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Fractional hybrid inclusion version of the Sturm–Liouville equation

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Abstract

The Sturm–Liouville equation is one of classical famous differential equations which has been studied from different of views in the literature. In this work, we are going to review its fractional hybrid inclusion version. In this way, we investigate two fractional hybrid Sturm–Liouville differential inclusions with multipoint and integral hybrid boundary conditions. Also, we provide two examples to illustrate our main results.

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Keywords: Fractional hybrid equations; Inclusion problem; Integral hybrid boundary condition; The Caputo derivative; The Sturm–Liouville equation

1 Introduction

Some differential equations such as that of Sturm–Liouville have established important relations between physics, mathematics, and other fields of engineering (see [1, 2]). During the last decades, many researchers have been studying some well-known problems involving differential equations such as Sturm–Lioville boundary value problems from different views (see, for example, [3–16]). It is important that researchers try to investigate distinct versions of famous and applicable differential equations (see, for example, [17–20]). On the other hand, some interesting integro-differential equations have been investigated by researchers. Among these interesting ones are hybrid differential equations (see, for example, [21–35]).

In 2010, Dhage and Lakshmikantham introduced hybrid differential equations [36]. In 2011, Zhao et al. extended Dhage's work to fractional order and investigated the hybrid fractional differential equations [25]. In 2012, Sun et al. studied a fractional hybrid two point boundary value problem [23]. In 2016, Baleanu et al. reviewed some existence results for the Caputo fractional hybrid inclusion problem

$$ {}^{c}D^{\alpha}\left(\frac{z(t)}{h(t,z(t),I^{\alpha_1}z(t),\dots,I^{\alpha_n}z(t))}\right)\in\mathcal{H}\big(t,z(t),I^{\beta_1}z(t),\dots,I^{\beta_k}z(t)\big)\quad\big(t\in[0,1]\big) $$

with boundary value conditions $z(0)=z_0^*$ and $z(1)=z_1^*$, where $p\in(1,2]$, ${}^{c}D^{\alpha}$ and I^{γ} denote the Caputo derivative operator of the fractional order α and the Riemann–Liouville integral operator of the fractional order $\gamma\in\{\alpha_i,\beta_j\}\subset(0,\infty)$ for $i=1,\dots,n$ and $j=1,\dots,k$, respectively [37]. In 2019, El-Sayed et al. investigated the fractional version of the Sturm–Liouville differential equation with multipoint boundary condition

$$\begin{cases} {}^cD^\alpha(p(t)u'(t)) + q(t)u(t) = h(t)f(u(t)), \\ u'(t) = 0, \qquad \sum_{i=1}^m \xi_i u(a_i) = \nu \sum_{j=1}^n \eta_j u(b_j), \end{cases}$$

where $\alpha \in (0,1]$, ${}^cD^\alpha$ denotes the Caputo fractional derivative, $p \in C^1(I,\mathbb{R})$, $q(t)$ and $h(t)$ are absolutely continuous functions on $I = [0,T]$ with $T < \infty$ and $p(t) \neq 0$ for all $t \in I$, $f : \mathbb{R} \to \mathbb{R}$ is defined and differentiable on the interval I, $0 \le a_1 < a_2 < \cdots < a_m < c$, $d \le b_1 < b_2 < \cdots < b_n < T$, $c < d$ and $\xi_1,\dots,\xi_m$, $\eta_1,\dots,\eta_n$, and ν are some real constants [6]. Since inclusion problems are really strong versions of the usual differential equations, by using and mixing the main ideas of these works, we are going to investigate the fractional hybrid inclusion version of the Sturm–Lioville equation given by

$$ {}^cD^\alpha\left(p(t)\left(\frac{z(t)}{g(t,z(t))}\right)' - \tilde{p}(t)\tilde{f}\big(z(t)\big)\right) \in \Psi\big(t,z(t)\big) \quad \big(t \in [0,1], 0 < \alpha \le 1\big) \tag{1}$$

with multipoint hybrid boundary conditions

$$\begin{cases} (\frac{z(t)}{g(t,z(t))})'_{t=0} = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(z(t)))_{t=0}, \\ \sum_{i=1}^m \xi_i(\frac{z(a_i)}{g(a_i,z(a_i))}) = \nu \sum_{j=1}^n \eta_j(\frac{z(b_j)}{g(b_j,z(b_j))}), \end{cases} \tag{2}$$

where $\alpha \in (0,1]$, ${}^cD^\alpha$ denotes the Caputo fractional derivative, $\Psi : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with some properties, $p,\tilde{p} \in C^1(I,\mathbb{R})$, $\tilde{p}(t)$ is absolutely continuous function on $[0,1]$, $p(t) \neq 0$ for all $t \in I$, $\inf_{t\in I}|p(t)| = p$, $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is defined and differentiable on the interval $[0,1]$, $0 \le a_1 < a_2 < \cdots < a_m < c$, $d \le b_1 < b_2 < \cdots < b_n < 1$, $c < d$, and $\xi_1,\dots,\xi_m$, $\eta_1,\dots,\eta_n$, and ν are some real constants with $\sum_{i=1}^m \xi_i - \nu\sum_{j=1}^n \eta_j \neq 0$. Moreover, we review the fractional hybrid Sturm–Liouville differential inclusion

$$ {}^cD^\alpha\left(p(t)\left(\frac{z(t) - f(t,z(t))}{g(t,z(t))}\right)'\right) \in \Psi\big(t,z(t)\big) \quad \big(t \in [0,1], 0 < \alpha \le 1\big) \tag{3}$$

with integral hybrid boundary conditions

$$\begin{cases} (\frac{z(t)-f(t,z(t))}{g(t,z(t))})'_{t=0} = 0, \\ \int_a^c(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\varpi(\theta) = \nu\int_d^e(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\upsilon(\theta), \end{cases} \tag{4}$$

where $\alpha \in (0,1]$, ${}^cD^\alpha$ denotes the Caputo fractional derivative, $\Psi : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with some properties, $f \in C([0,1] \times \mathbb{R},\mathbb{R})$, $g \in C([0,1] \times \mathbb{R},\mathbb{R} \setminus \{0\})$, $\nu \in \mathbb{R}$, $\varpi,\upsilon : [0,1] \to \mathbb{R}$ are two increasing functions, the integrals are in the Riemann–Stieltjes sense, and $0 \le a < c \le d < e \le 1$.

2 Preliminaries

We consider the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$ on the space $\mathcal{C}_{\mathbb{R}}([0,1])$ and $\|u\|_{\mathcal{L}^1} = \int_0^1 |u(s)|\,ds$ on $\mathcal{L}^1[0,1]$. The Riemann–Liouville fractional integral of order α for a function f is defined by $I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds$ ($\alpha > 0$) and the Caputo derivative of order α for a function f is defined by ${}^cD^{\alpha}f(t) = I^{n-\alpha}\frac{d^n}{dt^n}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds$, where $n = [\alpha] + 1$ (see [38, 39]).

Suppose that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed space. Denote by $\mathcal{P}(\mathcal{X})$, $\mathcal{P}_{cl}(\mathcal{X})$, $\mathcal{P}_b(\mathcal{X})$, $\mathcal{P}_{cp}(\mathcal{X})$, and $\mathcal{P}_{cv}(\mathcal{X})$ the set of all subsets of $\mathcal{X}$, the set of all closed subsets of $\mathcal{X}$, the set of all bounded subsets of $\mathcal{X}$, the set of all compact subsets of $\mathcal{X}$ and the set of all convex subsets of $\mathcal{X}$, respectively. We say that a set-valued map Ψ has convex values whenever the set $\Psi(z)$ is convex for each element $z \in \mathcal{X}$. A set-valued map Ψ is called upper semicontinuous (u.s.c.) whenever for each $z^* \in \mathcal{X}$ and open set $\hat{\mathcal{V}}$ containing $\Psi(z^*)$ there exists an open neighborhood $\hat{\mathcal{U}}_0$ of z^* such that $\Psi(\hat{\mathcal{U}}_0) \subseteq \hat{\mathcal{V}}$ [40]. An element $z^* \in \mathcal{X}$ is called a fixed point for the multivalued map $\Psi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ whenever $z^* \in \Psi(z^*)$. The set of all fixed points of the multifunction Ψ is denoted by $\mathrm{Fix}(\Psi)$ [40].

Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space. For each $A_1, A_2 \in \mathcal{P}(\mathcal{X})$, the Pompeiu–Hausdorff metric $PH_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$PH_d(A_1, A_2) = \max\Big\{\sup_{a_1\in A_1} d_{\mathcal{X}}(a_1, A_2), \sup_{a_2\in A_2} d_{\mathcal{X}}(A_1, a_2)\Big\},$$

where $d_{\mathcal{X}}(a_1, A_2) = \inf_{a_1\in A_1} d_{\mathcal{X}}(a_1, a_2)$ and $d_{\mathcal{X}}(A_1, a_2) = \inf_{a_2\in A_2} d_{\mathcal{X}}(a_1, a_2)$ [40]. A multivalued function $\Psi : \mathcal{X} \to \mathcal{P}_{cl}(\mathcal{X})$ is said to be Lipschitz with Lipschitz constant $k > 0$ whenever $PH_d(\Psi(z_1), \Psi(z_1)) \le kd_{\mathcal{X}}(z_1, z_2)$ holds for all $z_1, z_2 \in \mathcal{X}$. A Lipschitz map Ψ is called a contraction whenever $0 < k < 1$ [40]. A set-valued operator $\Psi : [0, 1] \to \mathcal{P}_{cl}(\mathcal{R})$ is called measurable whenever the function $t \to d_{\mathcal{X}}(\omega, \Psi(t)) = \inf\{|\omega - y| : y \in \Psi(t)\}$ is measurable for any real constant ω [40, 41]. The graph of a set-valued function $\Psi : \mathcal{X} \to \mathcal{P}_{cl}(\Omega)$ is defined by $\mathrm{Graph}(\Psi) = \{(z, \omega) \in \mathcal{X} \times \Omega : \omega \in \Psi(z)\}$ [40]. We say that the graph of Ψ is closed whenever for each sequence $\{z_n\}$ in $\mathcal{X}$ and $\{\omega_n\}$ in Ω with $z_n \to z_0$, $\omega_n \to \omega_0$ and $\omega_n \in \Psi(z_n)$, we have $\omega_0 \in \Psi(z_0)$ [41].

A multifunction Ψ is said to be a completely continuous operator whenever the set $\Psi(\mathcal{W})$ is relatively compact for all $\mathcal{W} \in \mathcal{P}_b(\mathcal{X})$. If the multifunction $\Psi : \mathcal{X} \to \mathcal{P}_{cl}(\Omega))$ is upper semicontinuous, then $\mathrm{Graph}(\Psi)$ is a subset of the product space $\mathcal{X} \times \Omega$ with the closedness property. Conversely, if the set-valued mapping Ψ is completely continuous and has a closed graph, then Ψ is upper semicontinuous (see [40], Proposition 2.1). A set-valued map $\Psi : [0, 1] \times \mathcal{R} \to \mathcal{P}(\mathbb{R})$ is said to be a Caratheodory multifunction whenever $t \to \Psi(t, z)$ is a measurable mapping for all $z \in \mathbb{R}$ and $z \to \Psi(t, z)$ is an upper semicontinuous mapping for almost all $t \in [0, 1]$ (see [40, 41]). Also, a Caratheodory multifunction $\Psi : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $\mathcal{L}^1$-Caratheodory whenever for each constant $\mu > 0$ there exists function $\phi_{\mu} \in \mathcal{L}^1([0, 1], \mathcal{R})$ such that

$$\big\|\Psi(t, z)\big\| = \sup_{t\in[0,1]}\big\{|s| : s \in \Psi(t, z)\big\} \le \phi_{\mu}(t)$$

for all $|z| \le \mu$ and for almost all $t \in [0, 1]$ (see [40, 41]). The set of selections of a multifunction Ψ at a point $z \in \mathcal{C}_{\mathbb{R}}([0, 1])$ is defined by $(\mathcal{SEL})_{\Psi,z} := \{y \in \mathcal{L}^1([0, 1], \mathcal{R}) : y(t) \in \Psi(t, z)\}$ for almost all $t \in [0, 1]$ (see [40, 41]). Let Ψ be a set-valued map. It is known that $(\mathcal{SEL})_{\Psi,z} \neq \emptyset$ for all $z \in \mathcal{C}_{\mathbb{R}}([0, 1])$ whenever $\dim \mathcal{X} < \infty$ [40]. We need the following results.

Theorem 1 ([42]) *Suppose that $\mathcal{X}$ is a separable Banach space, $\Psi : [0, 1] \times \mathcal{X} \to \mathcal{P}_{cp,cv}(\mathcal{X})$ is an $\mathcal{L}^1$-Carathéodory multifunction and $\hat{\Upsilon} : \mathcal{L}^1([0, 1], \mathcal{X}) \to C([0, 1], \mathcal{X})$ is a linear con-*

tinuous mapping. Then

$$\hat{\Upsilon} \circ (\mathcal{SEL})_\Psi : C\big([0,1],\mathcal{X}\big) \to \mathcal{P}_{cp,cv}\big(C\big([0,1],\mathcal{X}\big)\big)$$

is an operator in the product space $C([0,1],\mathcal{X}) \times C([0,1],\mathcal{X})$ *with the action* $z \to (\hat{\Upsilon} \circ (\mathcal{SEL})_\Psi)(z) = \hat{\Upsilon}((\mathcal{SEL})_{\Psi,z})$ *having the closed graph property.*

Theorem 2 (Theorem 4.8 of [43]) *Let* $V_\zeta(0)$ *and* $\bar{V}_\zeta(0)$ *denote respectively the open and closed balls centered at the origin* 0 *of radius* $\zeta > 0$ *in a Banach algebra* $\mathcal{X}$ *and let* $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_3 : \bar{V}_\zeta(0) \to \mathcal{X}$ *and* $\hat{\mathcal{B}}_2 : \bar{V}_\zeta(0) \to \mathcal{P}_{cp,cv}(\mathcal{X})$ *be three operators such that*

(i) $\hat{\mathcal{B}}_1$ *and* $\hat{\mathcal{B}}_3$ *are single-valued Lipschitz with the Lipschitz constants* ℓ_1^* *and* ℓ_2^*, *respectively*;

(ii) $\hat{\mathcal{B}}_2$ *is u.s.c. and compact*;

(iii) $\ell_1^* \hat{M} + \ell_2^* < \frac{1}{2}$ *where* $\hat{M} = \|\hat{\mathcal{B}}_2(\bar{V}_\zeta(0))\| = \sup\{\|\hat{\mathcal{B}}_2 z\| : z \in \bar{V}_\zeta(0)\}$.

Then either

(a) *the operator inclusion* $z \in \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ *has a solution, or*

(b) *there exists* $z \in \mathcal{X}$ *with* $\|z\| = \zeta$ *such that* $\mu z \in \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ *for some* $\mu > 1$.

Theorem 3 (Theorem 4.13 of [43]) *Let* $\mathcal{X}$ *be a Banach algebra. Let* $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_3 : \mathcal{X} \to \mathcal{X}$ *be two single-valued operators and* $\hat{\mathcal{B}}_2 : \mathcal{X} \to \mathcal{P}_{cp,cv}(\mathcal{X})$ *be a multivalued map such that*

(i) $\hat{\mathcal{B}}_1$ *and* $\hat{\mathcal{B}}_3$ *are single-valued Lipschitz with the Lipschitz constants* ℓ_1^* *and* ℓ_2^*, *respectively*;

(ii) $\hat{\mathcal{B}}_2$ *is u.s.c. and compact*;

(iii) $\ell_1^* \hat{M} + \ell_2^* < \frac{1}{2}$ *where* $\hat{M} = \|\hat{\mathcal{B}}_2(\bar{V}_\zeta(0))\| = \sup\{\|\hat{\mathcal{B}}_2 z\| : z \in \mathcal{X}\}$.

Then either

(a) *the operator inclusion* $z \in \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ *has a solution, or*

(b) *the set* $\sum^* = \{z \in \mathcal{X} : \mu z \in \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z, \mu > 1\}$ *is unbounded.*

3 Main results

Now, we investigate the fractional Sturm–Liouville differential inclusion (1)–(2).

Lemma 4 *Let* $y \in \mathcal{L}^1([0,1],\mathbb{R})$. *A function z is a solution for the fractional hybrid Sturm–Liouville differential equation*

$$^cD^\alpha\left(p(t)\left(\frac{z(t)}{g(t,z(t))}\right)' - \tilde{p}(t)\tilde{f}\big(z(t)\big)\right) = y(t) \tag{5}$$

with multipoint hybrid boundary conditions

$$\begin{cases} (\frac{z(t)}{g(t,z(t))})'_{t=0} = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(z(t)))_{t=0}, \\ \sum_{i=1}^m \xi_i(\frac{z(a_i)}{g(a_i,z(a_i))}) = \nu \sum_{j=1}^n \eta_j(\frac{z(b_j)}{g(b_j,z(b_j))}) \end{cases} \tag{6}$$

if and only if z is a solution for the integral equation

$$z(t) = g\big(t,z(t)\big)\Bigg[\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds + \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds\Bigg] + g\big(t,z(t)\big)\Bigg[\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\Bigg], \quad (7)$$

where $\mathcal{H} = \frac{1}{\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j}$.

Proof First assume that z is a solution for the hybrid fractional equation (5). Note that equation (5) can be written as

$$I^{1-\alpha}\left(\frac{d}{dt}\left[p(t)\left(\frac{z(t)}{g(t,z(t))}\right)' - \tilde{p}(t)\tilde{f}\big(z(t)\big)\right]\right) = y(t).$$

Then, $I^1(\frac{d}{dt}[p(t)(\frac{z(t)}{g(t,z(t))})' - \tilde{p}(t)\tilde{f}(z(t)]) = I^\alpha y(t)$ and so

$$p(t)\left(\frac{z(t)}{g(t,z(t))}\right)' - \tilde{p}(t)\tilde{f}\big(z(t)\big) - p(0)\left(\frac{z(t)}{g(t,z(t))}\right)'_{t=0} + \tilde{p}(0)\tilde{f}\big(z(0)\big) = I^\alpha y(t).$$

Since $(\frac{z(t)}{g(t,z(t))})'_{t=0} = (\frac{\tilde{p}(t)\tilde{f}(z(t))}{p(t)})_{t=0}$, one has $p(t)(\frac{z(t)}{g(t,z(t))})' = \tilde{p}(t)\tilde{f}(z(t)) + I^\alpha y(t)$, and so

$$\left(\frac{z(t)}{g(t,z(t))}\right)' = \frac{\tilde{p}(t)}{p(t)}\tilde{f}\big(z(t)\big) + \frac{1}{p(t)}I^\alpha y(t). \quad (8)$$

By integrating from 0 to t, we get

$$\frac{z(t)}{g(t,z(t))} - \ell = \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds + \int_0^t\frac{1}{p(s)}I^\alpha y(s)\,ds, \quad (9)$$

where $\ell = \frac{z(0)}{g(0,z(0))}$. Now, we can write

$$\sum_{i=1}^{m}\xi_i\left(\frac{z(a_i)}{g(a_i,z(a_i))}\right) - \sum_{i=1}^{m}\xi_i\ell = \sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds + \sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^\alpha y(s)\,ds \quad (10)$$

and

$$\nu\sum_{j=1}^{n}\eta_j\left(\frac{z(b_j)}{g(t,z(b_j))}\right) - \nu\sum_{j=1}^{n}\eta_j\ell = \nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds + \nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^\alpha y(s)\,ds. \quad (11)$$

Now by subtracting (10) from (11) and utilizing $\sum_{i=1}^{m}\xi_i(\frac{z(a_i)}{g(t,z(a_i))}) = \nu\sum_{j=1}^{n}\eta_j(\frac{z(b_j)}{g(t,z(b_j))})$, we conclude that

$$\begin{aligned}\ell &= \mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds\\ &\quad + \mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^\alpha y(s)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^\alpha y(s)\,ds,\end{aligned}$$

where $\mathcal{H} = \frac{1}{\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j}$. Now by substituting the value of ℓ in (9), we get

$$\begin{aligned}z(t) = g\big(t,z(t)\big)\bigg[&\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds\\ &+ \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds + \mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^\alpha y(s)\,ds\\ &- \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^\alpha y(s)\,ds + \int_0^t\frac{1}{p(s)}I^\alpha y(s)\,ds\bigg].\end{aligned}$$

Conversely, to complete the equivalence between integral equation (7) and problem (5)–(6), by using (8), we obtain

$$^{c}D^\alpha\left(p(t)\left(\frac{z(t)}{g(t,z(t))}\right)' - \tilde{p}(t)\tilde{f}\big(z(t)\big)\right) = {}^{c}D^\alpha I^\alpha y(t) = y(t)$$

and $(\frac{z(t)}{g(t,z(t))})'_{t=0} = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(z(t)))_{t=0}$. Also by using simple computations and (7), we obtain $\sum_{i=1}^{m}\xi_i(\frac{z(a_i)}{g(a_i,z(a_i))}) = \nu\sum_{j=1}^{n}\eta_j(\frac{z(b_j)}{g(b_j,z(b_j))})$. This completes the proof. □

Definition 5 We say that an absolutely continuous function $z:[0,1]\to\mathbb{R}$ is a solution for the fractional hybrid Sturm–Liouville differential inclusion (1)–(2) whenever there is an integrable function $y\in\mathcal{L}^1([0,1],\mathbb{R})$ with $y(t)\in\Psi(t,z(t))$ for almost all $t\in[0,1]$,

$$\left(\frac{z(t)}{g(t,z(t))}\right)'_{t=0} = \left(\frac{\tilde{p}(t)}{p(t)}\tilde{f}\big(z(t)\big)\right)_{t=0}, \sum_{i=1}^{m}\xi_i\left(\frac{z(a_i)}{g(a_i,z(a_i))}\right) = \nu\sum_{j=1}^{n}\eta_j\left(\frac{z(b_j)}{g(b_j,z(b_j))}\right),$$

and

$$\begin{aligned}z(t) = g\big(t,z(t)\big)\bigg[&\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds\\ &+ \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds\bigg] + g\big(t,z(t)\big)\bigg[\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\\ &- \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\bigg]\end{aligned}$$

for all $t\in[0,1]$.

Theorem 6 *Assume that*

($\mathcal{Q}_1$) *there exists a bounded mapping* $\chi : [0,1] \to \mathbb{R}^+$ *such that* $|g(t,x) - g(t,y)| \leq \chi(t)|x - y|$ *for all* $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$;

($\mathcal{Q}_2$) *the function* $\tilde{f} : \mathbb{R} \to \mathbb{R}$ *is differentiable on* $[0,1]$, *and* $\frac{\partial \tilde{f}}{\partial u}$ *is bounded on* $[0,1]$ *with* $\frac{\partial \tilde{f}}{\partial u} \leq \tilde{\mathcal{K}}$;

($\mathcal{Q}_3$) *the set-valued map* $\Psi : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ *has* $\mathcal{L}^1$*-Caratheodory property*;

($\mathcal{Q}_4$) *there exists a positive mapping* $\sigma \in \mathcal{C}([0,1], \mathbb{R}^+)$ *such that*

$$\left\| \Psi(t,x) \right\| = \sup\left\{ |y| : y \in \Psi(t,x) \right\} \leq \sigma(t)$$

for all $x \in \mathbb{R}$ *and almost all* $t \in [0,1]$;

($\mathcal{Q}_5$) *there exists* $\zeta > 0$ *such that* $\zeta > \bar{\Delta}(\chi^* \zeta + g_0)(\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0) + \frac{\|\sigma\|}{\Gamma(\alpha+2)})$ *and*

$$\left(\frac{\chi^* \|\sigma\|}{\Gamma(\alpha + 2)} + \|\tilde{p}\| \left(2\widetilde{\mathcal{K}}\chi^* \zeta + \chi^* \tilde{f}_0 + \widetilde{\mathcal{K}} g_0 \right) \right) \bar{\Delta} < \frac{1}{2},$$

where $\bar{\Delta} = \frac{1}{p}(|\mathcal{H}|(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j|) + 1)$, $\chi^* = \sup_{t\in[0,1]} \chi(t)$, $g_0 = \sup_{t\in[0,1]} g(t,0)$, *and* $\tilde{f}_0 = \tilde{f}(0)$. *Then the fractional hybrid Sturm–Liouville inclusion problem* (1)–(2) *has at least one solution.*

Proof Let $\mathcal{X} = \mathcal{C}_{\mathbb{R}}([0,1])$, and let $V_\zeta(0) := \{z \in \mathcal{X} : \|z\| < \zeta\}$ and $\bar{V}_\zeta(0) := \{z \in \mathcal{X} : \|z\| \leq \zeta\}$ be the open and closed balls centered at the origin 0 of radius ζ, respectively. Consider the operator $\hat{\mathcal{K}} : \bar{V}_\zeta(0) \to \mathcal{P}(\mathcal{X})$ defined by

$$\hat{\mathcal{K}}(z) = \left\{ \omega \in \mathcal{X} : \text{there exists } y \in (\mathcal{SEL})_{\Psi,z} \text{ such that } \omega(t) = \varsigma(t) \text{ for all } t \in [0,1] \right\},$$

where

$$\begin{aligned} \varsigma(t) = g\big(t,z(t)\big) \Bigg[&\mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds \\ &+ \int_0^t \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds \Bigg] + g\big(t,z(t)\big) \Bigg[\mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds \\ &- \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds \Bigg]. \end{aligned}$$

It is easy to check that fixed point of the set-valued map $\hat{\mathcal{K}}$ is solution for the fractional hybrid Sturm–Liouville inclusion problem (1)–(2). Define the maps $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_3 : \bar{V}_\zeta(0) \to \mathcal{X}$ and the set-valued-map $\hat{\mathcal{B}}_2 : \bar{V}_\zeta(0) \to \mathcal{P}(\mathcal{X})$ by $(\hat{\mathcal{B}}_1 z)(t) = g(t,z(t))$, $(\hat{\mathcal{B}}_3 z)(t) = g(t,z(t))(\vartheta z)(t)$,

$$(\hat{\mathcal{B}}_2 z)(t) = \left\{ \varphi \in \mathcal{X} : \text{there exists } y \in (\mathcal{SEL})_{\Psi,z} \text{ such that } \varphi(t) = \varrho(t) \text{ for all } t \in [0,1] \right\},$$

where

$$\varrho(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds$$

and

$$(\vartheta z)(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds + \int_0^t \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(z(s)\big)\, ds.$$

Note that $\hat{\mathcal{K}}(z) = \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ for all $z \in \bar{V}_\zeta(0)$. We show that the operators $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_2$, and $\hat{\mathcal{B}}_3$ satisfy the conditions of Theorem 2. First, we prove that the set-valued map $\hat{\mathcal{B}}_2$ is convex-valued. Let $\varphi_1, \varphi_2 \in \hat{\mathcal{B}}_2 z$. Choose $y_1, y_2 \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\varphi_k(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y_k(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y_k(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y_k(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds$$

for $k = 1, 2$. Let $\lambda \in (0,1)$. Then, we have

$$\begin{aligned} \lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) &= \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda) y_2(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds \\ &\quad - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda) y_2(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds \\ &\quad + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda) y_2(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds \end{aligned}$$

for almost all $t \in [0,1]$. Since Ψ is convex-valued, $(\mathcal{SEL})_{\Psi,z}$ is convex, that is,

$$\lambda y_1(\tau) + (1-\lambda) y_2(\tau)) \in (\mathcal{SEL})_{\Psi,z}.$$

Hence, $\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) \in \hat{\mathcal{B}}_2 z$, and so $\hat{\mathcal{B}}_2 z$ is a convex set for all $z \in \mathcal{X}$. Now, we show that the operator $\hat{\mathcal{B}}_2$ is completely continuous and upper semicontinuous on $\mathcal{X}$. To establish the complete continuity of the operator $\hat{\mathcal{B}}_2$, we should prove that $\hat{\mathcal{B}}_2(\mathcal{X})$ is an equicontinuous and uniformly bounded set. To do this, first we prove that $\hat{\mathcal{B}}_2$ maps all bounded sets into bounded subsets of $\mathcal{X}$. Let $\hat{\mathcal{V}}$ be a bounded subset of $\bar{V}_\zeta(0)$. Choose $0 < \kappa^* \leq \zeta$ such that $\|z\| \leq \kappa^*$ for all $z \in \hat{\mathcal{V}}$. For each $z \in \hat{\mathcal{V}}$ and $\varphi \in \hat{\mathcal{B}}_2(\hat{\mathcal{V}})$, there exists $y \in (\mathcal{SEL})_{\Psi,z}$

such that

$$\varphi(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds.$$

Hence,

$$\begin{aligned} |\varphi(t)| &\le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \\ &\quad + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \\ &\le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} \sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \\ &\quad + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} \sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} \sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds. \end{aligned}$$

Since $|\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \frac{\|\sigma\|}{p\Gamma(\alpha+2)}$,

$$|\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} \sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \le |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \frac{\|\sigma\|}{p\Gamma(\alpha+2)},$$

and $\int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \le \frac{\|\sigma\|}{p\Gamma(\alpha+2)}$, we get

$$\begin{aligned} |\varphi(t)| &\le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \frac{\|\sigma\|}{p\Gamma(\alpha+2)} + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \frac{\|\sigma\|}{p\Gamma(\alpha+2)} + \frac{\|\sigma\|}{p\Gamma(\alpha+2)} \\ &= \frac{\|\sigma\|}{p\Gamma(\alpha+2)} \left(|\mathcal{H}| \left(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j| \right) + 1 \right) = \frac{\|\sigma\|}{\Gamma(\alpha+2)} \bar{\Delta}, \end{aligned}$$

and so $\|\varphi\| \le \frac{\|\sigma\|}{\Gamma(\alpha+2)} \bar{\Delta}$. Thus, $\hat{\mathcal{B}}_2(\hat{\mathcal{V}})$ is a uniformly bounded. Now, we prove that the operator $\hat{\mathcal{B}}_2$ maps bounded sets onto equicontinuous sets. Let $z \in \hat{\mathcal{V}}$ and $\varphi \in \hat{\mathcal{B}}_2 z$. Choose $y \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\varphi(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds$$

for all $t \in [0,1]$. For each t_1, t_2 with $t_1 < t_2$, we have

$$\begin{aligned}\left|\varphi(t_2)-\varphi(t_1)\right| &= \left|\int_0^{t_2}\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds - \int_0^{t_1}\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right| \\ &= \left|\int_{t_1}^{t_2}\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right| \le \left|\int_{t_1}^{t_2}\int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right|.\end{aligned}$$

Since the right-hand side of the above inequality tends to zero as $t_1 \to t_2$, by using the Arzela–Ascoli theorem, the operator $\hat{\mathcal{B}}_2 : \bar{V}_\zeta(0) \to \mathcal{P}(\mathcal{X})$ is completely continuous. Here, we show that $\hat{\mathcal{B}}_2$ has a closed graph, and this implies that $\hat{\mathcal{B}}_2$ is upper semicontinuous. For this aim, suppose that $z_n \in \hat{\mathcal{V}}$ and $\varphi_n \in \hat{\mathcal{B}}_2 z_n$ with $z_n \to z^*$ and $\varphi_n \to \varphi^*$. We show that $\varphi^* \in \hat{\mathcal{B}}_2 z^*$. For each $\varphi_n \in \hat{\mathcal{B}}_2 z_n$, choose $y_n \in (\mathcal{SEL})_{\Psi,z_n}$ such that

$$\begin{aligned}\varphi_n(t) &= \mathcal{H}\nu\sum_{j=1}^n \eta_j \int_0^{b_j}\int_0^s \frac{(s-\tau)^{\alpha-1}y_n(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds - \mathcal{H}\sum_{i=1}^m \xi_i \int_0^{a_i}\int_0^s \frac{(s-\tau)^{\alpha-1}y_n(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &\quad + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y_n(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds.\end{aligned}$$

It is sufficient to be prove that there exists a function $y^* \in (\mathcal{SEL})_{\Psi,z^*}$ such that

$$\begin{aligned}\varphi^*(t) &= \mathcal{H}\nu\sum_{j=1}^n \eta_j \int_0^{b_j}\int_0^s \frac{(s-\tau)^{\alpha-1}y^*(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds - \mathcal{H}\sum_{i=1}^m \xi_i \int_0^{a_i}\int_0^s \frac{(s-\tau)^{\alpha-1}y^*(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &\quad + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y^*(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\end{aligned}$$

for all $t \in [0,1]$. Consider the continuous linear operator $\hat{\Upsilon} : \mathcal{L}^1([0,1],\mathbb{R}) \to \mathcal{X}$ defined by

$$\begin{aligned}\hat{\Upsilon}(y)(t) = z(t) &= \mathcal{H}\nu\sum_{j=1}^n \eta_j \int_0^{b_j}\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &\quad - \mathcal{H}\sum_{i=1}^m \xi_i \int_0^{a_i}\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &\quad + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\end{aligned}$$

for all $t \in [0,1]$. Then, we get

$$\begin{aligned}\left\|\varphi_n(t)-\varphi^*(t)\right\| &= \left\|\mathcal{H}\nu\sum_{j=1}^n \eta_j \int_0^{b_j}\int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau)-y^*(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right. \\ &\quad - \mathcal{H}\sum_{i=1}^m \xi_i \int_0^{a_i}\int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau)-y^*(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &\quad \left. + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau)-y^*(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right\| \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

Hence by using Theorem 1, we conclude that the operator $\hat{\Upsilon} \circ (\mathcal{SEL})_\Psi$ has a closed graph. In fact, since $\varphi_n \in \hat{\Upsilon}((\mathcal{SEL})_{\Psi,z_n})$ and $z_n \to z^*$, there exists $y^* \in (\mathcal{SEL})_{\Psi,z^*}$ such that

$$\varphi^*(t) = \mathcal{H}\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds - \mathcal{H} \sum_{i=1}^{m} \xi_i \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds$$

for all $t \in [0,1]$. Hence, $\varphi^* \in \hat{\mathcal{B}}_2 z^*$. This means that the graph of $\hat{\mathcal{B}}_2$ is closed. Thus, $\hat{\mathcal{B}}_2$ is upper semicontinuous. Furthermore, by using the assumptions, we know that the operator Ψ has compact values. Hence, $\hat{\mathcal{B}}_2$ is a compact and upper semicontinuous operator. Now, we show that $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{B}}_3$ are Lipschitz. Let $z_1, z_2 \in \bar{V}_\zeta(0)$. By using $(\mathcal{Q}_1)$, we get

$$\left|(\hat{\mathcal{B}}_1 z_1)(t) - (\hat{\mathcal{B}}_1 z_2)(t)\right| = \left|g(t,z_1) - g(t,z_2)\right| \le \chi(t)\left|z_1(t) - z_2(t)\right|,$$

and so $\|\hat{\mathcal{B}}_1 z_1 - \hat{\mathcal{B}}_1 z_2\| \le \chi^* \|z_1 - z_2\|$. Hence, $\hat{\mathcal{B}}_1$ is Lipschitz with Lipschitz constant $\ell_1^* = \chi^*$. Let $z \in \bar{V}_\zeta(0)$. By using $(\mathcal{Q}_2)$, we get $|\tilde{f}(z(s))| \le \widetilde{\mathcal{K}}\|z\| + \tilde{f}_0 \le \widetilde{\mathcal{K}}\zeta + \tilde{f}_0$. Similarly, one can show that $|g(s,z(s))| \le \chi^*\zeta + g_0$. Thus, we obtain

$$\begin{aligned}
\left|(\vartheta z)(t)\right| &\le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z(s)\big)\right| ds + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z(s)\big)\right| ds \\
&\quad + \int_0^t \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z(s)\big)\right| ds \\
&\le |\mathcal{H}||\nu| \left(\frac{\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)}{p}\right) \sum_{j=1}^{n} |\eta_j| + |\mathcal{H}| \left(\frac{\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)}{p}\right) \sum_{i=1}^{m} |\xi_i| \\
&\quad + \frac{\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)}{p} \\
&= \left(\frac{\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)}{p}\right)\left(|\mathcal{H}|\left(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j|\right) + 1\right) \\
&= \|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)\bar{\Delta}.
\end{aligned}$$

Let $z_1, z_2 \in \bar{V}_\zeta(0)$. Similarly, we get

$$\begin{aligned}
\left|(\vartheta z_1)(t) - (\vartheta z_2)(t)\right| &\le |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z_1(s)\big) - \tilde{f}\big(z_2(s)\big)\right| ds \\
&\quad + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z_1(s)\big) - \tilde{f}\big(z_2(s)\big)\right| ds \\
&\quad + \int_0^t \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\big(z_1(s)\big) - \tilde{f}\big(z_2(s)\big)\right| ds \\
&\le \left(\frac{\|\tilde{p}\|\widetilde{\mathcal{K}}}{p}\right)\left(|\mathcal{H}|\left(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j|\right) + 1\right)\|z_1 - z_2\| \\
&= \|\tilde{p}\|\widetilde{\mathcal{K}}\bar{\Delta}\|z_1 - z_2\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\big|(\hat{\mathcal{B}}_3 z_1)(t) - (\hat{\mathcal{B}}_3 z_2)(t)\big| \\
&\quad = \big|g\big(t,z_1(t)\big)(\vartheta z_1)(t) - g\big(t,z_2(t)\big)(\vartheta z_2)(t)\big| \\
&\quad = \big|g\big(t,z_1(t)\big)(\vartheta z_1)(t) - g\big(t,z_1(t)\big)(\vartheta z_2)(t) + g\big(t,z_1(t)\big)(\vartheta z_2)(t) - g\big(t,z_2(t)\big)(\vartheta z_2)(t)\big| \\
&\quad = \big|g\big(t,z_1(t)\big)\big((\vartheta z_1)(t) - (\vartheta z_2)(t)\big) + (\vartheta z_2)(t)\big(g\big(t,z_1(t)\big) - g\big(t,z_2(t)\big)\big)\big| \\
&\quad \leq \big|g\big(t,z_1(t)\big)\big|\big|(\vartheta z_1)(t) - (\vartheta z_2)(t)\big| + \big|(\vartheta z_2)(t)\big|\big|g\big(t,z_1(t)\big) - g\big(t,z_2(t)\big)\big|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big|(\hat{\mathcal{B}}_3 z_1)(t) - (\hat{\mathcal{B}}_3 z_2)(t)\big| \\
&\quad \leq \big|g\big(t,z_1(t)\big)\big|\big|(\vartheta z_1)(t) - (\vartheta z_2)(t)\big| + \big|(\vartheta z_2)(t)\big|\big|g\big(t,z_1(t)\big) - g\big(t,z_2(t)\big)\big| \\
&\quad \leq \|\tilde{p}\|\widetilde{\mathcal{K}}\big(\chi^*\zeta + g_0\big)\bar{\Delta}\|z_1 - z_2\| + \|\tilde{p}\|\chi^*(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0)\bar{\Delta}\|z_1 - z_2\| \\
&\quad = \|\tilde{p}\|\big(2\widetilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \widetilde{\mathcal{K}}g_0\big)\bar{\Delta}\|z_1 - z_2\|,
\end{aligned}$$

and so $\|\hat{\mathcal{B}}_3 z_1 - \hat{\mathcal{B}}_3 z_2\| \leq \|\tilde{p}\|(2\widetilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \widetilde{\mathcal{K}}g_0)\bar{\Delta}\|z_1 - z_2\|$. Hence, $\hat{\mathcal{B}}_3$ is Lipschitz with Lipschitz constant $\ell_2^* = \|\tilde{p}\|(2\widetilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \widetilde{\mathcal{K}}g_0)\bar{\Delta}$. Note that

$$\hat{M} = \big\|\hat{\mathcal{B}}_2\big(\bar{V}_\zeta(0)\big)\big\| = \sup\big\{|\hat{\mathcal{B}}_2 z| : z \in \bar{V}_\zeta(0)\big\} = \frac{\|\sigma\|}{\Gamma(\alpha+2)}\bar{\Delta}$$

and

$$\begin{aligned}
\ell_1^*\hat{M} + \ell_2^* &= \chi^*\frac{\|\sigma\|}{\Gamma(\alpha+2)}\bar{\Delta} + \|\tilde{p}\|\big(2\widetilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \widetilde{\mathcal{K}}g_0\big)\bar{\Delta} \\
&= \left(\frac{\chi^*\|\sigma\|}{\Gamma(\alpha+2)} + \|\tilde{p}\|\big(2\widetilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \widetilde{\mathcal{K}}g_0\big)\right)\bar{\Delta} < \frac{1}{2}.
\end{aligned}$$

Thus, the assumptions of Theorem 2 hold for $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_2$, and $\hat{\mathcal{B}}_3$. Hence, one of the conditions (a) or (b) holds. We show that condition (b) is impossible. Let $z \in \mathcal{X}$ with $\|z\| = \zeta$ be such that $\mu z \in \hat{\mathcal{B}}_1 z\hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ for some $\mu > 1$. Choose $y \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\begin{aligned}
z(t) = \frac{1}{\mu}g\big(t,z(t)\big)\Bigg[&\mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds - \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds \\
&+ \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(z(s)\big)\,ds + \mathcal{H}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\
&- \mathcal{H}\sum_{i=1}^{m}\xi_i\int_0^{a_i}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\Bigg].
\end{aligned}$$

Hence,

$$\begin{aligned} |z(t)| &\leq |g(t,z(t))| \\ &\quad \times \Bigg[|\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{|\tilde{p}(s)|}{|p(s)|} |\tilde{f}(z(s))| \, ds + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{|\tilde{p}(s)|}{|p(s)|} |\tilde{f}(z(s))| \, ds \\ &\quad + \int_0^t \frac{|\tilde{p}(s)|}{|p(s)|} |\tilde{f}(z(s))| \, ds + |\mathcal{H}||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{|p(s)|\Gamma(\alpha)} \, d\tau \, ds \\ &\quad + |\mathcal{H}| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{|p(s)|\Gamma(\alpha)} \, d\tau \, ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{p(s)\Gamma(\alpha)} \, d\tau \, ds \Bigg] \\ &\leq \bar{\Delta}(\chi^* \|z\| + g_0) \bigg[\|\tilde{p}\| (\widetilde{\mathcal{K}} \|z\| + \tilde{f}_0) + \frac{\|\sigma\|}{\Gamma(\alpha+2)} \bigg], \end{aligned}$$

and so $\zeta \leq \bar{\Delta}(\chi^* \zeta + g_0)[\|\tilde{p}\|(\widetilde{\mathcal{K}}\zeta + \tilde{f}_0) + \frac{\|\sigma\|}{\Gamma(\alpha+2)}]$, which is a contradiction. Hence, condition (b) is impossible, and so the fractional hybrid Sturm–Liouville inclusion problem (1)–(2) has at least one solution. □

Now, we investigate the fractional hybrid Sturm–Liouville inclusion problem (3)–(4).

Lemma 7 *Let $y \in \mathcal{L}^1([0,1],\mathbb{R})$. A function z is a solution for the fractional hybrid Sturm–Liouville differential equation*

$$ {}^{c}D^{\alpha}\left(p(t)\left(\frac{z(t)-f(t,z(t))}{g(t,z(t))}\right)'\right) = y(t) \quad \big(t \in [0,1], 0 < \alpha \leq 1\big) \tag{12}$$

with integral hybrid boundary conditions

$$\begin{cases} (\frac{z(t)-f(t,z(t))}{g(t,z(t))})'_{t=0} = 0, \\ \int_a^c (\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))}) \, d\varpi(\theta) = \nu \int_d^e (\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))}) \, d\upsilon(\theta), \end{cases} \tag{13}$$

if and only if z is a solution for the integral equation

$$\begin{aligned} z(t) &= g(t,z(t)) \bigg[\nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)} \, d\tau \, ds \, d\upsilon(\theta) \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)} \, d\tau \, ds \, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)} \, d\tau \, ds \bigg] \\ &\quad + f(t,z(t)), \end{aligned} \tag{14}$$

where $R = \frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e))-\upsilon(d))}$ with $\varpi(c) - \varpi(a) - \nu(\upsilon(e)) - \upsilon(d)) \neq 0$.

Proof First, assume that z is a solution for the fractional hybrid equation (12). Note that

$$I^{1-\alpha}\left(\frac{d}{dt}\left[p(t)\left(\frac{z(t)-f(t,z(t))}{g(t,z(t))}\right)'\right]\right) = y(t).$$

Hence, $I^1(\frac{d}{dt}[p(t)(\frac{z(t)-f(t,z(t))}{g(t,z(t))})'])=I^\alpha y(t)$, and so

$$p(t)\left(\frac{z(t)-f(t,z(t))}{g(t,z(t))}\right)'-p(0)\left(\frac{z(t)-f(t,z(t))}{g(t,z(t))}\right)'_{t=0}=I^\alpha y(t).$$

Since $(\frac{z(t)-f(t,z(t))}{g(t,z(t))})'_{t=0}=0$, we get

$$\left(\frac{z(t)-f(t,z(t))}{g(t,z(t))}\right)'=\frac{1}{p(t)}I^\alpha y(t). \tag{15}$$

By integrating from 0 to t, we obtain

$$\frac{z(t)-f(t,z(t))}{g(t,z(t))}-\ell^*=\int_0^t\frac{1}{p(s)}I^\alpha y(s)\,ds, \tag{16}$$

where $\ell^*=\frac{z(0)-f(0,z(0))}{g(0,z(0))}$. Thus,

$$\int_a^c\left(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))}\right)d\varpi(\theta)-\ell^*\int_a^c d\varpi(\theta)=\int_a^c\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\varpi(\theta)$$

and $\nu\int_d^e(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\upsilon(\theta)-\ell^*\nu\int_d^e d\upsilon(\theta)=\nu\int_d^e\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\upsilon(\theta)$. Hence,

$$\int_a^c\left(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))}\right)d\varpi(\theta)-\ell^*\big(\varpi(c)-\varpi(a)\big)=\int_a^c\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\varpi(\theta)$$

and

$$\nu\int_d^e\left(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))}\right)d\upsilon(\theta)-\ell^*\nu\big(\upsilon(e)-\upsilon(d)\big)=\nu\int_d^e\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\upsilon(\theta).$$

Since $\int_a^c(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\varpi(\theta)=\nu\int_d^e(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\upsilon(\theta)$, we have

$$\begin{aligned}\ell^*={}&\frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}\left(\nu\int_d^e\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\upsilon(\theta)\right.\\&\left.-\int_a^c\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\varpi(\theta)\right).\end{aligned}$$

Now by substituting the value of ℓ^* in (16), we obtain

$$\begin{aligned}z(t)={}&g\big(t,z(t)\big)\left[\frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}\left(\nu\int_d^e\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\upsilon(\theta)\right.\right.\\&\left.\left.-\int_a^c\int_0^\theta\frac{1}{p(s)}I^\alpha y(s)\,ds\,d\varpi(\theta)\right)+\int_0^t\frac{1}{p(s)}I^\alpha y(s)\,ds\right]+f\big(t,z(t)\big).\end{aligned}$$

For the converse part, from (15) we get ${}^cD^\alpha(p(t)(\frac{z(t)-f(t,z(t))}{g(t,z(t))})')={}^cD^\alpha I^\alpha y(t)=y(t)$ and $(\frac{z(t)-f(t,z(t))}{g(t,z(t))})'_{t=0}=0$. Also by using some simple computations and (14), we obtain $\int_a^c(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\varpi(\theta)=\nu\int_d^e(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\upsilon(\theta)$. □

Definition 8 We say that an absolutely continuous function $z:[0,1]\to\mathbb{R}$ is a solution for the fractional hybrid Sturm–Liouville inclusion problem (3)–(4) whenever there exists $y\in\mathcal{L}^1([0,1],\mathbb{R})$ such that $y(t)\in\Psi(t,z(t))$ for almost all $t\in[0,1]$, $(\frac{z(t)-f(t,z(t))}{g(t,z(t))})'_{t=0}=0$, $\int_a^c(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\varpi(\theta)=\nu\int_d^e(\frac{z(\theta)-f(\theta,z(\theta))}{g(\theta,z(\theta))})\,d\upsilon(\theta)$, and

$$\begin{aligned} z(t) = g\bigl(t,z(t)\bigr)\biggl[&\nu R\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &-R\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)+\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\biggr]\\ &+f\bigl(t,z(t)\bigr) \end{aligned}$$

for all $t\in[0,1]$.

Theorem 9 *Assume that*

($\mathcal{Q}_1^*$) *there exists a bounded mapping* $\chi_1:[0,1]\to\mathbb{R}^+$ *such that* $|g(t,x)-g(t,y)|\le\chi_1(t)|x-y|$ *for all* $(t,x,y)\in[0,1]\times\mathbb{R}\times\mathbb{R}$;

($\mathcal{Q}_2^*$) *there exists a bounded map* $\chi_2:[0,1]\to\mathbb{R}^+$ *such that* $|f(t,x)-f(t,y)|\le\chi_2(t)|x-y|$ *for all* $(t,x,y)\in[0,1]\times\mathbb{R}\times\mathbb{R}$;

($\mathcal{Q}_3^*$) *the set-valued map* $\Psi:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,cv}(\mathbb{R})$ *has* $\mathcal{L}^1$*-Caratheodory property*;

($\mathcal{Q}_4^*$) *there exists a positive map* $\sigma\in\mathcal{C}([0,1],\mathbb{R}^+)$ *such that*

$$\bigl\|\Psi(t,x)\bigr\|=\sup\bigl\{|y|:y\in\Psi(t,x)\bigr\}\le\sigma(t)$$

for all $x\in\mathbb{R}$ *and almost all* $t\in[0,1]$;

($\mathcal{Q}_5^*$) *the strict inequality* $\chi_1^*\Lambda^*+\chi_2^*<\frac{1}{2}$ *holds, where* $\chi_1^*=\sup_{t\in[0,1]}\chi_1(t)$, $\chi_2^*=\sup_{t\in[0,1]}\chi_2(t)$, *and* $\Lambda^*=\frac{\|\sigma\|}{p\Gamma(\alpha+1)}(|R|(\varpi(c)-\varpi(a)+|\nu|(\upsilon(e)-\upsilon(d)))+1)$.

Then the fractional hybrid Sturm–Liouville inclusion problem (3)–(4) *has a solution*.

Proof Let $\mathcal{X}=\mathcal{C}_{\mathbb{R}}([0,1])$. Consider the operator $\hat{\mathcal{K}}:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ defined by

$$\hat{\mathcal{K}}(z)=\bigl\{\omega\in\mathcal{X}:\text{there exists } y\in(\mathcal{SEL})_{\Psi,z}\text{ such that }\omega(t)=\varsigma(t)\text{ for all }t\in[0,1]\bigr\},$$

where

$$\begin{aligned} \varsigma(t) = g\bigl(t,z(t)\bigr)\biggl[&\nu R\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &-R\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)+\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\biggr]\\ &+f\bigl(t,z(t)\bigr). \end{aligned}$$

Note that each fixed point of the set-valued map $\hat{\mathcal{K}}$ is a solution for the fractional hybrid Sturm–Liouville inclusion problem (3)–(4). Define the single-valued maps $\hat{\mathcal{B}}_1,\hat{\mathcal{B}}_3:\mathcal{X}\to\mathcal{X}$ by $(\hat{\mathcal{B}}_1z)(t)=g(t,z(t))$ and $(\hat{\mathcal{B}}_3z)(t)=f(t,z(t))$, and the set-valued map $\hat{\mathcal{B}}_2:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ by

$$(\hat{\mathcal{B}}_2z)(t)=\bigl\{\varphi\in\mathcal{X}:\text{there exists } y\in(\mathcal{SEL})_{\Psi,z}\text{ such that }\varphi(t)=\varrho(t)\text{ for all }t\in[0,1]\bigr\},$$

where

$$\begin{aligned}\varrho(t) &= \nu R\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad - R\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)\\ &\quad + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds.\end{aligned}$$

Note that $\hat{\mathcal{K}}(z) = \hat{\mathcal{B}}_1 z\hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$. We prove that the operators $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_2$, and $\hat{\mathcal{B}}_3$ satisfy the conditions of Theorem 3. Now, we prove that the set-valued map $\hat{\mathcal{B}}_2$ is convex-valued. Let $z_1, z_2 \in \hat{\mathcal{B}}_2 z$. Choose $y_1, y_2 \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\begin{aligned}z_j(t) &= \nu R\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y_j(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad - R\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y_j(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y_j(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\end{aligned}$$

for $j = 1, 2$. Let $\lambda \in (0, 1)$. Then, we have

$$\begin{aligned}\lambda z_1(t) + (1-\lambda)z_2(t) &= \nu R\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda)y_2(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad - R\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda)y_2(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)\\ &\quad + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}(\lambda y_1(\tau) + (1-\lambda)y_2(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\end{aligned}$$

for almost all $t \in [0, 1]$. Since Ψ is convex-valued, $(\mathcal{SEL})_{\Psi,z}$ is convex. This implies that $\lambda y_1(\tau) + (1-\lambda)y_2(\tau)) \in (\mathcal{SEL})_{\Psi,z}$, and so $\hat{\mathcal{B}}_2 z$ is convex set for all $z \in \mathcal{X}$. Now, we show that the operator $\hat{\mathcal{B}}_2$ is completely continuous and upper semicontinuous on $\mathcal{X}$. To establish the complete continuity of the operator $\hat{\mathcal{B}}_2$, we should prove that $\hat{\mathcal{B}}_2(\mathcal{X})$ is an equicontinuous and uniformly bounded set. To do this, it is sufficient to prove that $\hat{\mathcal{B}}_2$ maps all bounded sets into bounded subsets of $\mathcal{X}$. Assume that $\mathcal{V}$ is a bounded subset of $\mathcal{X}$. Choose $\kappa^* > 0$ such that $\|z\| < \kappa^*$ for all $z \in \mathcal{V}$. For every $z \in \mathcal{V}$ and $\varphi \in \hat{\mathcal{B}}_2(\mathcal{V})$, there exists $y \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\begin{aligned}\varphi(t) &= \nu R\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad - R\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) + \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\varphi(t)\right| &\leq |\nu||R|\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|y(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad+|R|\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|y(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)+\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}|y(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\\ &\leq |\nu||R|\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad+|R|\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)+\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds.\end{aligned}$$

Since $|\nu||R|\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\leq\frac{|R|\|\sigma\||\nu|(\upsilon(e)-\upsilon(d))}{p\Gamma(\alpha+1)}$,

$$|R|\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)\leq\frac{|R|\|\sigma\|(\varpi(c)-\varpi(a))}{p\Gamma(\alpha+1)},$$

and $\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}|\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\leq\frac{\|\sigma\|}{p\Gamma(\alpha+1)}$, we get

$$\begin{aligned}\left|\varphi(t)\right| &\leq \frac{|R|\|\sigma\||\nu|(\upsilon(e)-\upsilon(d))}{p\Gamma(\alpha+1)}+\frac{|R|\|\sigma\|(\varpi(c)-\varpi(a))}{p\Gamma(\alpha+1)}+\frac{\|\sigma\|}{p\Gamma(\alpha+1)}\\ &\leq\frac{\|\sigma\|}{p\Gamma(\alpha+1)}\bigl(|R|\bigl(\varpi(c)-\varpi(a)+|\nu|\bigl(\upsilon(e)-\upsilon(d)\bigr)\bigr)+1\bigr)=\Lambda^*.\end{aligned}$$

Hence, $\|\varphi\|\leq\Lambda^*$. This means $\hat{\mathcal{B}}_2(\mathcal{V})$ is a uniformly bounded set. Here, we show that the operator $\hat{\mathcal{B}}_2$ maps bounded sets onto equicontinuous sets. Let $z\in\mathcal{V}$ and $\varphi\in\hat{\mathcal{B}}_2 z$. Choose $y\in(\mathcal{SEL})_{\Psi,z}$ such that

$$\begin{aligned}\varphi(t)&=\nu R\int_d^e\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta)\\ &\quad-R\int_a^c\int_0^\theta\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)+\int_0^t\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\end{aligned}$$

for all $t\in[0,1]$. For each t_1, t_2 with $t_1<t_2$, we can write

$$\begin{aligned}\left|\varphi(t_2)-\varphi(t_1)\right|&=\left|\int_0^{t_2}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds-\int_0^{t_1}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right|\\ &=\left|\int_{t_1}^{t_2}\int_0^s\frac{(s-\tau)^{\alpha-1}y(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right|\leq\left|\int_{t_1}^{t_2}\int_0^s\frac{(s-\tau)^{\alpha-1}\sigma(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\right|.\end{aligned}$$

Note that the right-hand side of the inequality tends to zero as $t_1\to t_2$. Now by using the Arzela–Ascoli theorem, the operator $\hat{\mathcal{B}}_2:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ is completely continuous. We show that $\hat{\mathcal{B}}_2$ has a closed graph, and this implies that $\hat{\mathcal{B}}_2$ is upper semicontinuous. For this, suppose that $z_n\in\mathcal{V}$ and $\varphi_n\in\hat{\mathcal{B}}_2 z_n$ with $z_n\to z^*$ and $\varphi_n\to\varphi^*$. We show that $\varphi^*\in\hat{\mathcal{B}}_2 z^*$.

For each $\varphi_n \in \hat{\mathcal{B}}_2 z_n$, choose $y_n \in (\mathcal{SEL})_{\Psi,z_n}$ such that

$$\begin{aligned}\varphi_n(t) &= \nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y_n(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y_n(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y_n(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds.\end{aligned}$$

It is sufficient to prove that there exists a function $y^* \in (\mathcal{SEL})_{\Psi,z^*}$ such that

$$\begin{aligned}\varphi^*(t) &= \nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\end{aligned}$$

for all $t \in [0,1]$. Consider the continuous linear operator $\hat{\Upsilon} : \mathcal{L}^1([0,1],\mathbb{R}) \to \mathcal{X}$ defined by

$$\begin{aligned}\hat{\Upsilon}(y)(t) &= z(t) \\ &= \nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\end{aligned}$$

for all $t \in [0,1]$. Then, we have

$$\begin{aligned}\left\|\varphi_n(t) - \varphi^*(t)\right\| &= \left\| \nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau) - y^*(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \right. \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau) - y^*(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) \\ &\quad \left. + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}(y_n(\tau) - y^*(\tau))}{p(s)\Gamma(\alpha)}\, d\tau\, ds \right\| \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

Now by using Theorem 1, we conclude that the operator $\hat{\Upsilon} \circ (\mathcal{SEL})_\Psi$ has a closed graph. In fact, since $\varphi_n \in \hat{\Upsilon}((\mathcal{SEL})_{\Psi,z_n})$ and $z_n \to z^*$, there exists $y^* \in (\mathcal{SEL})_{\Psi,z^*}$ such that

$$\begin{aligned}\varphi^*(t) &= \nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ &\quad - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y^*(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\end{aligned}$$

for all $t \in [0,1]$. Hence, $\varphi^* \in \hat{\mathcal{B}}_2 z^*$, and so the graph of $\hat{\mathcal{B}}_2$ is closed. Thus, $\hat{\mathcal{B}}_2$ is upper semicontinuous. Furthermore, by using the assumptions, we know that the operator Ψ has compact values. Hence, $\hat{\mathcal{B}}_2$ is a compact and upper semicontinuous operator. By a similar method as in proof of Theorem 6, we can prove that $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{B}}_3$ are Lipschitz operators on $\mathcal{X}$ with Lipschitz constants χ_1^* and χ_2^*, respectively. Now by using assumption $(\mathcal{Q}_5^*)$, we get $\ell_1^* \hat{M}^* + \ell_2^* = \chi_1^* \|\hat{\mathcal{B}}_2(\mathcal{X})\| + \chi_2^* = \chi_1^* \sup\{|\hat{\mathcal{B}}_2 z| : z \in \mathcal{X}\} + \chi_2^* \leq \chi_1^* \Lambda^* + \chi_2^* < \frac{1}{2}$. Thus, the assumptions of Theorem 3 hold for $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_2$, and $\hat{\mathcal{B}}_3$, and so one of the conditions (a) or (b) holds. We show that condition (b) is impossible. According to the definition of $\sum^*$, let z be an arbitrary element of $\sum^*$. Thus, $\mu z \in \hat{\mathcal{B}}_1 z \hat{\mathcal{B}}_2 z + \hat{\mathcal{B}}_3 z$ for some $\mu > 1$. Now, choose $y \in (\mathcal{SEL})_{\Psi,z}$ such that

$$\begin{aligned} z(t) = {}& \frac{1}{\mu} g\bigl(t, z(t)\bigr) \biggl[\nu R \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ & - R \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} y(\tau)}{p(s)\Gamma(\alpha)}\, d\tau\, ds \biggr] \\ & + \frac{1}{\mu} f\bigl(t, z(t)\bigr). \end{aligned}$$

Hence,

$$\begin{aligned} \bigl|z(t)\bigr| \le {}& \frac{1}{\mu} \bigl|g\bigl(t, z(t)\bigr)\bigr| \biggl[|\nu||R| \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ & + |R| \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) \\ & + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} |y(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \biggr] + \frac{1}{\mu} \bigl|f\bigl(t, z(t)\bigr)\bigr| \\ \le {}& \bigl|g\bigl(t, z(t)\bigr)\bigr| \biggl[|\nu||R| \int_d^e \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} |\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds\, d\upsilon(\theta) \\ & + |R| \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1} |\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds\, d\varpi(\theta) \\ & + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} |\sigma(\tau)|}{|p(s)|\Gamma(\alpha)}\, d\tau\, ds \biggr] + \bigl|f\bigl(t, z(t)\bigr)\bigr|. \end{aligned}$$

Since $|g(t, z(t))| \le \chi_1^* \|z\| + \mathcal{G}^*$ and $|f(t, z(t))| \le \chi_2^* \|z\| + \mathcal{F}^*$, where $\mathcal{G}^* = \sup_{t\in[0,1]} g(t, 0)$ and $\mathcal{F}^* = \sup_{t\in[0,1]} f(t, 0)$, we deduce that

$$\begin{aligned} \bigl|z(t)\bigr| \le {}& \bigl(\chi_1^* \|z\| + \mathcal{G}^*\bigr) \frac{\|\sigma\|}{p\Gamma(\alpha+1)} \bigl(|R|\bigl(\varpi(c) - \varpi(a) + |\nu|\bigl(\upsilon(e) - \upsilon(d)\bigr)\bigr) + 1\bigr) \\ & + \chi_2^* \|z\| + \mathcal{F}^* \\ = {}& \bigl(\chi_1^* \Lambda^* + \chi_2^*\bigr)\|z\| + \mathcal{G}^* \Lambda^* + \mathcal{F}^*. \end{aligned}$$

By taking the supremum over $t \in [0, 1]$ in the above inequality, we find a constant $\mathcal{M} > 0$ such that $\|z\| \le \mathcal{M} := \frac{\mathcal{G}^*\Lambda^* + \mathcal{F}^*}{1 - \chi_1^*\Lambda^* - \chi_2^*}$. Hence, $\sum^*$ is bounded. Thus, condition (b) is impossible. Now by using Theorem 3, the fractional hybrid Sturm–Liouville inclusion problem (3)–(4) has at least one solution. □

Now, we provide two examples to illustrate our main results.

Example 1 Consider the fractional hybrid Sturm–Liouville inclusion problem

$$\begin{aligned} & {}^cD^{\frac{4}{5}} \biggl(101\sqrt{1+t} \biggl(\frac{z(t)}{e^{-t^2-t-1} + \frac{2e^{-\sin \pi t}|z(t)|}{|z(t)|+1}} \biggr)' \\ & \quad - e^{-\cos \pi t} \tan^{-1}\bigl(z(t)+1\bigr) \biggr) \in \biggl[\frac{e^{-5\pi t} \ln(1 + e^{-\frac{|z(t)|}{1+|z(t)|}})}{3 + |z(t)|}, \frac{e^{-4\pi t} \ln(2 + e^{-\frac{|z(t)|}{1+|z(t)|}})}{1 + |z(t)|} \biggr] \end{aligned} \tag{17}$$

with multipoint hybrid boundary conditions

$$\begin{cases} \left(\frac{z(t)}{e^{-t^2-t-1}+\frac{2e^{-\sin\pi t}|z(t)|}{|z(t)|+1}}\right)'_{t=0} = \left(\frac{e^{-\cos\pi t}}{\sqrt{1+t}}\tan^{-1}(z(t)+1)\right)_{t=0}, \\ \sum_{i=1}^{2}\frac{1}{2^i}\left(\frac{z(\frac{1}{2^i})}{e^{-\frac{1}{2^{2i}}-\frac{1}{2^i}-1}+\frac{2e^{-\sin\frac{\pi}{2^i}}|z(\frac{1}{2^i})|}{|z(\frac{1}{2^i})|+1}}\right) \\ \quad = -\frac{1}{200}\sum_{j=1}^{3}\frac{1}{(-3)^j}\left(\frac{z(\frac{1}{(-3)^j})}{e^{-\frac{1}{(-3)^{2j}}-\frac{1}{(-3)^j}-1}+\frac{2e^{-\sin\pi\frac{1}{(-3)^j}}|z(\frac{1}{(-3)^j})|}{|z(\frac{1}{(-3)^j})|+1}}\right). \end{cases} \tag{18}$$

Put $\alpha = \frac{4}{5}$, $\zeta = 1$, $\xi_i = \frac{1}{2^i}$ $(i = 1,2)$, $\eta_j = \frac{1}{(-3)^j}$ $(j = 1,2,3)$, $\nu = -\frac{1}{200}$, $p(t) = 101\sqrt{1+t}$, $\tilde{p}(t) = e^{-\cos\pi t}$, $\tilde{f}(t) = \tan^{-1}(z(t)+1)$, $g(t,z(t)) = e^{-t^2-t-1} + \frac{2e^{-\sin\pi t}|z(t)|}{|z(t)|+1}$, and

$$\Psi(t,x) = \left[\frac{e^{-5\pi t}\ln(1+e^{-\frac{|x|}{1+|x|}})}{3+|x|}, \frac{e^{-4\pi t}\ln(2+e^{-\frac{|x|}{1+|x|}})}{1+|x|}\right].$$

Then, $|g(t,z_1) - g(t,z_2)| \le 2e^{-\sin\pi t}|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{R}$ and $t \in [0,1]$. If $\chi(t) = 2e^{\sin\pi t}$, then $\chi^* = 2$ and $g_0 = e^{-\frac{7}{4}}$. Also,

$$\begin{aligned} \|\Psi(t,x)\| &= \sup\{|y| : y \in \Psi(t,x)\} \\ &= \sup\left\{|y| : y \in \left[\frac{e^{-5\pi t}\ln(1+e^{-\frac{|x|}{1+|x|}})}{3+|x|}, \frac{e^{-4\pi t}\ln(2+e^{-\frac{|x|}{1+|x|}})}{1+|x|}\right]\right\} \\ &\le \ln(3)e^{-4\pi t} \end{aligned}$$

for all $x \in \mathbb{R}$ and almost all $t \in [0,1]$. Put $\sigma(t) = \frac{\ln(3)}{4\pi}(1 - e^{-4\pi t})$. Then, $\|\sigma\| = \frac{\ln(3)}{4\pi}(1 - e^{-4\pi})$, $p = 101$, $\|\tilde{p}\| = 1$, $|\frac{\partial\tilde{f}(z)}{\partial z}| \le 1 = \tilde{\mathcal{K}}$, $\tilde{f}_0 = \frac{\pi}{4}$, and $\bar{\Delta} = 0.0198816623$. Hence,

$$\zeta = 1 > 0.0794154803 \approx \bar{\Delta}(\chi^*\zeta + g_0)\left(\|\tilde{p}\|(\tilde{\mathcal{K}}\zeta + \tilde{f}_0) + \frac{\|\sigma\|}{\Gamma(\alpha+2)}\right)$$

and $(\frac{\chi^*\|\sigma\|}{\Gamma(\alpha+2)} + \|\tilde{p}\|(2\tilde{\mathcal{K}}\chi^*\zeta + \chi^*\tilde{f}_0 + \tilde{\mathcal{K}}g_0))\bar{\Delta} \approx 0.1162851566 < \frac{1}{2}$. Then by using Theorem 6, the fractional hybrid inclusion problem (17)–(18) has at least one solution.

Example 2 Consider the fractional hybrid Sturm–Liouville inclusion problem

$$^cD^{\frac{99}{100}}\left(150\sqrt{4+t}\left(\frac{z(t) - \frac{1}{60}e^{-\frac{t^2}{1+t^2}}(\frac{1}{40}z(t)+3)}{\frac{e^{-\ln^2(t+1)}|z(t)|}{1+|z(t)|} + \frac{1}{1+t}e^{-\pi t}}\right)'\right) \in \left[0, \frac{\cos\frac{\pi}{2}t\sin\frac{\pi|z(t)|}{2(1+|z(t)|)}}{1+\sin\frac{\pi|z(t)|}{2(1+|z(t)|)}}\right] \tag{19}$$

with integral hybrid boundary conditions

$$\begin{cases} \left(\frac{z(t)-\frac{1}{60}e^{-\frac{t^2}{1+t^2}}(\frac{1}{40}z(t)+3)}{\frac{e^{-\ln^2(t+1)}|z(t)|}{1+|z(t)|}+\frac{1}{1+t}e^{-\pi t}}\right)'_{t=0} = 0, \\ \int_0^{\frac{1}{4}} \left(\frac{z(\theta)-\frac{1}{60}e^{-\frac{\theta^2}{1+\theta^2}}(\frac{1}{40}z(\theta)+3)}{\frac{e^{-\ln^2(\theta+1)}|z(\theta)|}{1+|z(\theta)|}+\frac{1}{1+\theta}e^{-\pi\theta}}\right) d(4\theta+1) \\ \quad = -\frac{1}{111}\int_{\frac{1}{3}}^1 \left(\frac{z(\theta)-\frac{1}{60}e^{-\frac{\theta^2}{1+\theta^2}}(\frac{1}{40}z(\theta)+3)}{\frac{e^{-\ln^2(\theta+1)}|z(\theta)|}{1+|z(\theta)|}+\frac{1}{1+\theta}e^{-\pi\theta}}\right) d(6\theta+1). \end{cases} \tag{20}$$

Put $\alpha = \frac{99}{100}$, $\nu = -\frac{1}{111}$, $\tilde{\varepsilon} = 1$, $p(t) = 150\sqrt{4+t}$, $\varpi(\theta) = 4\theta + 1$, $\upsilon(\theta) = 6\theta + 1$,

$$g(t,x) = \frac{e^{-\ln^2(t+1)}|x|}{1+|x|} + \frac{1}{1+t}e^{-\pi t},$$

$f(t,x) = \frac{1}{60}e^{-\frac{t^2}{1+t^2}}(\frac{1}{40}z(t)+3)$, and $\Psi(t,x) = [0, \frac{\cos\frac{\pi}{2}t\sin\frac{\pi|x|}{2(1+|x|)}}{1+\sin\frac{\pi|x|}{2(1+|x|)}}]$. Then $p = 300$, $\varpi(\frac{1}{4}) - \varpi(0) = 1$, $\upsilon(1) - \upsilon(\frac{1}{3}) = 4$, $|g(t,z_1) - g(t,z_2)| \le e^{-\ln^2(t+1)}|z_1 - z_2|$, and

$$\left|f(t,z_1) - f(t,z_2)\right| \le \frac{1}{2400}e^{-\frac{t^2}{1+t^2}}|z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{R}$ and $t \in [0,1]$. If $\chi_1(t) = e^{-\ln^2(t+1)}$ and $\chi_2(t) = \frac{1}{2400}e^{-\frac{t^2}{1+t^2}}$, then $\chi_1^* = 1$ and $\chi_1^* = \frac{1}{2400}$. Also,

$$\left\|\Psi(t,x)\right\| = \sup\left\{|y| : y \in \Psi(t,x)\right\} = \sup\left\{|y| : y \in \left[0, \frac{\cos\frac{\pi}{2}t\sin\frac{\pi|x|}{2(1+|x|)}}{1+\sin\frac{\pi|x|}{2(1+|x|)}}\right]\right\} \le \cos\frac{\pi}{2}t$$

for all $x \in \mathbb{R}$ and almost all $t \in [0,1]$. Put $\sigma(t) = \frac{2}{\pi}\cos^2\frac{\pi}{2}t$. Then, $\|\sigma\| = \frac{2}{\pi}$ and so

$$\Lambda^* \approx 0.0042619756 \quad \text{and so} \quad \chi_1^*\Lambda^* + \chi_2^* \approx 0.0046786423 < \frac{1}{2}.$$

Now by using Theorem 9, the hybrid inclusion problem (19)–(20) has at least one solution.

4 Conclusion

Many natural processes are modeled by some types of fractional differential equations. This diversity factor in studying complicated fractional integro-differential equations increases our ability for exact modeling of more phenomena. We know that inclusion problems are real generalizations for differential equations and some economic phenomena could be model by inclusions. Thus, it is important that we study different inclusion problems, especially those related to well-known differential equations such as Sturm–Liouville. In this work, we review fractional hybrid inclusion version of the Sturm–Liouville equation. In this way, we investigate two fractional hybrid Sturm–Liouville differential inclusions with multipoint and integral hybrid boundary conditions. Also, we provide two examples to illustrate our main results.

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