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Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators

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Abstract

In this article, we are concerned with the existence of mild solutions and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators and nonlocal conditions. The existence results are obtained by first defining Green's function and approximate controllability by specifying a suitable control function. These results are established with the help of Schauder's fixed point theorem and theory of almost sectorial operators in a Banach space. An example is also presented for the demonstration of obtained results.

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1 Introduction

In current times, the rising interest of researchers in fractional calculus reflects the popularity of this branch [1–3, 5–13, 15, 19, 30–32]. Differential equations of fractional order are widely applicable in the areas of physics, chemistry, electromagnetics, and mechanics. Fractional differential equations (FDEs) have been used widely in the identification of different physical systems, in control theory, in simulating viscoelastic materials, and in the modeling of different complex phenomena [22–29, 39, 42]. The concept of exact controllability and approximate controllability of FDEs is an active field of investigation because of its major applications in physical sciences. Under some admissible control inputs, exact controllability steers the system to arbitrary final state, while approximate controllability steers the system to the small neighborhood of arbitrary final state. In the published works, there are numerous articles focussing on the exact or approximate controllability of systems represented by FDEs, neutral FDEs, FDEs with impulsive inclusions, and FDEs with delay functions [4, 18, 40, 44].

In particular, approximate controllability of Hilfer FDEs under different conditions has been discussed widely. Mahmudov et al. [36] investigated the exact controllability of Hilfer FDEs in a Hilbert space under the assumption that a linear system of the given equation is approximate controllable. In 2017, Yang et al. [41] discussed the approximate controllabil-

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ity of Hilfer FDEs with nonlocal conditions in a Banach space with the help of semigroup theory, fixed point techniques, and multivalued analysis. Later on, Debbouche et al. [14] and Du et al. [17] studied the approximate controllability of Hilfer FDEs and semilinear Hilfer FDEs with impulsive control inclusions in Banach spaces, respectively. In 2018, Lv and Yang [34] investigated the approximate controllability of neutral Hilfer FDEs by applying the techniques of stochastic analysis theory and semigroup operator theory in a Hilbert space. In recent works, Lv and Yang [35], with the help of Banach contraction principle, discussed the approximate controllability for a class of Hilfer FDEs of order $1 < \alpha < 2$ and type $0 \leq \beta \leq 1$. It is noted that, in almost all the problems discussed above, a linear operator generates the strongly continuous semigroup of bounded linear operators.

In [37] Periago and Straub formulated theory to analyze almost sectorial operators. Here authors also mentioned the suitable assumptions required to establish the existence of mild solutions and classical solutions of FDEs with almost sectorial operator. There are numerous works focussing on the existence of mild solutions and analytical solutions of fractional evolution equations with almost sectorial operators [16, 33, 43]. Recently, Jaiswal et al. [21] proved the existence of mild solutions of Hilfer FDEs with almost sectorial operators. We found that in the available literature, approximate controllability of Hilfer FDEs with almost sectorial operators has not been discussed yet. Thus, motivated by the above-discussed works, we consider here the following system of Hilfer FDEs for investigating proposed results:

$$\begin{cases} {}^H\mathcal{D}_{0^+}^{\nu,\mu} v(t) = \mathcal{A}v(t) + \Upsilon w(t) + \Psi(t, v(t), \int_0^t q(t, s)\varphi(t, s, v(s)) ds), & t \in \mathfrak{J}, \\ \mathcal{I}_{0^+}^{(1-\nu)(1-\mu)} v(0) = \sum_{\kappa=1}^m c_\kappa v(t_\kappa), \end{cases} \tag{1.1}$$

where ${}^H\mathcal{D}_{0^+}^{\nu,\mu}$ is the Hilfer fractional derivative of order $0 < \nu < 1$ and type $0 \leq \mu \leq 1$. $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{V} \rightarrow \mathcal{V}$ is an almost sectorial operator in Banach space \mathcal{V} and $\mathfrak{J} = [0, b]$, $b > 0$. The control function $w(t)$ takes value in $\mathcal{L}^2(\mathfrak{J}, \mathcal{W})$. $\mathcal{L}^2(\mathfrak{J}, \mathcal{W})$ denotes the space of admissible control functions for Banach space \mathcal{W} . Υ is a bounded linear operator from \mathcal{W} into \mathcal{V} . $\Psi : \mathfrak{J} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ is a continuous linear mapping, $0 < t_1 < t_2 < \dots < t_m < b$, $m \in \mathbb{N}$, and c_κ are real numbers such that $c_\kappa \neq 0$. The characteristics functions $q(t, s) : \Delta \rightarrow \mathbb{R}$ and $\varphi : \Delta \times \mathcal{V} \rightarrow \mathcal{V}$, $\Delta = (t, s) \in [0, b] \times [0, b]$ are specified in the next section.

2 Preliminaries

Let us consider \mathcal{V} and \mathcal{W} as real Banach spaces with respective norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$. $\mathcal{C}(\mathfrak{J}, \mathcal{V})$ denotes the Banach space formed by all continuous functions from \mathfrak{J} into \mathcal{V} with corresponding norm function $\|w\| = \sup_{t \in \mathfrak{J}} \|w(t)\|$. $\mathcal{L}^2(\mathfrak{J}, \mathcal{W})$ denotes the Banach space of all \mathcal{W} -valued Bochner square integrable functions defined on \mathfrak{J} w.r.t. norm function

$$\|w\| = \left(\int_0^b \|w(t)\|^2 dt \right)^{\frac{1}{2}}, \quad w \in \mathcal{L}^2(\mathfrak{J}, \mathcal{W}).$$

Definition 2.1 ([20]) Hilfer fractional derivative of a continuously differentiable function f of order $0 < \nu < 1$ and $0 \leq \mu \leq 1$ is defined as

$$\mathcal{D}_{0^+}^{\nu,\mu} f(t) = \mathcal{I}_{0^+}^{\nu(1-\mu)} \frac{d}{dt} \mathcal{I}_{r^+}^{(1-\nu)(1-\mu)} f(t) = \mathcal{I}_{0^+}^{\nu(1-\mu)} \mathcal{D}_{0^+}^{\nu+\mu-\nu\mu} f(t),$$

where $\mathcal{I}_{0^+}^{\nu(1-\mu)}$ is a Riemann–Liouville fractional integral and $\mathcal{D}_{0^+}^{\nu+\mu-\nu\mu}$ is a Riemann–Liouville fractional derivative.

Definition 2.2 ([38]) For $\alpha > 0$, the Riemann–Liouville fractional integral of a continuously differentiable f of order α is defined as follows:

$$\mathcal{I}_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Almost sectorial operator ([21, 37]) Let $0 < \beta < \pi$ and $-1 < \gamma < 0$. We define $S_\beta^o = \{\omega \in \mathbb{C} \setminus \{0\} : |\arg \omega| < \beta\}$ and $S_\beta = \bar{S}_\beta^o = \{\omega \in \mathbb{C} \setminus \{0\} : |\arg \omega| \leq \beta\} \cup \{0\}$. A closed linear operator $A : D(A) \subset \mathcal{V} \rightarrow \mathcal{V}$ is called an almost sectorial operator if the following hold:

1. $\sigma(A)$ is contained in S_ω .
 2. $\forall \beta \in (\omega, \pi)$ there exists a constant $\mathcal{M}_\beta > 0$ such that $\|\mathcal{R}(\mathfrak{z}, A)\|_{L(\mathcal{V})} \leq \mathcal{M}_\beta |\mathfrak{z}|^\gamma$,
- where $\mathcal{R}(\mathfrak{z}, A) = (\mathfrak{z}I - A)^{-1}$ is the resolvent operator of A for $\mathfrak{z} \in \rho(A)$.

Let us define Θ_ω^γ as a family of almost sectorial operators.

Proposition 2.1 ([21]) Let $\mathcal{T}(t)$ be the compact semigroup defined in [37] and $A \in \Theta_\omega^\gamma$ for $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Then following holds for $\mathcal{T}(t)$:

- (i) $\mathcal{T}(t)$ is analytic and $\frac{d^n}{dt^n} \mathcal{T}(t) = (-A)^n \mathcal{T}(t)$, $t \in S_{\frac{\pi}{2}-\omega}^o$.
- (ii) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s) \forall t, s \in S_{\frac{\pi}{2}-\omega}^o$.
- (iii) $\|\mathcal{T}(t)\|_{L(\mathcal{V})} \leq c_0 t^{-\gamma-1}$, $t > 0$; where $c_0 > 0$ is a constant.
- (iv) Let $\Sigma_{\mathcal{T}} = \{v \in \mathcal{V} : \lim_{t \rightarrow 0^+} \mathcal{T}(t)v = v\}$, then $D(A^\theta) \subset \Sigma_{\mathcal{T}}$ if $\theta > 1 + \gamma$.

Let us define the operator families $\{S_\nu(t)\}_{t \in S_{\frac{\pi}{2}-\omega}^o}$ and $\{Q_\nu(t)\}_{t \in S_{\frac{\pi}{2}-\omega}^o}$ as follows:

$$S_\nu(t) = \int_0^\infty \mathcal{M}_\nu(\xi) \mathcal{T}(t^\nu \xi) d\xi,$$

$$Q_\nu(t) = \int_0^\infty \nu \xi \mathcal{M}_\nu(\xi) \mathcal{T}(t^\nu \xi) d\xi,$$

where $\mathcal{M}_\nu(\theta)$ is a Wright-type function defined as follows:

$$\mathcal{M}_\nu(\theta) = \sum_{n \in \mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1-\nu n)(n-1)!}, \quad \theta \in \mathbb{C}.$$

Proposition 2.2 ([21]) (i) For each fixed $t \in S_{\frac{\pi}{2}-\omega}^o$, $S_\nu(t)$ and $Q_\nu(t)$ are bounded linear operators in \mathcal{V} . Also

$$\|S_\nu(t)\| \leq C_S t^{-\nu(1+\gamma)}, \quad \|Q_\nu(t)\| \leq C_P t^{-\nu(1+\gamma)}, \quad t > 0,$$

where C_S and C_P are constants depending only on ν and γ . (ii) $S_\nu(t)$ and $Q_\nu(t)$ are continuous in the uniform operator topology for $t > 0$.

Lemma 2.1 ([2, 20]) *The mild solution for the system of equations Eq. (1.1) is defined as follows:*

$$v(t) = S_{v,\mu}(t) \sum_{\kappa=1}^m c_{\kappa} v(t_{\kappa}) + \int_0^t \mathcal{K}_v(t-s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds, \tag{2.1}$$

where $S_{v,\mu}(t) = \mathcal{I}_{0^+}^{\mu(1-\nu)}$ $\mathcal{K}_v(t)$ and $\mathcal{K}_v(t) = t^{\nu-1} Q_v(t)$.

Proposition 2.3 ([21]) (i) *For every fixed $t \in S_{\frac{\pi}{2}-\omega}^0$, $S_{v,\mu}(t)$ and $\mathcal{K}_v(t)$ are bounded linear operators on \mathcal{V} . For $t > 0$,*

$$\| \mathcal{K}_v(t)v \| \leq C_P t^{-1-\nu\gamma} \| v \|, \quad \| S_{v,\mu}(t)v \| \leq \frac{\Gamma(-\gamma\nu)}{\Gamma(\mu(1-\nu) - \nu\gamma)} C_P t^{\mu(1-\nu) - \nu\gamma - 1} \| v \|.$$

(ii) $S_{v,\mu}(t)$ and $\mathcal{K}_v(t)$ are strongly continuous operators.

Let $\mathcal{M}_S = \sup_{t \in \mathcal{J}'} \| S_{v,\mu}(t) \|$, $\mathcal{J}' = (0, b]$. Assume that $\sum_{\kappa=1}^m |c_{\kappa}| \leq \frac{1}{\mathcal{M}_S}$. We have

$$\left\| \sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t_{\kappa}) \right\| \leq \mathcal{M}_S \cdot \frac{1}{\mathcal{M}_S} < 1.$$

By the operator spectrum theorem, $\mathcal{O} = (I - \sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t_{\kappa}))^{-1}$ exists bounded and $D(\mathcal{O}) = \mathcal{V}$.

By the Neumann series expression, \mathcal{O} can be expressed as $\mathcal{O} = \sum_{n=0}^{\infty} [\sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t_{\kappa})]^n$. Therefore

$$\| \mathcal{O} \| \leq \frac{1}{1 - \sum_{\kappa=1}^m |c_{\kappa}| \mathcal{M}_S}.$$

By Lemma 2.1 mild solutions of Eq. (1.1) are given by

$$v(t) = S_{v,\mu}(t) \hat{v} + \int_0^t \mathcal{K}_v(t-s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds,$$

where $\hat{v} = \sum_{\kappa=1}^m c_{\kappa} v(t_{\kappa})$.

At $t = t_{\kappa}$,

$$v(t_{\kappa}) = S_{v,\mu}(t) \hat{v} + \int_0^{t_{\kappa}} \mathcal{K}_v(t_{\kappa} - s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds,$$

$$\sum_{\kappa=1}^m c_{\kappa} v(t_{\kappa}) = \sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t) \hat{v} + \sum_{\kappa=1}^m c_{\kappa} \int_0^{t_{\kappa}} \mathcal{K}_v(t_{\kappa} - s) \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds$$

$$\begin{aligned} \Rightarrow & \hat{v} \left[I - \sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t_{\kappa}) \right] \\ &= \sum_{\kappa=1}^m c_{\kappa} \int_0^{t_{\kappa}} \mathcal{X}_v(t_{\kappa} - s) \\ &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \\ \Rightarrow & \hat{v} = \sum_{\kappa=1}^m c_{\kappa} \mathcal{O} \int_0^{t_{\kappa}} \mathcal{X}_v(t_{\kappa} - s) \\ &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds. \end{aligned}$$

Thus the mild solutions of Eq. (1.1) are defined as follows:

$$\begin{aligned} v(t) &= \sum_{\kappa=1}^m c_{\kappa} S_{v,\mu}(t) \mathcal{O} \int_0^{t_{\kappa}} \mathcal{X}_v(t_{\kappa} - s) \\ &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \\ &\quad + \int_0^t \mathcal{X}_v(t - s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds. \end{aligned}$$

Now we introduce the Green’s function $\mathcal{G}(t, s)$ as follows:

$$\mathcal{G}(t, s) = \sum_{\kappa=1}^m \chi_{t_{\kappa}}(s) S_{v,\mu}(t) \mathcal{O} K_v(t_{\kappa} - s) + \chi_t(s) \mathcal{X}_v(t - s),$$

where

$$\chi_{t_{\kappa}}(s) = \begin{cases} c_{\kappa}, & s \in [0, t_{\kappa}), \\ 0, & s \in [t_{\kappa}, b), \end{cases} \quad \chi_t(s) = \begin{cases} 1, & s \in [0, t), \\ 0, & s \in [t, b). \end{cases}$$

The mild solutions of Eq. (1.1) in terms of the Green’s function are expressed as

$$v(t) = \int_0^b \mathcal{G}(t, s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds, \quad t \in \mathfrak{J}. \tag{2.2}$$

In addition to the above-mentioned propositions and lemma, we assume here the following assumptions to establish the proposed results:

- (A1) For each $t \in \mathfrak{J}'$, $\Psi : \mathfrak{J}' \times \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ is a Caratheodory function.
- (A2) There exist $\psi_1 \in L^1(\mathfrak{J}, \mathbb{R}^+)$ and a continuous function ψ_2 such that, for $v_1, v_2 \in \mathcal{V}$ and $t \in \mathfrak{J}$, $\Psi(t, v_1, v_2) \leq \psi_1(t) \|v_1\| + \psi_2(t) \|v_2\|$.
- (A3) $\varphi(t, s, \cdot) : \mathcal{V} \rightarrow \mathcal{V}$ is a Caratheodory function and there exists $w : \Delta \rightarrow \mathbb{R}$ with

$$\bar{w} = \sup_{t \in \mathfrak{J}} \int_0^t w(t, s) ds < \infty \quad \text{such that } \|\varphi(t, s, v)\| \leq w(t, s) \|v\|, v \in \mathcal{V}.$$

- (A4) $q(t, s)$ is bounded on $\mathfrak{J} = [0, b]$ with $q^* = \sup\{q(t, s) : 0 \leq s \leq t\} < \infty$.
- (A5) There exists $\kappa \in L^1(\mathfrak{J}, \mathbb{R}^+)$ such that $\|\Upsilon w(t)\| \leq \kappa(t), \forall t \in \mathfrak{J}$ and $w \in \mathcal{L}^2(\mathfrak{J}, \mathcal{W})$.

3 Existence result

For $\mathfrak{J} = [0, b]$ and $\mathfrak{J}' = [0, b)$, we introduce

$$\mathcal{Y} = \left\{ v \in C(\mathfrak{J}', \mathcal{V}) : \lim_{t \rightarrow 0} t^{(1+\nu\gamma)(1-\mu)} v(t) \text{ exists and is finite} \right\}$$

with the norm defined as $\|v\|_{\mathcal{Y}} = \sup_{t \in \mathfrak{J}'} \{t^{(1+\nu\gamma)(1-\mu)} \|v(t)\|\}$.

Let $y(t) = t^{(1+\nu\gamma)(1-\mu)} v(t)$. Then $v \in \mathcal{Y}$ iff $y \in C(\mathfrak{J}, \mathcal{V})$ and $\|v\|_{\mathcal{Y}} = \|y\|$.

We define

$$\mathcal{B}_{\delta}(\mathfrak{J}) = \{y \in C(\mathfrak{J}, \mathcal{V}) : \|y\| \leq \delta\} \quad \text{and} \quad \mathcal{B}_{\delta}^{\mathcal{Y}}(\mathfrak{J}') = \{v \in \mathcal{Y} : \|v\|_{\mathcal{Y}} \leq \delta\}.$$

Define an operator $\mathcal{P} : \mathcal{B}_{\delta}(\mathfrak{J}) \rightarrow \mathcal{B}_{\delta}(\mathfrak{J})$ such that

$$\mathcal{P}y(t) = \begin{cases} 0, & t = 0, \\ t^{(1+\nu\gamma)(1-\mu)} \Pi v(t), & t \in \mathfrak{J}' = (0, b], \end{cases}$$

where $\Pi : C(\mathfrak{J}', \mathcal{V}) \rightarrow C(\mathfrak{J}', \mathcal{V})$ is defined as

$$\Pi v(t) = \int_0^b \mathcal{G}(t, s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds.$$

Let us denote $\mathcal{N}_1 = \sup_{t \in \mathfrak{J}'} \{\psi_1(t)\}$, $\mathcal{N}_2 = \sup_{t \in \mathfrak{J}'} \{\psi_2(t)\}$, $\hat{m} = \sup_{t \in \mathfrak{J}'} \{\kappa(t)\}$, and $\wedge_o = \frac{\sum_{\kappa=1}^m |c_{\kappa}|}{1 - \sum_{\kappa=1}^m |c_{\kappa}| \mathcal{M}_S}$.

Lemma 3.1 *Let $\mathcal{A} \in \Theta_{\omega}^{\gamma}$ for $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Assume that assumptions (A1)–(A5) and Proposition 2.3 hold. Then the operator $\{\mathcal{P}y : y \in \mathcal{B}_{\delta}(\mathfrak{J})\}$ is bounded and continuous if $C_{\mathcal{P}}(\mathcal{N}_1 + \mathcal{N}_2 \bar{w} q^*)(1 + \mathcal{M}_S \wedge_o) b^{-\nu\gamma} < 1$.*

Proof For $y \in \mathcal{B}_{\delta}(\mathfrak{J})$, we have

$$\begin{aligned} \|\mathcal{P}y\| &= \left\| t^{(1+\nu\gamma)(1-\mu)} \int_0^b \mathcal{G}(t, s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\| \\ &\leq \left\| t^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_{\kappa} \mathcal{S}_{\nu, \mu}(t) \mathcal{O} \int_0^{t_{\kappa}} \mathcal{K}_{\mathcal{W}}(t_{\kappa} - s) \right. \\ &\quad \times \left. \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\| \\ &\quad + \left\| t^{(1+\nu\gamma)(1-\mu)} \int_0^t \mathcal{K}_{\mathcal{W}}(t - s) \right. \\ &\quad \times \left. \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\| \\ &\leq \mathcal{M}_S \wedge_o C_{\mathcal{P}} t^{(1+\nu\gamma)(1-\mu)} \int_0^{t_{\kappa}} (t_{\kappa} - s)^{-1-\nu\gamma} \{ \psi_1(s) \|v\| + \psi_2(s) \bar{w} q^* \|v\| + \kappa(s) \} ds \end{aligned}$$

$$\begin{aligned}
 &+ C_P t^{(1+\nu\gamma)(1-\mu)} \int_0^t (t-s)^{-1-\nu\gamma} \{ \psi_1(s) \|v\| + \psi_2(s) \bar{w}q^* \|v\| + \kappa(s) \} ds \\
 &\leq C_P \delta (\mathcal{N}_1 + \mathcal{N}_2 \bar{w}q^*) (1 + \mathcal{M}_S \wedge_o) \frac{b^{-\nu\gamma}}{(-\nu\gamma)} + C_P \hat{\mathcal{M}} (1 + \mathcal{M}_S \wedge_o) \frac{b^{(1+\nu\gamma)(1-\mu)-\nu\gamma}}{(-\nu\gamma)} \\
 &\leq C_P \delta (\mathcal{N}_1 + \mathcal{N}_2 \bar{w}q^*) (1 + \mathcal{M}_S \wedge_o) b^{-\nu\gamma} \\
 &\quad + C_P \hat{\mathcal{M}} (1 + \mathcal{M}_S \wedge_o) b^{(1+\nu\gamma)(1-\mu)-\nu\gamma} \quad \text{as } (-\nu\gamma) > 0.
 \end{aligned}$$

If possible, suppose $\|Py\| > \delta$.

This implies

$$C_P (\mathcal{N}_1 + \mathcal{N}_2 \bar{w}q^*) (1 + \mathcal{M}_S \wedge_o) b^{-\nu\gamma} + \frac{C_P \hat{\mathcal{M}} (1 + \mathcal{M}_S \wedge_o) b^{(1+\nu\gamma)(1-\mu)-\nu\gamma}}{\delta} > 1.$$

Applying $\lim_{\delta \rightarrow \infty}$ on both sides, we have

$$C_P (\mathcal{N}_1 + \mathcal{N}_2 \bar{w}q^*) (1 + \mathcal{M}_S \wedge_o) b^{-\nu\gamma} > 1,$$

which is a contradiction.

So our supposition is wrong. Therefore $\|Py\| \leq \delta$. This implies the boundedness of operator \mathcal{P} .

Now, to show that $\mathcal{P} : \mathcal{B}_\delta(\mathfrak{J}) \rightarrow \mathcal{B}_\delta(\mathfrak{J})$ is a continuous operator, let y_n and $y \in \mathcal{B}_\delta(\mathfrak{J})$ with $\lim_{n \rightarrow \infty} y_n = y, n \in \mathbb{N}$.

By the continuity of function Ψ w.r.t. to the second and third variable

$$\begin{aligned}
 &\Psi \left(t, v_n(t), \int_0^t q(t,s) \varphi(t,s, v_n(s)) ds \right) \\
 &\rightarrow \Psi \left(t, v(t), \int_0^t q(t,s) \varphi(t,s, v(s)) ds \right) \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

For $t \in \mathfrak{J}$,

$$\begin{aligned}
 &\|Py_n(t) - Py(t)\| \\
 &\leq \mathcal{M}_s \wedge_o t^{(1+\nu\gamma)(1-\mu)} \int_0^{t_\kappa} \| \mathcal{K}_v(t_\kappa - s) \| \left\| \Psi \left(s, v_n(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v_n(\zeta)) d\zeta \right) \right. \\
 &\quad \left. - \Psi \left(s, v(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v(\zeta)) d\zeta \right) \right\| ds \\
 &\quad + t^{(1+\nu\gamma)(1-\mu)} \int_0^t \| \mathcal{K}_v(t-s) \| \left\| \Psi \left(s, v_n(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v_n(\zeta)) d\zeta \right) \right. \\
 &\quad \left. - \Psi \left(s, v(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v(\zeta)) d\zeta \right) \right\| ds \\
 &\leq \mathcal{M}_s \wedge_o t^{(1+\nu\gamma)(1-\mu)} C_P \int_0^{t_\kappa} (t_\kappa - s)^{-1-\nu\gamma} \left\| \Psi \left(s, v_n(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v_n(\zeta)) d\zeta \right) \right. \\
 &\quad \left. - \Psi \left(s, v(s), \int_0^s q(s,\zeta) \varphi(s,\zeta, v(\zeta)) d\zeta \right) \right\| ds
 \end{aligned}$$

$$\begin{aligned}
 &+ t^{(1+\nu\gamma)(1-\mu)} C_p \int_0^t (t-s)^{-1-\nu\gamma} \left\| \Psi \left(s, v_n(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v_n(\zeta)) d\zeta \right) \right. \\
 &\left. - \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\| ds.
 \end{aligned}$$

Since

$$\begin{aligned}
 &t^{(1+\nu\gamma)(1-\mu)} \left\| \Psi \left(s, v_n(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v_n(\zeta)) d\zeta \right) \right. \\
 &\left. - \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\| \leq 2\delta(\mathcal{N}_1 + \mathcal{N}_2 \bar{w}q^*), \quad \text{which is finite,}
 \end{aligned}$$

by the Lebesgue dominated convergence theorem and the continuity of function Ψ ,

$$\|\mathcal{P}y_n(t) - \mathcal{P}y(t)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 *Let $\mathcal{A} \in \Theta'_\omega$ for $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. If assumptions (A1)–(A5) and Proposition 2.3 are satisfied, the operator $\{\mathcal{P}y : y \in \mathcal{B}_\delta(\mathfrak{J})\}$ is equicontinuous for $t \in \mathfrak{J}$.*

Proof For $y \in \mathcal{B}_\delta(\mathfrak{J})$ and $0 \leq t_1 < t_2 < b$,

$$\begin{aligned}
 &\|\mathcal{P}y(t_2) - \mathcal{P}y(t_1)\| \\
 &= \left\| t_2^{(1+\nu\gamma)(1-\mu)} \Pi v(t_2) - t_1^{(1+\nu\gamma)(1-\mu)} \Pi v(t_1) \right\| \\
 &= \left\| t_2^{(1+\nu\gamma)(1-\mu)} \int_0^b \mathcal{G}(t_2, s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right. \\
 &\quad \left. - t_1^{(1+\nu\gamma)(1-\mu)} \int_0^b \mathcal{G}(t_1, s) \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\|. \\
 &\leq \left\| t_2^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_\kappa \mathcal{S}_{v,\mu}(t_2) \mathcal{O} \int_0^{t_\kappa} \mathcal{X}_\omega(t_\kappa - s) \right. \\
 &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \\
 &\quad \left. - t_1^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_\kappa \mathcal{S}_{v,\mu}(t_1) \mathcal{O} \int_0^{t_\kappa} \mathcal{X}_\omega(t_\kappa - s) \right. \\
 &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \Big\| \\
 &+ \left\| t_2^{(1+\nu\gamma)(1-\mu)} \int_0^{t_1} \mathcal{X}_\omega(t_2 - s) \right. \\
 &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \\
 &\quad \left. - t_1^{(1+\nu\gamma)(1-\mu)} \int_0^{t_1} \mathcal{X}_\omega(t_1 - s) \right.
 \end{aligned}$$

$$\begin{aligned} & \times \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \Big\| \\ & + \left\| t_2^{(1+\nu\gamma)(1-\mu)} \int_{t_1}^{t_2} \mathcal{K}_\nu(t_2 - \mathfrak{s}) \right. \\ & \times \left. \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right\| \\ & = I_1 + I_2 + I_3. \end{aligned}$$

Here,

$$\begin{aligned} I_1 = & \left\| t_2^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_{\kappa, S_{\nu, \mu}}(t_2) \mathcal{O} \int_0^{t_\kappa} \mathcal{K}_\nu(t_\kappa - \mathfrak{s}) \right. \\ & \times \left. \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right. \\ & - t_1^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_{\kappa, S_{\nu, \mu}}(t_1) \mathcal{O} \int_0^{t_\kappa} \mathcal{K}_\nu(t_\kappa - \mathfrak{s}) \\ & \times \left. \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right\|. \end{aligned}$$

The strong continuity of operator $S_{\nu, \mu}(t)$ yields $I_1 \rightarrow 0$ as $t_1 \rightarrow t_2$.

$$\begin{aligned} I_2 = & \left\| t_2^{(1+\nu\gamma)(1-\mu)} \int_0^{t_1} \mathcal{K}_\nu(t_2 - \mathfrak{s}) \right. \\ & \times \left. \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right. \\ & - t_1^{(1+\nu\gamma)(1-\mu)} \int_0^{t_1} \mathcal{K}_\nu(t_1 - \mathfrak{s}) \\ & \times \left. \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right\|. \end{aligned}$$

Following Lemma 3.1 and assumptions (A1)–(A5), it is easy to see that

$$\left\| \int_0^{t_1} \mathcal{K}_\nu(t_i - \mathfrak{s}) \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right\|, \quad i = 1, 2,$$

exists and is bounded. Thus $I_2 \rightarrow 0$ as $t_2 \rightarrow t_1$.

$$I_3 = \left\| t_2^{(1+\nu\gamma)(1-\mu)} \int_{t_1}^{t_2} \mathcal{K}_\nu(t_2 - \mathfrak{s}) \left\{ \Upsilon w(\mathfrak{s}) + \Psi \left(\mathfrak{s}, v(\mathfrak{s}), \int_0^{\mathfrak{s}} q(\mathfrak{s}, \zeta) \varphi(\mathfrak{s}, \zeta, v(\zeta)) d\zeta \right) \right\} d\mathfrak{s} \right\|.$$

After following the given assumptions and performing some steps of calculation, we have

$$I_3 \leq t_2^{(1+\nu\gamma)(1-\mu)} C_P \delta (\mathcal{N}_1 + \mathcal{N}_2 \bar{w} q^*) (t_2 - t_1)^{-\gamma\nu} + t_2^{(1+\nu\gamma)(1-\mu)} C_P \hat{\mathcal{M}} (t_2 - t_1)^{-\gamma\nu}.$$

Clearly, $I_3 \rightarrow 0$ as $t_2 \rightarrow t_1$.

So, we have

$$\| \mathcal{P}y(t_2) - \mathcal{P}y(t_1) \| \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1.$$

This proves the equicontinuity of operator $\mathcal{P}y(t)$. □

Theorem 3.1 *Let $-1 < \gamma < 0, 0 < \omega < \frac{\pi}{2}$, and $\mathcal{A} \in \Theta'_\omega$. The system of equations Eq. (1.1) has at least one mild solution in $\mathcal{B}_\delta^{\mathcal{J}'}$ if assumptions (A1)–(A5) hold along with Lemma 3.1 and Lemma 3.2.*

Proof The mild solution of Eq. (1.1) is equivalent to the fixed point of operator $\mathcal{P} : \mathcal{B}_\delta(\mathcal{J}) \rightarrow \mathcal{B}_\delta(\mathcal{J})$. Here we prove that the operator \mathcal{P} has at least one fixed point.

In the following, we show that the operator \mathcal{P} is relatively compact in \mathcal{V} for every $t \in \mathcal{J}$.

We prove this by showing that there is a relatively compact set $\{\mathcal{P}^\epsilon y(t) : y \in \mathcal{B}_\delta(\mathcal{J})\}$ arbitrarily close to the set $\{\mathcal{P}y(t) : y \in \mathcal{B}_\delta(\mathcal{J})\}$ in \mathcal{V} for $0 < t \leq b$.

Define the operator $\mathcal{P}^\epsilon y$ by

$$\begin{aligned} \mathcal{P}^\epsilon y(t) &= t^{(1+\nu\gamma)(1-\mu)} \sum_{\kappa=1}^m c_\kappa S_{\nu,\mu}(t) \mathcal{O} \int_0^{t_\kappa} \mathcal{X}_\nu(t_\kappa - s) \\ &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \\ &\quad + t^{(1+\nu\gamma)(1-\mu)} \mathcal{X}_\nu(\epsilon) \int_0^{t-\epsilon} \mathcal{X}_\nu(t - \epsilon - s) \\ &\quad \times \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds. \end{aligned}$$

Since $S_{\nu,\mu}(t)$ and $\mathcal{X}_\nu(t)$ are compact for $t > 0$, the set $\mathcal{P}^\epsilon y(t)$ is compact for $t \in \mathcal{J}'$. In the following

$$\begin{aligned} &\| \mathcal{P}^\epsilon y(t) - \mathcal{P}y(t) \| \\ &\leq \left\| t^{(1+\nu\gamma)(1-\mu)} \int_0^{t-\epsilon} \{ \mathcal{X}_\nu(t - \epsilon - s) \mathcal{X}_\nu(\epsilon) - \mathcal{X}_\nu(t - s) \} \right. \\ &\quad \times \left. \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\| \\ &\quad + \left\| t^{(1+\nu\gamma)(1-\mu)} \int_{t-\epsilon}^t \mathcal{X}_\nu(t - s) \right. \\ &\quad \times \left. \left\{ \Upsilon w(s) + \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \right\| \\ &\leq \int_0^{t-\epsilon} \| \mathcal{X}_\nu(t - \epsilon - s) \mathcal{X}_\nu(\epsilon) - \mathcal{X}_\nu(t - s) \| (\mathcal{N}_1 + \mathcal{N}_2 \bar{w} q^*) \delta ds \\ &\quad + b^{(1+\nu\gamma)(1-\mu)} \hat{M} \int_0^{t-\epsilon} \| \mathcal{X}_\nu(t - \epsilon - s) \mathcal{X}_\nu(\epsilon) - \mathcal{X}_\nu(t - s) \| ds \\ &\quad + C_p \delta (\mathcal{N}_1 + \mathcal{N}_2 \bar{w} q^*) \frac{\epsilon^{-\nu\gamma}}{(-\nu\gamma)} + \hat{M} C_p b^{(1+\nu\gamma)(1-\mu)} \frac{\epsilon^{-\nu\gamma}}{(-\nu\gamma)}, \quad (-\nu\gamma) > 0. \end{aligned}$$

Clearly,

$$\| \mathcal{P}^\epsilon y(t) - \mathcal{P}y(t) \| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

We have shown that the set $\{ \mathcal{P}y(t) : y \in \mathcal{B}_\delta(\mathfrak{J}) \}$ is arbitrarily close to the relatively compact set $\{ \mathcal{P}^\epsilon y(t) : y \in \mathcal{B}_\delta(\mathfrak{J}) \}$. This implies $\{ \mathcal{P}y(t) : y \in \mathcal{B}_\delta(\mathfrak{J}') \}$ is relatively compact in \mathcal{V} . Also, by Lemma 3.1 and Lemma 3.2, the operator $\mathcal{P} : \mathcal{B}_\delta(\mathfrak{J}') \rightarrow \mathcal{B}_\delta(\mathfrak{J}')$ is bounded, continuous, and equicontinuous in \mathcal{V} . So by the Arzela–Ascoli theorem, $\{ \mathcal{P}y(t) : y \in \mathcal{B}_\delta(\mathfrak{J}') \}$ is a compact operator. Hence, by Schauder’s fixed point theorem, \mathcal{P} has at least one fixed point $y^* \in \mathcal{B}_\delta(\mathfrak{J}')$.

Let $v^*(t) = t^{(1+\nu\gamma)(\mu-1)} y^*(t)$. Then v^* is a mild solution of Eq. (1.1). This completes the proof. \square

4 Approximate controllability

In this section, we discuss the approximate controllability of Eq. (1.1).

The system of equations Eq. (1.1) is said to be approximate controllable on $[0, b]$ if, for every desired final state $v_b \in \mathcal{V}$ and $\epsilon > 0$, there exists a control function $w \in L^2(\mathfrak{J}, \mathcal{W})$ such that the mild solution v' of Eq. (1.1) satisfies

$$\| v(b) - v_b \| < \epsilon.$$

Following this, we first introduce the following two operators:

$$\begin{aligned} \Gamma_0^b &= \int_0^b \mathcal{G}(b, s) \Upsilon \Upsilon^* \mathcal{G}^*(b, s) ds, \\ \mathcal{R}(\lambda, \Gamma_0^b) &= (\lambda I + \Gamma_0^b)^{-1}, \quad \lambda > 0, \end{aligned} \tag{4.1}$$

where Υ^* , \mathcal{O}^* , $\mathcal{X}_\psi^*(t)$, and $\mathcal{S}_{\nu, \mu}^*(t)$ characterize the adjoint operators of Υ , \mathcal{O} , $\mathcal{X}_\nu(t)$, and $\mathcal{S}_{\nu, \mu}(t)$ respectively,

$$\mathcal{G}^*(b, s) = \sum_{\kappa=1}^m \chi_{t_\kappa}(s) \mathcal{S}_{\nu, \mu}^*(b) \mathcal{O}^* \mathcal{X}_\psi^*(t_\kappa - s) + \chi_t(s) \mathcal{X}_\psi^*(b - s), \quad s \in [0, b].$$

Theorem 4.1 *Let $\mathcal{A} \in \Theta_\omega^\gamma$ for $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Assume that assumptions (A1)–(A5) and Proposition 2.3 hold. The system of equations Eq. (1.1) is approximate controllable on \mathfrak{J} if $\lambda \mathcal{R}(\lambda, \Gamma_0^b) \rightarrow 0$ as $\lambda \rightarrow 0^+$ in the strong operator topology.*

Proof By Theorem 3.1, the system of equations Eq. (1.1) has at least one mild solution $v_\lambda \in \mathcal{B}_\delta$ given by

$$v_\lambda(t) = \int_0^b \mathcal{G}(t, s) \left\{ \Upsilon w_\lambda(s) + \Psi \left(s, v_\lambda(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v_\lambda(\zeta)) d\zeta \right) \right\} ds, \quad t \in \mathfrak{J}, \tag{4.2}$$

where we define the control function w_λ as

$$w_\lambda(t) = \Upsilon^* \mathcal{G}^*(b, t) \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v(\cdot)) \tag{4.3}$$

with

$$\vartheta(v(\cdot)) = v_b - \int_0^b \mathcal{G}(b, s) \Psi \left(s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) ds. \tag{4.4}$$

At $t = b$,

$$\begin{aligned} v_\lambda(b) &= \int_0^b \mathcal{G}(b, s) \left\{ \Upsilon w_\lambda(s) + \Psi \left(s, v_\lambda(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v_\lambda(\zeta)) d\zeta \right) \right\} ds \\ &= v_b - \vartheta(v_\lambda(\cdot)) + \int_0^b \mathcal{G}(b, s) \Upsilon w_\lambda(s) ds \\ &= v_b - \vartheta(v_\lambda(\cdot)) + \int_0^b \mathcal{G}(b, s) \Upsilon \Upsilon^* \mathcal{G}^*(b, s) \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda(\cdot)) ds \\ &= v_b - (\lambda I + \Gamma_0^b) \cdot \frac{1}{(\lambda I + \Gamma_0^b)} \vartheta(v_\lambda(\cdot)) + \Gamma_0^b \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda(\cdot)) \\ &= v_b - (\lambda I + \Gamma_0^b) \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda(\cdot)) + \Gamma_0^b \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda(\cdot)) \\ \implies v_\lambda(b) &= v_b - \lambda \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda(\cdot)), \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\left(\int_0^b \left\| \Psi \left(s, v_\lambda(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\|^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^b (\psi_1(s) \|v_\lambda\| + \psi_2(s) \bar{w} q^* \|v_\lambda\|)^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^b (\psi_1(s) \delta + \psi_2(s) \bar{w} q^* \delta)^2 ds \right)^{\frac{1}{2}} \\ &\leq b^{\frac{1}{2}} \delta (\mathcal{N}_1^2 + \mathcal{N}_2^2 \bar{w}^2 q^{*2} + 2\mathcal{N}_1 \mathcal{N}_2 \bar{w} q^*)^{\frac{1}{2}}, \end{aligned}$$

which implies that the sequence $\{\Psi(\cdot, v_\lambda(\cdot), \cdot) : \lambda > 0\}$ is bounded in the Hilbert space $L^2(\mathfrak{J}, \mathcal{W})$. Therefore, there exists a subsequence of the sequence $\{\Psi(\cdot, v_\lambda(\cdot), \cdot) : \lambda > 0\}$ converging weakly to some point $\varrho(\cdot) \in L^2(\mathfrak{J}, \mathcal{W})$.

Let us write

$$\varpi = v_b - \int_0^b \mathcal{G}(b, s) \varrho(s) ds. \tag{4.6}$$

Hence, by Eqs. (4.4) and (4.6), we have

$$\|\vartheta(v_\lambda) - \varpi\| \leq \left\| \int_0^b \mathcal{G}(b, s) \left[\Psi \left(s, v_\lambda(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) - \varrho(s) \right] ds \right\|. \tag{4.7}$$

By the compactness of operators $\mathcal{K}_v(t)$ and $\mathcal{S}_{v,\mu}(t)$ for $t > 0$, one gets the compactness of Green's function $\mathcal{G}(t, s)$ for $t, s > 0$, which implies that

$$\begin{aligned} &\left\| \int_0^b \mathcal{G}(b, s) \left[\Psi \left(s, v_\lambda(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) - \varrho(s) \right] ds \right\| \\ &\longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+. \end{aligned} \tag{4.8}$$

Thus from Eq. (4.7) and Eq. (4.8) we get

$$\| \vartheta(v_\lambda) - \varpi \| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+. \tag{4.9}$$

Equation (4.5) implies that

$$\begin{aligned} \| v_\lambda(b) - v_b \| &\leq \| \lambda \mathcal{R}(\lambda, \Gamma_0^b) \vartheta(v_\lambda) \| \\ &\leq \| \lambda \mathcal{R}(\lambda, \Gamma_0^b) \varpi \| + \| \lambda \mathcal{R}(\lambda, \Gamma_0^b) \| \cdot \| \vartheta(v_\lambda) - \varpi \| \\ &\rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+ \text{ (by the stated condition and Eq. (4.9)).} \end{aligned}$$

Hence the approximate controllability of Eq. (1.1). □

5 Applications

Here we investigate the proposed results for the following Cauchy problem:

$$\begin{cases} {}^H D_{0^+}^{v, \mu} v(t, x) = -\frac{\partial^2 v(t, x)}{\partial x^2} + \wp w(t, x) + \frac{\cos^2 tv(t, x)}{(20+e^t)} \\ \quad + \frac{e^{-\frac{t}{15}}}{15} \int_0^t e^{-s} \sin(v(s, x)) ds, \quad x \in [0, \pi], t \in \mathfrak{J}, \\ v(t, 0) = v(t, \pi) = 0, \quad t \in \mathfrak{J}, \\ \mathcal{I}_{0^+}^{(1-\nu)(1-\mu)} v(0, x) = \sum_{\kappa=1}^m c_\kappa v(t_\kappa, x), \quad x \in [0, \pi], \end{cases} \tag{5.1}$$

in the Banach space $\mathcal{V} = C^\eta[0, \pi]$, $0 < \eta < 1$, where $\nu = \frac{1}{4}$, $\mu = \frac{1}{2}$, $\wp > 0$ and $c_\kappa \in \mathbb{R}$, $\kappa = 1, 2, \dots, m$, are such that $\sum_{\kappa=1}^m |c_\kappa| < \frac{1}{\mathfrak{M}_\zeta}$. On substituting $v(t, x) = v(t)$, Eq. (5.1) reduces to Eq. (1.1) with $\mathcal{A} = \frac{-\partial^2 v}{\partial x^2}$ and $D(\mathcal{A}) = \{v \in C^{\eta+2}[0, \pi] \text{ such that } v(t, 0) = v(t, \pi) = 0\}$. It follows from article [37] that there exist constants $\rho, \epsilon > 0$ such that $\mathcal{A} + \rho \in \Theta_{\frac{\eta}{2}-\epsilon}^{-1}(\mathcal{V})$. The compactness of semigroup $\{\mathfrak{T}(t)\}$ follows from (Lemma 4.66) [42]. Since $D(\mathcal{A}) \in C^{2+\eta}[0, \pi]$ and $C^{2+\eta}[0, \pi]$ is embedded in $C^\eta[0, \pi]$, the compactness of resolvent operators follows for every $\eta > 0$.

$$\Psi \left(t, v(t), \int_0^t \mathfrak{q}(t, s) \varphi(t, s, v(s)) ds \right) = \frac{\cos^2 tv(t)}{(20 + e^t)} + \frac{e^{-\frac{t}{15}}}{15} \int_0^t e^{-s} \sin v(s) ds.$$

The bounded linear operator $\Upsilon : \mathcal{W} = \mathcal{V} \rightarrow \mathcal{V}$ is defined as $\Upsilon w(t) = \wp w(t)$.

Following this discussion and the definition of function Ψ and bounded operator Υ , it is easy to verify that assumptions (A1)–(A5) hold with

$$\mathfrak{N}_1 = \frac{1}{20 + e^b}, \quad \mathfrak{N}_2 = \frac{e^{-\frac{b}{15}}}{15}, \quad \bar{w} = (1 - e^{-b}), \quad \text{and} \quad \mathfrak{q}^* = 1.$$

Hence the existence and approximate controllability of Eq. (5.1) follow from Theorem 3.1 and Theorem 4.1 respectively.

6 Conclusion

In this paper, we discussed the approximate controllability of Hilfer fractional differential equations with almost sectorial operators. We first prove the existence of mild solutions for similar equations by applying fixed point theory. We will try to investigate the exact controllability and stability of a similar problem in our future research work.

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