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Further results on existence of positive solutions of generalized fractional boundary value problems

Hojjat Afshari[1*], Mohammed S. Abdo[2] and Jehad Alzabut[3]

*Correspondence:
hojat.afshari@yahoo.com
[1]Department of Mathematics, Faculty of Basic Science, University of Bonab, Bonab, Iran
Full list of author information is available at the end of the article

Abstract

This paper studies two classes of boundary value problems within the generalized Caputo fractional operators. By applying the fixed point result of α-ϕ-Geraghty contractive type mappings, we derive new results on the existence and uniqueness of the proposed problems. Illustrative examples are constructed to demonstrate the advantage of our results. The theorems reported not only provide a new approach but also generalize existing results in the literature.

Keywords: Generalized Caputo differential equation; α-ϕ-Geraghty contractive; Positive solution

1 Introduction

Recently, it has been realized that fractional calculus (FC) has played a very important role in different areas of research; see [26, 32] and the references cited therein. Consequently, fractional differential equations (FDEs) have grasped the interest of many researchers working in diverse applications [22, 39]. Most relevant results have been obtained in terms of the classical fractional derivatives (FDs) of Riemann–Liouville (RL), Caputo (Ca), Katugampola (Ka), Hadamard (Ha), Hilfer (Hi) FDs etc.

Generalized fractional derivatives (GFDs) with respect to another function κ have been considered in [32, 41] as a generalization of RL fractional operator (FO). The GFD is different from the other classical FD because the kernel appears in terms of another function κ. Recently, Almeida in [13] presented a generalized version of Ca with some advantageous properties. Many properties of the generalized FO can be found in [11, 32, 33, 41, 45]. The advantage of studying the generalized FD lay in providing a general platform that includes all particular derivatives. For some special cases of a function κ, one can realize that κ-Ca FD can be reduced to the (Ca, when $\kappa(t) \to t$ see [32], Ca-Ha, when $\kappa(t) \to \log t$ [28], Ca-Ka, when $\kappa(t) \to t^{\rho} (\rho > 0)$ [30, 31]) FD.

On the other hand, the investigation of existence and uniqueness of solutions to several types of fractional (impulsive, functional, evolution, etc.) differential equations is the main topic of applied mathematics research. Many interesting results with regard the existence, uniqueness, and stability of solutions or positive solutions by using some fixed point (FP) theorem have been discussed in Refs. [1, 12, 18, 19, 21, 23, 24, 27, 35, 37, 38, 44].

For the purpose of completeness, we refer thereafter to some relevant papers that deal with the existence of positive solutions involving classical Ca and RL derivatives. More precisely, the authors in [17] studied the existence and multiplicity of positive solutions for the following problem:

$$\begin{cases} {}^{\mathrm{RL}}D_{0+}^{\iota}\varpi(\varkappa) + g(\varkappa,\varpi(\varkappa)) = 0, & \varkappa \in (0,1), \\ \varpi(0) = \varpi(1) = 0, \end{cases}$$

where $1 < \iota \leq 2$, and ${}^{\mathrm{RL}}D_{0+}^{\iota}$ is RL FO. Also, the problem

$$\begin{cases} {}^{C}D_{0+}^{\iota}\varpi(\varkappa) + g(\varkappa,\varpi(\varkappa)) = 0, & \varkappa \in (0,1), \\ \varpi(0) + \varpi'(0) = 0, \qquad \varpi(1) + \varpi'(1) = 0, \end{cases}$$

was discussed in [48], where $1 < \iota \leq 2$, and ${}^{C}D_{0+}^{\iota}$ is the CF operator.

For some recent findings on GFDs with respect to another function κ, see [2, 3, 14, 15, 25, 34, 36, 42, 43, 46, 47].

In other direction, Karapinar and Samet introduced the notion of generalized α-ψ-Geraghty contractive (α-ψ-GC) type mappings (see [29]). The generalized α-ψ-GC in complete b-metric spaces (b-MS) and their applications in b-metric spaces b-MS was introduced in [4–10, 16, 40].

To the best of our observation, the investigation of positive solutions to fractional BVP has not been studied within κ-Ca and κ-RL FOs yet. Moreover, the FP technique based on α-ψ-GC has never been applied to such problems.

Inspired by the above results and motivated by the recent evolutions in κ-fractional calculus, in this paper, we apply the FP technique of α-ψ-GC type mappings to investigate the existence of positive solutions for the following fractional BVPs:

$$\begin{cases} {}^{C}D_{0+}^{\iota,\kappa}\varpi(\varkappa) + g(\varkappa,\varpi(\varkappa)) = 0, & \varkappa \in (0,1), \\ \varpi(0) = \varpi(1) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{C}D_{0+}^{\iota,\kappa}\varpi(\varkappa) + g(\varkappa,\varpi(\varkappa)) = 0, & \varkappa \in (0,1), \\ \varpi(0) + \varpi'(0) = 0, \qquad \varpi(1) + \varpi'(1) = 0. \end{cases} \tag{2}$$

where $1 < \iota \leq 2$, and ${}^{C}D_{0+}^{\iota,\kappa}$ is κ-FD of order ι in the sense the κ-Ca operator, and $g : \digamma \times \mathbb{R} \to \mathbb{R}^{+}$ is a continuous function. Throughout the article $\digamma = [0,1]$ and $\mathbb{R}^{+} = [0,\infty)$.

We claim that our approach is new and the reported results are different form existing ones in the literature.

The remaining parts of the paper are outlined as follows: Some preliminary facts needed for the proofs of the main results are recalled in Sect. 2. In Sect. 3, we prove the existence of positive solutions for problems (1) and (2) by the aid of the FP result of α-ψ-GC type mappings. Examples are given in Sect. 4 to check the applicability of the theoretical findings. We end the paper by a conclusion.

2 Preliminaries

Definition 2.1 ([32]) Let $\iota > 0$ and κ be an increasing function, having a continuous derivative κ' on (a,b). The left-sided κ-RL fractional integral of a function h with respect to κ on $[a,b]$ is defined by

$$I_{a^+}^{\iota,\kappa} h(\varrho) = \frac{1}{\Gamma(\iota)} \int_a^{\varrho} \kappa'(\varsigma)\big[\kappa(\varrho) - \kappa(\varsigma)\big]^{\iota-1} h(\varsigma)\,d\varsigma, \quad \varrho > a, \iota > 0,$$

provided that $I_{a^+}^{\iota,\kappa}$ exists. Note that when $\kappa(\varrho) = \varrho$, we obtain the known classical RL fractional integral.

Definition 2.2 ([32, 41]) Let $\iota > 0$, n be the smallest integer greater than or equal to ι and $h \in L^p[a,b]$, $p \geq 1$ let $\kappa \in C^n[a,b]$ an increasing function such that $\kappa'(\varrho) \neq 0$, for all $\varrho \in [a,b]$. The left-sided κ-RL FD of h of order ι is given by

$$D_{a^+}^{\iota;\kappa} h(\varrho) = \left(\frac{1}{\kappa'(\varrho)}\frac{d}{d\varrho}\right)^n I_{a^+}^{n-\iota,\kappa} h(\varrho), \quad n-1 < \iota < n, n \in \mathbb{N}.$$

Definition 2.3 ([13, 14]) Let $n-1 < \iota < n$, $h \in C^n[a,b]$, and let $\kappa \in C^n[a,b]$ an increasing function such that $\kappa'(\varrho) \neq 0$, for all $\varrho \in [a,b]$. The left-sided κ-Ca FD of h of order ι is given by

$${}^C D_{a^+}^{\iota;\kappa} h(\varrho) = I_{a^+}^{n-\iota,\kappa} D^{n,\kappa} h(\varrho),$$

where $D^{n,\kappa} := (\frac{1}{\kappa'(\varrho)}\frac{d}{d\varrho})^n$, and $n = [\iota] + 1$.

Definition 2.4 ([20]) Let M be a nonempty set and $r \geq 1$. A mapping $d \colon M \times M \to \mathbb{R}^+$ is said to be a b-metric if for $\varrho, \varsigma, \varpi \in M$;

(bM_1) $d(\varrho,\varsigma) = 0$ if and only if $\varsigma = \varrho$;
(bM_2) $d(\varrho,\varsigma) = d(\varsigma,\varrho)$;
(bM_3) $d(\varrho,\varpi) \leq r[d(\varrho,\varsigma) + d(\varsigma,\varpi)]$.
The pair (M,d) is called a b-MS with constant r.

Let Φ be set of all increasing and continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the property: $\phi(c\varrho) \leq c\phi(\varrho) \leq c\varrho$ for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}^+ \to [0, \frac{1}{r^2})$ for some $r \geq 1$.

Definition 2.5 ([8]) Let (M,d) be a b-MS and $T : M \to M$, we say that T is a generalized α-ϕ-GC type mapping whenever there exists $\alpha : M \times M \to \mathbb{R}^+$ such that

$$\alpha(\varrho,\varsigma)\phi\big(r^3 d(T\varrho, T\varsigma)\big) \leq \lambda\big(\phi\big(d(\varrho,\varsigma)\big)\big)\phi\big(d(\varrho,\varsigma)\big),$$

for $\varrho, \varsigma \in M$, where $\lambda \in \mathcal{F}$ and $\phi \in \Phi$.

Definition 2.6 ([40]) For $M(\neq \emptyset)$, let $T : M \to M$ and $\alpha : M \times M \to \mathbb{R}^+$ be given mappings. We say that T is α-admissible if, for $\varrho, \varsigma \in M$, we have

$$\alpha(\varrho,\varsigma) \geq 1 \quad \Longrightarrow \quad \alpha(T\varrho, T\varsigma) \geq 1. \tag{3}$$

Theorem 2.7 ([8]) *Let (M,d) be a complete b-MS and $T:M\to M$ be a generalized α-ϕ-GC type mapping such that*

(i) *T is α-admissible;*
(ii) *there exists $\varrho_0\in M$ such that $\alpha(\varrho_0,T\varrho_0)\geq 1$;*
(iii) *if $\{\varrho_n\}\subseteq M$ with $\varrho_n\to\varrho$ and $\alpha(\varrho_n,\varrho_{n+1})\geq 1$, then $\alpha(\varrho_n,\varrho)\geq 1$.*

Then T has a FP.

Lemma 2.8 ([46]) *Let $g\in C(\digamma)$ and $1<\iota\leq 2$. Then the FBVP*

$$\begin{cases} {}^{C}D_{a+}^{\iota,\kappa}\varpi(\varkappa)+g(\varkappa,\varpi(\varkappa))=0, & \varkappa\in(0,1),\\ \varpi(0)=0, & \varpi(1)=0, \end{cases} \tag{4}$$

is equivalent to

$$\varpi(\varkappa)=\int_0^1\mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)g\big(\vartheta,\varpi(\vartheta)\big)\,d\vartheta,$$

where

$$\mathcal{G}(\varkappa,\vartheta)=\frac{(\kappa(\varkappa)-\kappa(0))^{\iota-1}}{(\kappa(1)-\kappa(0))^{\iota-1}\Gamma(\iota)}\begin{cases}(\kappa(1)-\kappa(\vartheta))^{\iota-1}-\frac{(\kappa(1)-\kappa(0))^{\iota-1}}{(\kappa(\varkappa)-\kappa(0))^{\iota-1}}(\kappa(\varkappa)-\kappa(\vartheta))^{\iota-1},\\ \quad 0\leq\vartheta\leq\varkappa\leq 1,\\ (\kappa(1)-\kappa(\vartheta))^{\iota-1},\\ \quad 0\leq\varkappa\leq\vartheta\leq 1.\end{cases} \tag{5}$$

Lemma 2.9 ([46]) *For the function $\mathcal{G}$ defined by (5) we have;*

(i) *$\mathcal{G}(\varkappa,\vartheta)>0$, for all $\varkappa,\vartheta\in(0,1)$.*
(ii) *For $\vartheta\in(0,1)$, there exists a positive function γ such that*

$$\min_{\varkappa\in[1/4,3/4]}\mathcal{G}(\varkappa,\vartheta)\geq\gamma(\vartheta)\max_{\varkappa\in\digamma}\mathcal{G}(\varkappa,\vartheta).$$

Lemma 2.10 ([46]) *Let $g\in C[a,b]$, and $1<\iota\leq 2$, then the FBVP*

$$\begin{cases} {}^{C}D_{0+}^{\iota,\kappa}\varpi(\varkappa)-g(\varkappa)=0, & \varkappa\in(0,1),\\ \varpi(0)+\varpi'(0)=0, & \varpi(1)+\varpi'(1)=0, \end{cases} \tag{6}$$

has a solution

$$\varpi(\varkappa)=\int_0^1\Lambda(\varkappa,\vartheta)\kappa'(\vartheta)g(\vartheta)\,d\vartheta,$$

where

$$\begin{aligned}\Lambda(\varkappa,\vartheta)&=\big(\Gamma(\iota-1)\big[\big(\kappa(1)-\kappa(0)\big)+\big(\kappa'(1)-\kappa'(0)\big)\big]\big)^{-1}\\ &\quad\times\begin{cases}((\kappa'(0)+\kappa(0)-\kappa(\varkappa))[((\kappa(1)-\kappa(\vartheta))^{\iota-2}+\frac{1}{\iota-1}(\kappa(1)-\kappa(\vartheta))^{\iota-1}]\\ \quad+\frac{(\kappa(1)-\kappa(0))+(\kappa'(1)-\kappa'(0))}{\iota-1}(\kappa(\varkappa)-\kappa(\vartheta))^{\iota-1}, \quad 0\leq\vartheta\leq\varkappa\leq 1,\\ ((\kappa'(0)+\kappa(0)-\kappa(\varkappa))[((\kappa(1)-\kappa(\vartheta))^{\iota-2}+\frac{1}{\iota-1}(\kappa(1)-\kappa(\vartheta))^{\iota-1}],\\ \quad 0\leq\varkappa\leq\vartheta\leq 1.\end{cases}\end{aligned} \tag{7}$$

Lemma 2.11 ([46]) *Let* $\kappa(\varkappa) \leq \kappa(0) + \kappa'(0)$, *then* (7) *satisfies* $\Lambda(\varkappa, \vartheta) > 0$, *for all* $\vartheta, \varkappa \in (0,1)$. *Besides, there exists a positive function* $\upsilon \in (0,1)$, *such that*

$$\min_{\varkappa\in[1/4,3/4]} \Lambda(\varkappa, \vartheta) \geq \upsilon(\vartheta) M(\vartheta), \quad \vartheta \in (0,1),$$

and

$$\max_{\varkappa\in F} \Lambda(\varkappa, \vartheta) \leq M(\vartheta),$$

where

$$m(\vartheta) = \left(\kappa'(0) + \kappa(0) - \kappa(3/4)\right)\left[\left(\kappa(1) - \kappa(\vartheta)\right)^{\iota-2} + \frac{1}{\iota-1}\left(\kappa(1) - \kappa(\vartheta)\right)^{\iota-1}\right],$$
$$M(\vartheta) = \left(\kappa(1) + \kappa'(1)\right)\left[\left(\kappa(1) - \kappa(\vartheta)\right)^{\iota-2} + \frac{2}{\iota-1}(\left(\kappa(1) - \kappa(\vartheta)\right)^{\iota-1}\right],$$

and

$$\upsilon(\vartheta) = \frac{m(\vartheta)}{M(\vartheta)} = \frac{\kappa'(0) + \kappa(0) - \kappa(3/4)}{\kappa'(1) + \kappa(1)} \frac{(\iota-1)(\kappa(1) - \kappa(\vartheta))^{\iota-2} + (\kappa(1) - \kappa(\vartheta))^{\iota-1}}{(\iota-1)(\kappa(1) - \kappa(\vartheta))^{\iota-2} + 2(\kappa(1) - \kappa(\vartheta))^{\iota-1}},$$
$$\vartheta \in (0,1).$$

3 Main results

Let $M = C(F, \mathbb{R}^+)$ and $d : M \times M \to \mathbb{R}^+$ be given by

$$d(\varpi, w) = \left\|(\varpi - w)^2\right\|_\infty = \sup_{\vartheta\in[0,1]} \left(\varpi(\vartheta) - w(\vartheta)\right)^2.$$

Then, (M, d) is a complete b-MS with $r = 2$.

Theorem 3.1 *Suppose that*

(i) $\exists\, g : F \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\left|g\left(\vartheta, \varpi(\vartheta)\right) - g\left(\vartheta, w(\vartheta)\right)\right|$$
$$\leq \frac{1}{2\sqrt{2}} \frac{\Gamma(\iota+1)}{(\kappa(1) - \kappa(0))^\iota} \sqrt{\phi\left(\left\|(\varpi - w)^2\right\|_\infty\right)\lambda\left(\phi\left(\left\|(\varpi - w)^2\right\|_\infty\right)\right)},$$

where $\phi \in \Phi$ *and* $\lambda \in \mathcal{F}$;

(ii) $\exists\, \varpi_0 \in C(F)$ *and* $\mu : \mathbb{R}^2 \to \mathbb{R}$ *with* $\mu(\varpi_0(\vartheta), \int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g(\vartheta, \varpi_0(\vartheta))\, d\vartheta) \geq 0$, $\vartheta \in F$;

(iii) *for* $\vartheta \in F$ *and* $\varpi, w \in C(F)$, $\mu(\varpi(\vartheta), w(\vartheta)) \geq 0$ *implies*

$$\mu\left(\int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g\left(\vartheta, \varpi(\vartheta)\right) d\vartheta, \int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g\left(\vartheta, w(\vartheta)\right) d\vartheta\right) \geq 0;$$

(iv) *If* $\{\varpi_n\} \subseteq C(F)$ *with* $\varpi_n \to \varpi$ *and* $\mu(\varpi_n, \varpi_{n+1}) \geq 0$, *then* $\mu(\varpi_n, \varpi) \geq 0$.

Then the problem (4) *has at least one solution.*

Proof By Lemma 2.8, $\varpi \in C(\digamma)$ is a solution of (6) if and only if ϖ is a solution of the integral equation $\varpi(\varkappa) = \int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)g(\vartheta,\varpi(\vartheta))\,d\vartheta$, $\varkappa \in \digamma$. Define, $O : C(\digamma) \to C(\digamma)$ by $O\varpi(\varkappa) = \int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)g(\vartheta,\varpi(\vartheta))\,d\vartheta$. We find a FP of O. Now, let $\varpi, w \in C(\digamma)$ be such that $\mu(\varpi(\varkappa), w(\varkappa)) \geq 0$. On one hand we have

$$\begin{aligned}
&\bigl|O\varpi(\varkappa) - Ow(\varkappa))\bigr|^2 \\
&\quad = \left|\int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)g\bigl(\vartheta,\varpi(\vartheta)\bigr)\,d\vartheta - \int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)g\bigl(\vartheta,w(\vartheta)\bigr)\,d\vartheta\right|^2 \\
&\quad \leq \left[\int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)\bigl|g\bigl(\vartheta,\varpi(\vartheta)\bigr) - g\bigl(\vartheta,w(\vartheta)\bigr)\bigr|\,d\vartheta\right]^2.
\end{aligned}$$

By Lemma 2.8, for $0 < \varkappa < \vartheta < 1$ we have

$$\begin{aligned}
\int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)\,d\vartheta &= \frac{(\kappa(\varkappa) - \kappa(0))^{\iota-1}}{(\kappa(1) - \kappa(0))^{\iota-1}\Gamma(\iota)} \int_0^1 \bigl(\kappa(1) - \kappa(\vartheta)\bigr)^{\iota-1}\kappa'(\vartheta)\,d\vartheta \\
&= \frac{(\kappa(\varkappa) - \kappa(0))^{\iota-1}}{(\kappa(1) - \kappa(0))^{\iota-1}\Gamma(\iota)} \left[\frac{-(\kappa(1) - \kappa(\vartheta))^{\iota}}{\iota}\right]_0^1 \\
&\leq \frac{(\kappa(1) - \kappa(0))^{\iota-1}}{(\kappa(1) - \kappa(0))^{\iota-1}\Gamma(\iota+1)} \bigl(\kappa(1) - \kappa(0)\bigr)^{\iota} \\
&= \frac{(\kappa(1) - \kappa(0))^{\iota}}{\Gamma(\iota+1)}.
\end{aligned}$$

For $0 < \vartheta < \varkappa < 1$, the same estimates can be proved in analogous way to the previous one. So we will omit it.

Using (i), we get

$$\begin{aligned}
&\int_0^1 \mathcal{G}(\varkappa,\vartheta)\kappa'(\vartheta)\bigl|g\bigl(\vartheta,\varpi(\vartheta)\bigr) - g\bigl(\vartheta,w(\vartheta)\bigr)\bigr|\,d\vartheta \\
&\quad \leq \frac{(\kappa(1) - \kappa(0))^{\iota}}{\Gamma(\iota+1)} \bigl|g\bigl(\vartheta,\varpi(\vartheta)\bigr) - g\bigl(\vartheta,w(\vartheta)\bigr)\bigr| \\
&\quad \leq \frac{1}{2\sqrt{2}}\sqrt{\phi\bigl(\bigl\|(\varpi - w)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(\varpi - w)^2\bigr\|_\infty\bigr)\bigr)}.
\end{aligned}$$

Thus,

$$\bigl|O\varpi(\varkappa) - Ow(\varkappa))\bigr|^2 \leq \frac{1}{8}\phi\bigl(\bigl\|(\varpi - w)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(\varpi - w)^2\bigr\|_\infty\bigr)\bigr).$$

Put $\alpha : C(\digamma) \times C(\digamma) \to \mathbb{R}^+$ by

$$\alpha(\varpi, w) = \begin{cases} 1 & \mu(\varpi(\vartheta), w(\vartheta)) \geq 0, \vartheta \in \digamma, \\ 0 & \text{else.} \end{cases}$$

So for $\varpi, w \in C(\digamma)$ with $\mu(\varpi(\vartheta), w(\vartheta)) \geq 0$, we have

$$\alpha(\varpi, w)8d(O\varpi, Ow) \leq 8d(O\varpi, Ow) \leq \lambda\bigl(\phi\bigl(d(\varpi, w)\bigr)\bigr)\phi\bigl(d(\varpi, w)\bigr), \quad \lambda \in \digamma.$$

So, we conclude that O is a α-ϕ-GC type mapping.

From (iii), we get

$$\begin{aligned} \alpha(\varpi, w) \geq 1 \quad &\Rightarrow \quad \mu\big(\varpi(\vartheta), w(\vartheta)\big) \geq 0 \quad \Rightarrow \quad \mu\big(O(\varpi), O(w)\big) \geq 0 \\ &\Rightarrow \quad \alpha\big(O(\varpi), O(w)\big) \geq 1, \end{aligned}$$

for $\varpi, w \in C(F)$. Thus, O is α-admissible. From (ii), there exists $\varpi_0 \in C(F)$ with $\alpha(\varpi_0, O\varpi_0) \geq 1$. By (iv) and Theorem 2.7, we find ϖ^* with $\varpi^* = O\varpi^*$, that is, a positive solution of (4). □

Theorem 3.2 *Suppose that*

(i) $\exists\, g : F \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned} &\big|g\big(\vartheta, \varpi(\vartheta)\big) - g\big(\vartheta, w(\vartheta)\big)\big| \\ &\quad \leq \frac{1}{2\sqrt{2}}\Bigg(\big(\Gamma(\iota-1)\big[\kappa(1)-\kappa(0)+\kappa'(1)-\kappa'(0)\big]\big)^{-1}\frac{\kappa'(0)}{\iota-1} \\ &\qquad \times \big(\kappa(1)-\kappa(0)\big)^{\iota-1}\bigg(1+\frac{1}{\iota}\big(\kappa(1)-\kappa(0)\big)\bigg) + \frac{1}{\Gamma(\iota+1)}\big(\kappa(1)-\kappa(0)\big)^{\iota}\Bigg)^{-1} \\ &\qquad \times \sqrt{\phi\big(\big\|(\varpi-w)^2\big\|_\infty\big)\lambda\big(\phi\big(\big\|(\varpi-w)^2\big\|_\infty\big)\big)}, \end{aligned}$$

where $\phi \in \Phi$ *and* $\lambda \in \mathcal{F}$;

(ii) $\exists\, \varpi_0 \in C(F)$ *and* $\mu : \mathbb{R}^2 \to \mathbb{R}$ *with* $\mu(\varpi_0(\vartheta), \int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta)g(\vartheta, \varpi_0(\vartheta))\, d\vartheta) \geq 0$, $\vartheta \in F$;

(iii) *for* $\vartheta \in F$ *and* $\varpi, w \in C(F)$, $\mu(\varpi(\vartheta), w(\vartheta)) \geq 0$ *implies*

$$\mu\bigg(\int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)g\big(\vartheta,\varpi(\vartheta)\big)\,d\vartheta, \int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)g\big(\vartheta,w(\vartheta)\big)\,d\vartheta\bigg) \geq 0;$$

(iv) *if* $\{\varpi_n\} \subseteq C(F)$ *with* $\varpi_n \to \varpi$ *and* $\mu(\varpi_n, \varpi_{n+1}) \geq 0$, *then* $\mu(\varpi_n, \varpi) \geq 0$.

Then (6) *has at least one solution.*

Proof By Lemma 2.10, $\varpi \in C(F)$ is a solution of (6) if and only if ϖ is a solution of the integral equation $\varpi(\varkappa) = \int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta)g(\vartheta, \varpi(\vartheta))\, d\vartheta$, $\varkappa \in F$. Define $O : C(F) \to C(F)$ by $O\varpi(\varkappa) = \int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta)g(\vartheta, \varpi(\vartheta))\, d\vartheta$. Let $\varpi, w \in C(F)$ be such that $\mu(\varpi(\varkappa), w(\varkappa)) \geq 0$. On the one hand we have

$$\begin{aligned} &\big|O\varpi(\varkappa) - Ow(\varkappa))\big|^2 \\ &\quad = \bigg|\int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)g\big(\vartheta,\varpi(\vartheta)\big)\,d\vartheta - \int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)g\big(\vartheta,w(\vartheta)\big)\,d\vartheta\bigg|^2 \\ &\quad \leq \bigg[\int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)\big|g\big(\vartheta,\varpi(\vartheta)\big) - g\big(\vartheta,w(\vartheta)\big)\big|\,d\vartheta\bigg]^2. \end{aligned}$$

By Lemma 2.10, for $0 < \vartheta < \varkappa < 1$ we have

$$\int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)\,d\vartheta$$

$$\begin{aligned}
&\leq \bigl(\Gamma(\iota-1)\bigl[\kappa(1)-\kappa(0)+\kappa'(1)-\kappa'(0)\bigr]\bigr)^{-1}\bigl(\kappa'(0)+\kappa(0)-\kappa(\varkappa)\bigr) \\
&\quad\times \int_0^1 \biggl[\bigl(\kappa(1)-\kappa(\vartheta)\bigr)^{\iota-2}+\frac{1}{\iota-1}\bigl(\kappa(1)-\kappa(\vartheta)\bigr)^{\iota-1}\biggr]\kappa'(\vartheta)\,d\vartheta \\
&\quad+\frac{1}{\Gamma(\iota)}\int_0^1 \bigl(\kappa(1)-\kappa(\vartheta)\bigr)^{\iota-1}\kappa'(\vartheta)\,d\vartheta \\
&\leq \bigl(\Gamma(\iota-1)\bigl[\kappa(1)-\kappa(0)+\kappa'(1)-\kappa'(0)\bigr]\bigr)^{-1}\frac{\kappa'(0)}{\iota-1} \\
&\quad\times \biggl\{-\bigl[\kappa(1)-\kappa(\vartheta))^{\iota-1}\bigr]_0^1+\frac{1}{\iota}\bigl[-\bigl(\kappa(1)-\kappa(\vartheta)\bigr)^{\iota}\bigr]_0^1\biggr\} \\
&\quad+\frac{1}{\Gamma(\iota+1)}\bigl[-\bigl(\kappa(1)-\kappa(\vartheta)\bigr)^{\iota}\bigr]_0^1 \\
&\leq \bigl(\Gamma(\iota-1)\bigl[\kappa(1)-\kappa(0)+\kappa'(1)-\kappa'(0)\bigr]\bigr)^{-1}\frac{\kappa'(0)}{\iota-1} \\
&\quad\times \bigl(\kappa(1)-\kappa(0)\bigr)^{\iota-1}\biggl(1+\frac{1}{\iota}\bigl(\kappa(1)-\kappa(0)\bigr)\biggr)+\frac{1}{\Gamma(\iota+1)}\bigl(\kappa(1)-\kappa(0)\bigr)^{\iota}.
\end{aligned}$$

For $0<\varkappa<\vartheta<1$, we obtain the same estimates as analogous way to the previous one.

From (i), we get

$$\begin{aligned}
&\int_0^1 \Lambda(\varkappa,\vartheta)\kappa'(\vartheta)\bigl|g\bigl(\vartheta,\varpi(\vartheta)\bigr)-g\bigl(\vartheta,w(\vartheta)\bigr)\bigr|\,d\vartheta \\
&\quad\leq \biggl[\bigl(\Gamma(\iota-1)\bigl[\kappa(1)-\kappa(0)+\kappa'(1)-\kappa'(0)\bigr]\bigr)^{-1}\frac{\kappa'(0)}{\iota-1} \\
&\qquad\times \bigl(\kappa(1)-\kappa(0)\bigr)^{\iota-1}\biggl(1+\frac{1}{\iota}\bigl(\kappa(1)-\kappa(0)\bigr)\biggr)+\frac{1}{\Gamma(\iota+1)}\bigl(\kappa(1)-\kappa(0)\bigr)^{\iota}\biggr] \\
&\qquad\times \bigl|g\bigl(\vartheta,\varpi(\vartheta)\bigr)-g\bigl(\vartheta,w(\vartheta)\bigr)\bigr| \\
&\quad\leq \frac{1}{2\sqrt{2}}\sqrt{\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\bigr)}.
\end{aligned}$$

Thus,

$$\bigl|O\varpi(\varkappa)-Ow(\varkappa))\bigr|^2\leq\frac{1}{8}\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\bigr).$$

Put, $\alpha: C(\digamma)\times C(\digamma)\to\mathbb{R}^+$ by

$$\alpha(\varpi,w)=\begin{cases}1 & \mu(\varpi(\vartheta),w(\vartheta))\geq 0, \vartheta\in\digamma,\\ 0 & \text{else}.\end{cases}$$

Hence, for $\varpi,w\in C(\digamma)$ with $\mu(\varpi(\vartheta),w(\vartheta))\geq 0$, we have

$$\alpha(\varpi,w)8d(O\varpi,Ow)\leq 8d(O\varpi,Ow)\leq\lambda\bigl(\phi\bigl(d(\varpi,w)\bigr)\bigr)\phi\bigl(d(\varpi,w)\bigr),\quad \lambda\in\mathcal{F}.$$

From (iii),

$$\alpha(\varpi,w)\geq 1 \quad\Rightarrow\quad \mu\bigl(\varpi(\vartheta),w(\vartheta)\bigr)\geq 0 \quad\Rightarrow\quad \mu\bigl(O(\varpi),O(w)\bigr)\geq 0$$

$$\Rightarrow \quad \alpha\big(O(\varpi), O(w)\big) \geq 1,$$

for $\varpi, w \in C(\digamma)$. Thus, O is α-admissible. From (ii), there exists $\varpi_0 \in C(\digamma)$ with $\alpha(\varpi_0, O\varpi_0) \geq 1$. By (iv) and Theorem 2.7, we find ϖ^* with $\varpi^* = O\varpi^*$, that is, a positive solution of the problem (6). □

Setting $\phi(t) = t$ and $\lambda(t) = \frac{\cos^2 t}{4}$ in Theorems 3.1 and 3.2 we deduce the following corollaries.

Corollary 3.3 *Suppose that*

(i) $\exists\, g : \digamma \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned} &\big|g\big(\vartheta, \varpi(\vartheta)\big) - g\big(\vartheta, w(\vartheta)\big)\big| \\ &\quad \leq \frac{1}{2\sqrt{2}} \frac{\Gamma(\iota+1)}{(\kappa(1)-\kappa(0))^{\iota}} \frac{\sqrt{\|(\varpi - w)^2\|_\infty \cos^2 \|(\varpi - w)^2\|_\infty}}{2}; \end{aligned}$$

(ii) $\exists\, \varpi_0 \in C(\digamma)$ *and* $\mu : \mathbb{R}^2 \to \mathbb{R}$ *with* $\mu(\varpi_0(\vartheta), \int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g(\vartheta, \varpi_0(\vartheta))\, d\vartheta) \geq 0$, $\vartheta \in \digamma$;

(iii) *for* $\vartheta \in \digamma$ *and* $\varpi, w \in C(\digamma)$, $\mu(\varpi(\vartheta), w(\vartheta)) \geq 0$ *implies*

$$\mu\left(\int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g\big(\vartheta, \varpi(\vartheta)\big)\, d\vartheta, \int_0^1 \mathcal{G}(\varkappa, \vartheta)\kappa'(\vartheta) g\big(\vartheta, w(\vartheta)\big)\, d\vartheta\right) \geq 0;$$

(iv) *if* $\{\varpi_n\} \subseteq C(\digamma)$ *with* $\varpi_n \to \varpi$ *and* $\mu(\varpi_n, \varpi_{n+1}) \geq 0$, *then* $\mu(\varpi_n, \varpi) \geq 0$.

Then (4) *has at least one solution.*

Corollary 3.4 *Suppose that*

(i) $\exists\, g : \digamma \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned} &\big|g\big(\vartheta, \varpi(\vartheta)\big) - g\big(\vartheta, w(\vartheta)\big)\big| \\ &\quad \leq \frac{1}{2\sqrt{2}} \left(\big(\Gamma(\iota-1)\big[\kappa(1) - \kappa(0) + \kappa'(1) - \kappa'(0)\big]\big)^{-1} \frac{\kappa'(0)}{\iota - 1} \right. \\ &\qquad \left. \times \big(\kappa(1)-\kappa(0)\big)^{\iota-1}\left(1 + \frac{1}{\iota}\big(\kappa(1)-\kappa(0)\big)\right) + \frac{1}{\Gamma(\iota+1)}\big(\kappa(1)-\kappa(0)\big)^{\iota}\right)^{-1} \\ &\qquad \times \frac{\sqrt{\|(\varpi - w)^2\|_\infty \cos^2 \|(\varpi - w)^2\|_\infty}}{2}, \end{aligned}$$

where $\phi \in \Phi$ *and* $\lambda \in \mathcal{F}$;

(ii) $\exists\, \varpi_0 \in C(\digamma)$ *and* $\mu : \mathbb{R}^2 \to \mathbb{R}$ *with* $\mu(\varpi_0(\vartheta), \int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta) g(\vartheta, \varpi_0(\vartheta))\, d\vartheta) \geq 0$, $\vartheta \in \digamma$;

(iii) *for* $\vartheta \in \digamma$ *and* $\varpi, w \in C(\digamma)$, $\mu(\varpi(\vartheta), w(\vartheta)) \geq 0$ *implies*

$$\mu\left(\int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta) g\big(\vartheta, \varpi(\vartheta)\big)\, d\vartheta, \int_0^1 \Lambda(\varkappa, \vartheta)\kappa'(\vartheta) g\big(\vartheta, w(\vartheta)\big)\, d\vartheta\right) \geq 0;$$

(iv) *if* $\{\varpi_n\} \subseteq C(\digamma)$ *with* $\varpi_n \to \varpi$ *and* $\mu(\varpi_n, \varpi_{n+1}) \geq 0$, *then* $\mu(\varpi_n, \varpi) \geq 0$.

Then (6) *has at least one solution.*

4 Examples

Example 4.1 Consider the κ-Ca fractional integral BVP

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2},\frac{e^{\varkappa}}{3}}\varpi(\varkappa) = g(\varkappa,\varpi(\varkappa)), & \varkappa\in(0,1), \|\varpi\|_\infty < \frac{\pi}{4}, \\ \varpi(0)=\varpi(1)=0, \end{cases} \tag{8}$$

where $\iota = \frac{3}{2}$, $\kappa(\varkappa) = e^{\frac{\varkappa}{3}}$ and $g(\varkappa,\varpi(\varkappa)) = \frac{3\sqrt{\pi}}{128\sqrt{2}}\frac{(\varkappa+3)}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\sin(2\|\varpi\|_\infty)$. Let $\vartheta\in\digamma$, and $\varpi, w\in\mathbb{R}$, we have

$$\begin{aligned}
&\bigl|g(\vartheta,\varpi(\vartheta)) - g(\vartheta,w(\vartheta))\bigr| \\
&= \frac{3\sqrt{\pi}}{128\sqrt{2}}\frac{(\vartheta+3)}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\bigl|\sin 2\|\varpi\|_\infty - \sin 2\|w\|_\infty\bigr| \\
&= \frac{3\sqrt{\pi}}{128\sqrt{2}}\frac{(\vartheta+3)}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\bigl|2\sin\bigl(\|\varpi\|_\infty - \|w\|_\infty\bigr)\cos\bigl(\|\varpi\|_\infty + \|w\|_\infty\bigr)\bigr| \\
&\le \frac{3\sqrt{\pi}}{128\sqrt{2}}\frac{(\vartheta+3)}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\bigl|2\|\varpi - w\|_\infty\cos\bigl(\|\varpi\|_\infty + \|w\|_\infty\bigr)\bigr| \\
&\le \frac{3\sqrt{\pi}}{128\sqrt{2}}\frac{(\vartheta+3)}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\bigl|2\|\varpi - w\|_\infty\cos\bigl(\|\varpi - w\|_\infty\bigr)\bigr| \\
&\le \frac{3\sqrt{\pi}}{8\sqrt{2}}\frac{1}{\sqrt{(e^{\frac{1}{3}}-1)^3}}\frac{\sqrt{\|(\varpi-w)^2\|_\infty\cos^2\|(\varpi-w)^2\|_\infty}}{2} \\
&\le \frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{5}{2})}{(e^{\frac{1}{3}}-1)^{\frac{3}{2}}}\frac{\sqrt{\|(\varpi-w)^2\|_\infty\cos^2\|(\varpi-w)^2\|_\infty}}{2} \\
&\le \frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{5}{2})}{(e^{\frac{1}{3}}-1)^{\frac{3}{2}}}\frac{\sqrt{\|(\varpi-w)^2\|_\infty\cos^2\|(\varpi-w)^2\|_\infty}}{2} \\
&= \frac{1}{2\sqrt{2}}\frac{\Gamma(\iota+1)}{(\kappa(1)-\kappa(0))^\iota}\sqrt{\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(\varpi-w)^2\bigr\|_\infty\bigr)\bigr)},
\end{aligned}$$

where $\phi(t) = t$ and $\lambda(t) = \frac{\cos^2 t}{4}$ for $t\in\digamma$.

Hence, all assumptions of Corollary 3.3 hold. Therefore, (8) has a solution on $\digamma$.

Example 4.2 Consider the κ-Ca fractional integral BVP

$$\begin{cases} {}^{c}D_{0^+}^{\frac{3}{2},e^{\varkappa}}\varpi(\varkappa) = g(\varkappa,\varpi(\varkappa)), & \varkappa\in(0,1), \|\varpi\|_\infty < \frac{\pi}{4}, \\ \varpi(0)+\varpi'(0)=0, \qquad \varpi(1)+\varpi'(1)=0, \end{cases} \tag{9}$$

where $\iota = \frac{3}{2}$, $\kappa(\varkappa) = e^{\varkappa}$ such that for

$$g(\varkappa,\varpi(\varkappa)) = \frac{1}{8\sqrt{2}}\biggl|\Bigl(\bigl(\sqrt{\pi}\bigl[e^{\varkappa}+e-1\bigr]\bigr)^{-1}2\bigl(e^{\varkappa}\bigr)^{\frac{1}{2}}\bigl(1+e^{\varkappa}\bigr)$$

$$+ 2\big[e^{\varkappa} + e - 1\big]\frac{2}{3}\big(e^{\varkappa}\big)^{\frac{3}{2}}\Big)^{-1}\Big|\sin\big(2\|\varpi\|_\infty\big).$$

Let $\vartheta \in \digamma$, and $\varpi, w \in \mathbb{R}$, we have

$$\begin{aligned}
&\big|g\big(\vartheta, \varpi(\vartheta)\big) - g\big(\vartheta, w(\vartheta)\big)\big| \\
&\le \frac{1}{8\sqrt{2}}\bigg|\Big(\big(\sqrt{\pi}\big[e^{\vartheta} + e - 1\big]\big)^{-1} 2\big(e^{\vartheta}\big)^{\frac{1}{2}}\big(1 + e^{\vartheta}\big) \\
&\quad + 2\big(e^{\vartheta} + e - 1\big)\frac{2}{3}\big(e^{\vartheta}\big)^{\frac{3}{2}}\Big)^{-1}\bigg|\big|\sin 2\|\varpi\|_\infty - \sin 2\|w\|_\infty\big| \\
&= \frac{1}{8\sqrt{2}}\bigg|\Big(\big(\sqrt{\pi}\big[e^{\vartheta} + e - 1\big]\big)^{-1} 2\big(e^{\vartheta}\big)^{\frac{1}{2}}\big(1 + e^{\vartheta}\big) \\
&\quad + 2\big(e^{\vartheta} + e - 1\big)\frac{2}{3}\big(e^{\vartheta}\big)^{\frac{3}{2}}\Big)^{-1}\bigg|\big|2\sin\big(\|\varpi\|_\infty - \|w\|_\infty\big)\cos\big(\|\varpi\|_\infty + \|w\|_\infty\big)\big| \\
&\le \frac{1}{8\sqrt{2}}\bigg|\Big(\big(\sqrt{\pi}\big[e^{\vartheta} + e - 1\big]\big)^{-1} 2\big(e^{\vartheta}\big)^{\frac{1}{2}}\big(1 + e^{\vartheta}\big) \\
&\quad + 2\big(e^{\vartheta} + e - 1\big)\frac{2}{3}\big(e^{\vartheta}\big)^{\frac{3}{2}}\Big)^{-1}\bigg|\big|2\|\varpi - w\|_\infty\cos\big(\|\varpi\|_\infty + \|w\|_\infty\big)\big| \\
&\le \frac{1}{8\sqrt{2}}\bigg|\Big(\big(\sqrt{\pi}e^{\vartheta} + e - 1\big]\big)^{-1} 2\big(e^{\vartheta}\big)^{\frac{1}{2}}\big(1 + e^{\vartheta}\big) \\
&\quad + 2\big(e^{\vartheta} + e - 1\big)\frac{2}{3}\big(e^{\vartheta}\big)^{\frac{3}{2}}\Big)^{-1}\bigg|\frac{\sqrt{\|(\varpi - w)^2\|_\infty\cos^2\|(\varpi - w)^2\|_\infty}}{2} \\
&= \frac{1}{2\sqrt{2}}\bigg(\big(\Gamma(\iota - 1)\big[\kappa(1) - \kappa(0) + \kappa'(1) - \kappa'(0)\big]\big)^{-1}\frac{\kappa'(0)}{\iota - 1} \\
&\quad \times \big(\kappa(1) - \kappa(0)\big)^{\iota - 1}\bigg(1 + \frac{1}{\iota}\big(\kappa(1) - \kappa(0)\big)\bigg) + \frac{1}{\Gamma(\iota + 1)}\big(\kappa(1) - \kappa(0)\big)^{\iota}\bigg)^{-1} \\
&\quad \times \sqrt{\phi\big(\big\|(\varpi - w)^2\big\|_\infty\big)\lambda\big(\phi\big(\big\|(\varpi - w)^2\big\|_\infty\big)\big)}.
\end{aligned}$$

Hence, assumptions of Corollary 3.4 hold. So (9) has a solution on $\digamma$.

Remark 4.3 One can easily see that Eqs. (8) and (9) considered in the above examples cannot be addressed via methods in the current literature. This in a certain sense confirms the superiority of the results of this paper over previous approaches.

5 Conclusion

In recent years and with the explosive growth of studies of derivatives of fractional order, there have appeared tremendous numbers of papers that reported their results by using the classical FDs and FP theorems. Meanwhile, interested researchers have raised the question of the possibility of introducing a different approach that covers all classical cases.

In this paper, we provided an affirmative answer to this inquiry by investigating the notion of existence of solutions for BVPs defined within κ-generalized FD and with the help of the FP technique based on α-ϕ-GC type mappings. The results reported in this paper generalize existing results in the literature. Two examples are presented as particular cases for our proposed theorems. It is proved that the results obtained are consistent with our theoretical findings.

We believe that the investigation of this problem in terms of a general approach will provide an effective platform for the study of BVPs via generalized FOs.

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Author details
[1]Department of Mathematics, Faculty of Basic Science, University of Bonab, Bonab, Iran. [2]Department of Mathematics, Hodiedah University, Al-Hodeidah, Yemen. [3]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia.

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