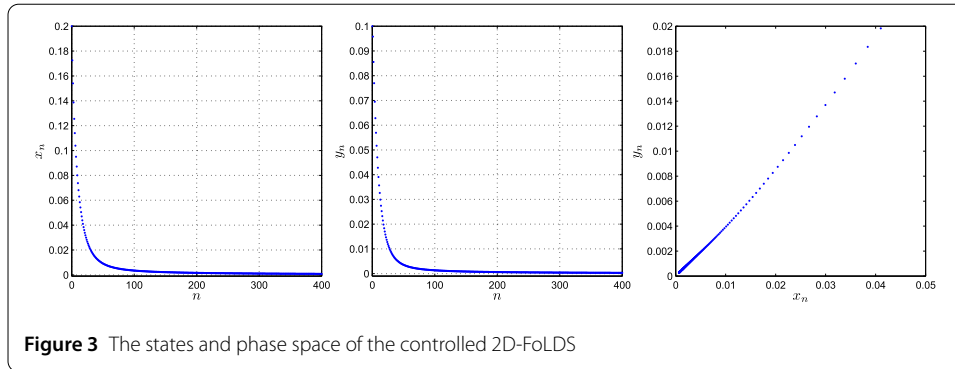
$${}^C_h \Delta_a^\nu V(t) = \frac{1}{2h} {}^C_h \Delta_a^\nu x^2(t) + \frac{1}{2h} {}^C_h \Delta_a^\nu y^2(t). \tag{13}$$

Using Lemma 2 leads to the following steps:

$$\begin{aligned} {}^C_h \Delta_a^\nu V &\leq x(t + \nu h) {}^C_h \Delta_a^\nu x(t) + y(t + \nu h) {}^C_h \Delta_a^\nu y(t) \\ &= -x^2(t + \nu h) - \beta y(t + \nu h)x^2(t + \nu h) \\ &\quad + \beta y(t + \nu h)x(t + \nu h)^2 - \beta y^2(t + \nu h) \\ &= -x^2(t + \nu h) - \beta y^2(t + \nu h) < 0 \quad (\text{because } \beta = 0.73). \end{aligned}$$

Hence, it can be deduced, based on Theorem 1, that the equilibrium point at zero of system (11) is asymptotically stable. Therefore, it has been indeed shown that the dynamics of the proposed 2D-FoLDS given in (5) can be stabilized by controller (9). \square

In order to highlight the potency of the proposed controller, we illustrate the evolution of all the states and the phase-space plots of the controlled system (10) in Fig. 3 when $\alpha = 1$, $\beta = 0.73$, and $\nu = 0.9$. Obviously, all these plots clearly reflect that the proposed 2D-FoLDS has been completely controlled. Next, an additional control law related to the 3D-FoWDS given in (7) is, moreover, established in identical fashion of the preceding discussion.



Theorem 4 *The 3D-FoWDS given in (7) can be controlled under the following two-dimensional control law:*

$$\begin{cases} \mathbf{L}_1(t) = -(\alpha_3 + \alpha_1)y(t) - (\alpha_4 + 1)x(t), \\ \mathbf{L}_2(t) = -\alpha_5 - (|\alpha_6 b|)z(t) - \alpha_2 y(t), \end{cases} \tag{14}$$

where $|y(t)| \leq b$ and $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Proof By considering both (7) and (14), the following controlled map will be deduced:

$$\begin{cases} {}^C_h \Delta_a^\nu x(t) = \alpha_3 y(t + \nu h) + \alpha_4 x(t + \nu h) + \mathbf{L}_1, \\ {}^C_h \Delta_a^\nu y(t) = \alpha_1 x(t + \nu h) + \alpha_2 z(t + \nu h), \\ {}^C_h \Delta_a^\nu z(t) = \alpha_7 z(t + \nu h) + \alpha_6 y(t + \nu h)z(t + \nu h) + \alpha_5 + \mathbf{L}_2. \end{cases} \tag{15}$$

Substituting (14) into (15) yields the following system:

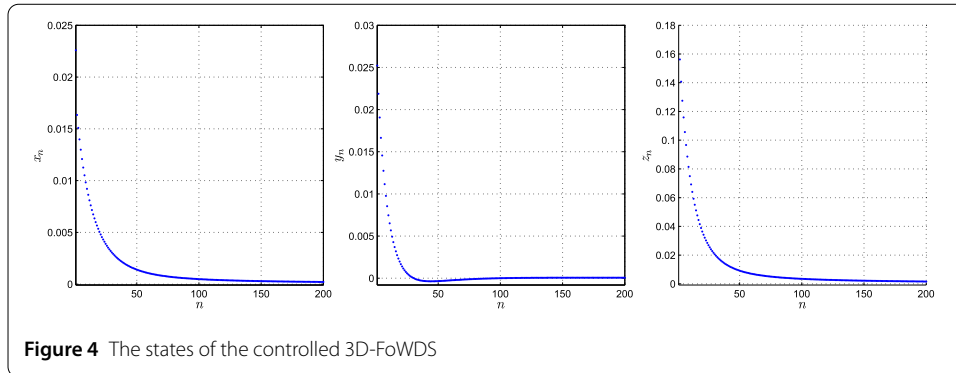
$$\begin{cases} {}^C_h \Delta_a^\nu x(t) = -\alpha_1 y(t + \nu h) - x(t + \nu h), \\ {}^C_h \Delta_a^\nu y(t) = \alpha_1 x(t + \nu h) + \alpha_2 z(t + \nu h), \\ {}^C_h \Delta_a^\nu z(t) = (\alpha_7 - |\alpha_6 b|)z(t + \nu h) + \alpha_6 y(t + \nu h)z(t + \nu h) - \alpha_2 y(t + \nu h). \end{cases} \tag{16}$$

Now, assume the Lyapunov function has the form

$$V = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t) + \frac{1}{2}z^2(t). \tag{17}$$

This implies ${}^C_h \Delta_a^\nu V = {}^C_h \Delta_a^\nu x^2(t) + {}^C_h \Delta_a^\nu y^2(t) + {}^C_h \Delta_a^\nu z^2(t)$, and then by applying Lemma 2, one obtains

$$\begin{aligned} {}^C_h \Delta_a^\nu V &\leq x(t + \nu h) {}^C_h \Delta_a^\nu x(t + \nu h) + y(t + \nu h) {}^C_h \Delta_a^\nu y(t + \nu h) + z(t + \nu h) {}^C_h \Delta_a^\nu z(t + \nu h) \\ &= -\alpha_1 x(t + \nu h)y(t - 1 + \nu) - x^2(t + \nu h) + \alpha_1 y(t - 1 + \nu)x(t + \nu h) \\ &\quad + \alpha_2 y(t + \nu h)z(t + \nu h) + (\alpha_7 - |\alpha_6 b|)z^2(t + \nu h) + \alpha_6 y(t + \nu h)z^2(t + \nu h) \\ &\quad - \alpha_2 z(t + \nu h)y(t + \nu h) \\ &\leq -x^2(t + \nu h) + (\alpha_7 - |\alpha_6 b|)z^2(t + \nu h) + \alpha_6 y(t + \nu h)z^2(t + \nu h) \end{aligned}$$



$$\begin{aligned}
 &\leq -x^2(t + \nu h) + (\alpha_7 - |\alpha_6|b)z^2(t + \nu h) + |\alpha_6 y(t + \nu h)|z^2(t + \nu h) \\
 &\leq -x^2(t + \nu h) + (\alpha_7 - |\alpha_6|b)z^2(t + \nu h) + |\alpha_6|bz^2(t + \nu h) \\
 &= -x^2(t + \nu h) + \alpha_7 z^2(t + \nu h) < 0 \quad (\text{because } \alpha_7 = -1.9).
 \end{aligned}$$

Again, it can be concluded, based also on Theorem 1, that the equilibrium point at zero of system (16) is asymptotically stable. In this case as well, it has been shown that the dynamics of the other proposed 3D-FoWDS given in (7) could be stabilized by controller (14). □

Remark 2 The existence of the upper-bound constant b , identified in Theorem 4, is justified by the boundedness property that characterizes all chaotic maps' states.

With the aim of showing some findings associated with Theorem 4, a numerical simulation has been illustrated in Fig. 4. In this figure, the evolution of all the states and the phase-space plots of the controlled system (15) have been exhibited when $\nu = 0.9$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$. Based on such simulation, one can definitely observe that the 3D-FoWDS has been also completely controlled.

Remark 3 Observe that the two controllers established in this section demand a minor control effort due to their linearity.

5 Hybrid synchronization scheme

In this part, the two fractional-order maps (2D-FoLDS & 3D-FoWDS) will be investigated, despite their various dimensions, for the possibility to be synchronized via a suitable synchronization scheme within a certain time. One might suppose the 2D-FoLDS as a master system, and indicate to its states by the subscript, m , for each of them. That is,

$$\begin{cases}
 {}^C_h \Delta_a^\nu x_m(t) = \alpha \beta x_m(t + \nu h) - \beta y_m(t + \nu h)x_m(t + \nu h), \\
 {}^C_h \Delta_a^\nu y_m(t) = \beta (x_m(t + \nu h))^2 - y_m(t + \nu h).
 \end{cases} \tag{18}$$

At the same time, the 3D-FoWDS is supposed to be a slave system and all its states are indicated by another subscript, say s , for each of them, i.e.,

$$\begin{cases} {}^C_h \Delta_a^\nu x_s(t) = \alpha_3 y_s(t + \nu h) + \alpha_4 x_s(t + \nu h) + \mathbf{U}_1, \\ {}^C_h \Delta_a^\nu y_s(t) = \alpha_1 x_s(t + \nu h) + \alpha_2 z_s(t + \nu h) + \mathbf{U}_2, \\ {}^C_h \Delta_a^\nu z_s(t) = \alpha_7 z_s(t + \nu h) + \alpha_6 y_s(t + \nu h) z_s(t + \nu h) + \alpha_5 + \mathbf{U}_3, \end{cases} \tag{19}$$

where $\mathbf{U}_1, \mathbf{U}_2$, and \mathbf{U}_3 are the synchronization controllers that need to be established. Actually, the process of picking up an adaptive control law $(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3)^T$ aims to compel the following synchronization errors:

$$\begin{cases} e_1 = x_s - x_m, \\ e_2 = y_s + y_m, \\ e_3 = z_s - (x_m + y_m), \end{cases} \tag{20}$$

to be asymptotically close to the origin, i.e.,

$$\lim_{t \rightarrow +\infty} |e_i(t)| = 0, \quad \text{for } i = 1, 2, 3. \tag{21}$$

Remark 4 In light of the error system (20), it is apparent that the two states x_s and x_m are completely synchronized, while the state y_s is antisynchronized with its corresponding state y_m , and finally, the state z_s appears full-state synchronized with two other states, x_m and y_m . Such three types of synchronization (complete, anti-, and full-state synchronization) show coexisting between the master and slave systems given in (18) and (19), respectively.

For highlighting other significant results related to the proposed synchronization scheme, we state the following theorem which is considered one of the main results of this work.

Theorem 5 *The two master and slave systems given in (18) and (19), respectively, achieve synchronized dynamics under the following control law:*

$$\begin{cases} \mathbf{U}_1(t) = -\beta y_m(t) x_m(t) - \alpha_3 y_s(t) + (\alpha\beta - \alpha_4) x_m(t), \\ \mathbf{U}_2(t) = -\beta x_m^2(t) - \alpha_2 z_s(t) - \alpha_1 x_s(t) - \beta y_s(t), \\ \mathbf{U}_3(t) = -(\alpha_7 + 1) z_s(t) - \alpha_6 y_s(t) z_s(t) - \beta y_m(t) x_m(t) - \alpha_5 + \beta x_m^2(t) \\ \quad + (\beta + 1) y_m(t) + (\alpha\beta + 1) x_m(t), \end{cases} \tag{22}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Proof For the purpose of establishing asymptotic convergence of the synchronization errors given in (20) to zero according to (22), we start applying the Caputo h -DDO on (20),

which yields:

$$\begin{cases} {}_h^C \Delta_a^\nu e_1 = \alpha_3 y_s(t + \nu h) + \alpha_4 x_s(t + \nu h) - \alpha \beta x_m(t + \nu h) \\ \quad + \beta y_m(t + \nu h) x_m(t + \nu h) + \mathbf{U}_1, \\ {}_h^C \Delta_a^\nu e_2 = \alpha_1 x_s(t + \nu h) + \alpha_2 z_s(t + \nu h) + \beta x_m^2(t + \nu h) - \beta y_m(t + \nu h) + \mathbf{U}_2, \\ {}_h^C \Delta_a^\nu e_3 = \alpha_7 z_s(t + \nu h) + \alpha_6 y_s(t + \nu h) z_s(t + \nu h) + \alpha_5 - \alpha \beta x_m(t + \nu h) \\ \quad + \beta y_m(t + \nu h) x_m(t + \nu h) - \beta x_m^2(t + \nu h) - \beta y_m(t + \nu h) + \mathbf{U}_3. \end{cases} \quad (23)$$

Substituting the proposed control law given in (22) into (23) leads to the following new discrete system:

$$\begin{cases} {}_h^C \Delta_a^\nu e_1 = \alpha_4 e_1, \\ {}_h^C \Delta_a^\nu e_2 = -\beta e_2, \\ {}_h^C \Delta_a^\nu e_3 = -e_3. \end{cases} \quad (24)$$

Now, letting $V = \frac{1}{2}e_1^2(t) + \frac{1}{2}e_2^2(t) + \frac{1}{2}e_3^2(t)$ implies ${}_h^C \Delta_a^\nu V = {}_h^C \Delta_a^\nu e_1^2(t) + {}_h^C \Delta_a^\nu e_2^2(t) + {}_h^C \Delta_a^\nu e_3^2(t)$, and, by using Lemma 2, we obtain

$$\begin{aligned} {}_h^C \Delta_a^\nu V &\leq e_1(t + \nu h) {}_h^C \Delta_a^\nu e_1(t + \nu h) + e_2(t + \nu h) {}_h^C \Delta_a^\nu e_2^2(t) + e_3(t + \nu h) {}_h^C \Delta_a^\nu e_3(t) \\ &= \alpha_4 e_1^2 - \beta e_2^2 - e_3^2 < 0. \end{aligned}$$

In the light of Theorem 1, it can be deduced that the dynamics of the error system (20) have been stabilized at the origin. As a consequence, the master and slave systems given in (18) and (19), respectively, have achieved the synchronized dynamics via non-control laws. \square

In order to show the effectiveness of the proposed approach, Fig. 5 displays the synchronization errors. These plots clearly show that the two fractional-order maps achieve hybrid synchronization.

6 Conclusion and future work

This work has established two new versions of the Fractional-order Discrete Chaotic Systems (FoDCSs), namely the Two-Dimensional Fractional-order Lorenz Discrete System (2D-FoLDS) and Three-Dimensional Fractional-order Wang Discrete System (2D-FoWDS). Using the Caputo h -Difference Discrete Operator (h -DDO), all the states of such two versions have been demonstrated to contain chaos. Despite all this, these states could still be controlled through quite simple linear controllers as is demonstrated in some parts of this work. Besides, we have constructed a suitable synchronization scheme which has allowed us to establish a proper controller that has the ability to synchronize the two fractional-order maps under consideration. It has been further shown that all the trajectories of such two maps, together with their proposed controller, converge asymptotically to zero using Lyapunov approach. Finally, several numerical simulations have been performed to highlight the potency of all proposed theoretical findings.

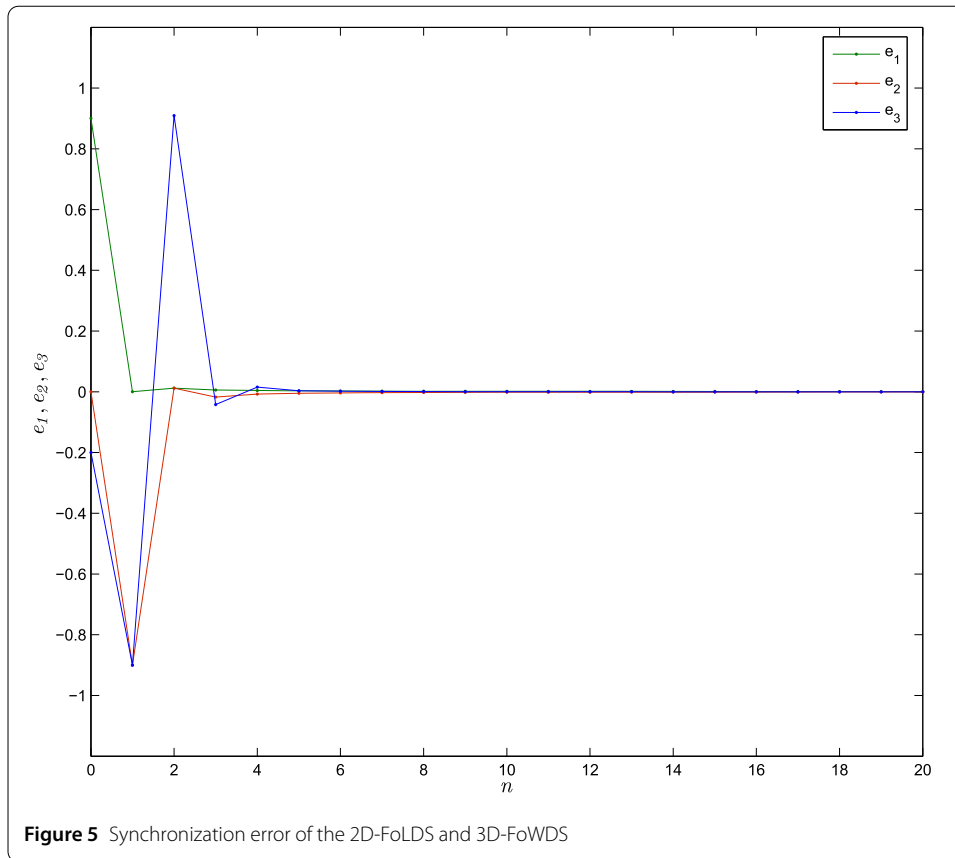


Figure 5 Synchronization error of the 2D-FoLDS and 3D-FoWDS

All the results of this work, without doubt, will inspire us to go deeper into this subject by focusing on two essential steps. The first is related to further experimental implementations of such two maps with the aim of reaching the highest degree of their complexity for their trajectories, whereas the second revolves around the execution of the proposed linear controllers to the field of image encryption. In other words, our future strategy can be described by entering and implementing the proposed master–slave synchronization scheme in a hardware device. This contribution, together with an appropriate encryption algorithm, will allow us to experimentally generate and recover the secret keys.

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Authors' contributions

Conceptualization, AO, A-AK, IT and IMB; Data curation, GG, AB, AO and V-TP; Investigation, GG, IT, V-TP and IMB; Methodology, AO; Supervision, V-TP; Validation, AO, GG. All authors read and approved the final manuscript.

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