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Some new Simpson-type inequalities for generalized p -convex function on fractal sets with applications

Thabet Abdeljawad^{1,2,3}, Saima Rashid⁴, Zakia Hammouch⁵, İmdat İşcan⁶ and Yu-Ming Chu^{7,8*}

*Correspondence:

chuyuming2005@126.com

⁷Department of Mathematics, Huzhou University, Huzhou, China

⁸Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China

Full list of author information is available at the end of the article

Abstract

The present article addresses the concept of p -convex functions on fractal sets. We are able to prove a novel auxiliary result. In the application aspect, the fidelity of the local fractional is used to establish the generalization of Simpson-type inequalities for the class of functions whose local fractional derivatives in absolute values at certain powers are p -convex. The method we present is an alternative in showing the classical variants associated with generalized p -convex functions. Some parts of our results cover the classical convex functions and classical harmonically convex functions. Some novel applications in random variables, cumulative distribution functions and generalized bivariate means are obtained to ensure the correctness of the present results. The present approach is efficient, reliable, and it can be used as an alternative to establishing new solutions for different types of fractals in computer graphics.

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1 Introduction

The following inequality is known in the literature as a Simpson-type inequality:

$$\left| \frac{1}{6} \left[\chi(\varkappa_1) + 4\chi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) + \chi(\varkappa_2) \right] - \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \chi(u) du \right| \leq \frac{1}{2880} \|\chi^{(4)}\|_{\infty} (\varkappa_2 - \varkappa_1)^4, \quad (1.1)$$

where the mapping $\chi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ is assumed to be four times continuously differentiable on the interval $(\varkappa_1, \varkappa_2)$ and $\|\chi^{(4)}\|_{\infty} = \sup_{\xi \in (\varkappa_1, \varkappa_2)} |\chi^{(4)}(\xi)| < \infty$. The inequality (1.1) has gained considerable attention, as convex analysis and fractional calculus operators involving several classes of convex functions is an uphill task. Therefore many authors proposed different numerical techniques to find Simpson-type inequalities, arising in the substantial literature of numerical analysis and engineering, and many other fields of sci-

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ences [1–12]. Profusely novel versions of Simpson-type inequalities for the class of convex functions have been modified and generalized by numerous researchers [13–24]. Recently, many investigations about (1.1) can be found by Rashid et al. [25] for preinvex functions, Li and Du [26] for (α, m) -GA-convex functions, Xi and Qi [27] for logarithmically convex functions, Sarikaya et al. [28] for s -convex functions and İşcan et al. [29] for p -convex function.

With the development of the inequality theory, the inequalities for various kinds of convex functions have a rapid blossom in the area of pure and applied mathematics [30–42]. As mentioned above, many articles are all involved with Hermite–Hadamard type and trapezoid type, midpoint type inequalities [43–49]. However, to the best of our knowledge, Simpson-type inequalities for functions whose first local α -derivatives in absolute value are the class of generalized p -convex functions have not been reported. So we turn our attention to this new research.

One of the aspects which are nowadays particularly well known among researchers is the integral inequalities with applications. In this field, the majority of the authors are generalizing the standard results in the accessible literature by utilizing various sorts/definitions of the fractional integral operators [4, 25, 50, 51]. An enormous heft of fractional differential problems and partial differential equations can be converted into problems of comprehending some estimated integral equations.

The early research motivations in the area of the local fractional theory were for solving a bulk of initial and boundary value associated with differential equations [52, 53]. In [53], Yang introduced a contemporary study used to tackle nondifferentiable problems that incorporate in complex systems of real-world phenomena. The fractal sets in science have introduced some fascinating complex graphs and picture compressions to computer graphics. The expression “fractal” was first utilized by a young mathematician, Julia [54] when he was considering Cayley’s problem identified with the conduct of Newton’s method in a complex plane. The fractal is frequently utilized in real-world studies, involving fractal antennas, fractal transistors, and fractal heat exchangers. It has applications in the music industry, the creation of photography, soil mechanics, small-angle scattering theory, and many more fields. It is to be emphasized that fractal theory assumes an essential job in the improvement of picturing of fractal sets. The utilizations of fractal sets in cryptography and other useful areas of research have increased the interest of researchers to broaden the utilization in mathematical inequalities. Fractals are particular arbitrary examples descriptions addressing the erratic developments of the disorderly world at work. The most significant utilization of fractals in software engineering is fractal picture compression. This sort of compression utilizes the way that this present reality is very well portrayed by fractal geometry [52, 55, 56]. Interestingly, the author of [53] investigated the local fractional functions on fractal space deliberately, which comprises of local fractional calculus and the monotonicity of functions. Numerous analysts contemplated the characteristics of functions on fractal space and built numerous sorts of fractional calculus by utilizing various strategies [57–59]. Additionally, integral inequalities in the context of local fractional calculus have a significant role in all fields of pure and applied mathematics. For example, Chen [60] derived a novel version of Hölder inequality on fractals. In [58], Mo et al. established the fractal version of Hermite–Hadamard inequality by the use of generalized convex functions. In [61], Du et al. contemplated the novel generalizations for Simpson’s, Hermite Hadamard and Hermite–Hadamard–Fejér type inequalities

for generalized m -convex functions concerning to local fractional calculus. In addition to these results, Luo et al. [62], deduced several new Fejér–Hermite–Hadamard inequalities for a class of h -convex functions with applications. For some useful and recent studies on fractional calculus and its applications in different fields of mathematics [63–68].

Owing to the above phenomena, the key aim of this research is to introduce a new auxiliary result depending on local fractal sets are presented. With the aid of novel identity, we derived numerous novel generalizations of Simpson-type for mappings whose powers contain local fractional derivatives in modulus are generalized p -convex. The main impetus of this study to capture new estimates for generalized convex functions and generalized harmonically convex functions. In addition, the application of the proved results in a random variable, cumulative distribution functions, and the generalized bivariate mean formula is also presented. We hope that the new strategy formulated in the present paper is more invigorating than the accessible one.

2 Preliminaries

Now, we mention the preliminaries from the theory of local fractional calculus. These ideas and important consequences associated with the local fractional derivative and local fractional integral are mainly due to Yang et al. [53].

Let $\varsigma_1^{\dot{\alpha}}, \varsigma_2^{\dot{\alpha}}$ and $\varsigma_3^{\dot{\alpha}}$ belong to the set $\mathbb{R}^{\dot{\alpha}}$ ($0 < \dot{\alpha} \leq 1$). Then

- (1) $\varsigma_1^{\dot{\alpha}} + \varsigma_2^{\dot{\alpha}}$ and $\varsigma_1^{\dot{\alpha}} \varsigma_2^{\dot{\alpha}}$ belong to the set $\mathbb{R}^{\dot{\alpha}}$;
- (2) $\varsigma_1^{\dot{\alpha}} + \varsigma_2^{\dot{\alpha}} = \varsigma_2^{\dot{\alpha}} + \varsigma_1^{\dot{\alpha}} = (\varsigma_1 + \varsigma_2)^{\dot{\alpha}} = (\varsigma_2 + \varsigma_1)^{\dot{\alpha}}$;
- (3) $\varsigma_1^{\dot{\alpha}} + (\varsigma_2^{\dot{\alpha}} + \varsigma_3^{\dot{\alpha}}) = (\varsigma_1^{\dot{\alpha}} + \varsigma_2^{\dot{\alpha}}) + \varsigma_3^{\dot{\alpha}}$;
- (4) $\varsigma_1^{\dot{\alpha}} \varsigma_2^{\dot{\alpha}} = \varsigma_2^{\dot{\alpha}} \varsigma_1^{\dot{\alpha}} = (\varsigma_1 \varsigma_2)^{\dot{\alpha}} = (\varsigma_2 \varsigma_1)^{\dot{\alpha}}$;
- (5) $\varsigma_1^{\dot{\alpha}} (\varsigma_2^{\dot{\alpha}} \varsigma_3^{\dot{\alpha}}) = (\varsigma_1^{\dot{\alpha}} \varsigma_2^{\dot{\alpha}}) \varsigma_3^{\dot{\alpha}}$;
- (6) $\varsigma_1^{\dot{\alpha}} (\varsigma_2^{\dot{\alpha}} + \varsigma_3^{\dot{\alpha}}) = \varsigma_1^{\dot{\alpha}} \varsigma_2^{\dot{\alpha}} + \varsigma_1^{\dot{\alpha}} \varsigma_3^{\dot{\alpha}}$;
- (7) $\varsigma_1^{\dot{\alpha}} + 0^{\dot{\alpha}} = 0^{\dot{\alpha}} + \varsigma_1^{\dot{\alpha}} = \varsigma_1^{\dot{\alpha}}$ and $\varsigma_1^{\dot{\alpha}} 1^{\dot{\alpha}} = 1^{\dot{\alpha}} \varsigma_1^{\dot{\alpha}} = \varsigma_1^{\dot{\alpha}}$.

Definition 2.1 A nondifferentiable mapping $\chi : \mathbb{R} \rightarrow \mathbb{R}^{\dot{\alpha}}, \theta \rightarrow \chi(\epsilon)$ is said to be local fractional continuous at ϵ_0 , if, for any $\epsilon > 0$, there exists $\xi > 0$ such that

$$|\chi(\epsilon) - \chi(\epsilon_0)| < \epsilon^{\dot{\alpha}}$$

for $|\epsilon - \epsilon_0| < \kappa$. If $\chi(\epsilon)$ is local continuous on $(\varkappa_1, \varkappa_2)$, then we denote it by $\chi(\epsilon) \in \mathbb{C}_{\dot{\alpha}}(\varkappa_1, \varkappa_2)$.

Definition 2.2 The local fractional derivative of $\chi(\epsilon)$ of order $\dot{\alpha}$ at $\epsilon = \epsilon_0$ is defined by the expression

$$\begin{aligned} \chi^{(\dot{\alpha})}(\epsilon_0) &= {}_{\epsilon_0} \mathcal{D}_{\epsilon}^{\dot{\alpha}} \chi(\epsilon) = \left. \frac{d^{\dot{\alpha}} \chi(\epsilon)}{d\epsilon^{\dot{\alpha}}} \right|_{\epsilon=\epsilon_0} \\ &= \lim_{\epsilon \rightarrow \epsilon_0} \frac{\Delta^{\dot{\alpha}}(\chi(\epsilon) - \chi(\epsilon_0))}{(\epsilon - \epsilon_0)^{\dot{\alpha}}}, \end{aligned}$$

where $\Delta^{\dot{\alpha}}(\chi(\epsilon) - \chi(\epsilon_0)) = \Gamma(\dot{\alpha} + 1)(\chi(\epsilon) - \chi(\epsilon_0))$. Let $\chi^{(\dot{\alpha})}(\epsilon) = \mathcal{D}_{\epsilon}^{\dot{\alpha}} \chi(\epsilon)$. If there exists $\chi^{(k+1)\dot{\alpha}}(\epsilon) = \overbrace{\mathcal{D}_{\epsilon}^{\dot{\alpha}} \dots \mathcal{D}_{\epsilon}^{\dot{\alpha}}}^{(k+1) \text{ times}} \chi(\epsilon)$ for any $\epsilon \in \Omega \subseteq \mathbb{R}$, then it is denoted by $\chi \in \mathcal{D}_{(k+1)\dot{\alpha}}(\mathcal{I})$, where $k = 0, 1, 2, \dots$

Definition 2.3 Let $\chi(\epsilon) \in \mathbb{C}_\alpha[\varkappa_1, \varkappa_2]$ and $\Delta = \{\xi_0, \xi_1, \dots, \xi_N\}$ ($N \in \mathbb{N}$) be a partition of $[\varkappa_1, \varkappa_2]$ with $\varkappa_1 = \xi_0 < \xi_1 < \dots < \xi_N = \varkappa_2$. Then the local fractional integral of χ on $[\varkappa_1, \varkappa_2]$ of order α is defined by

$$\begin{aligned} {}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(\epsilon) &= \frac{1}{\Gamma(1 + \alpha)} \int_{\varkappa_1}^{\varkappa_2} \chi(\xi)(d\xi)^\alpha \\ &:= \frac{1}{\Gamma(1 + \alpha)} \lim_{\delta\xi \rightarrow 0} \sum_{j=0}^{N-1} \chi(\xi_j)(\Delta\xi_j), \end{aligned}$$

where $\delta\xi := \max\{\Delta\xi_1, \Delta\xi_2, \dots, \Delta\xi_{N-1}\}$ and $\Delta\xi_j := \xi_{j+1} - \xi_j, j = 0, \dots, N - 1$.

We clearly see that ${}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(\epsilon) = 0$ if $\varkappa_1 = \varkappa_2$ and ${}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(\epsilon) = -{}_{\varkappa_2}I_{\varkappa_1}^{(\alpha)}\chi(\epsilon)$ if $\varkappa_1 < \varkappa_2$. For any $\epsilon \in [\varkappa_1, \varkappa_2]$, if there exists ${}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(\epsilon)$, then it is denoted by $\chi(\epsilon) \in \mathcal{I}_\epsilon^\alpha[\varkappa_1, \varkappa_2]$.

Lemma 2.4 (See [53]) *The following statements are true:*

- (1) *If $\chi(u) = \mathcal{G}^{(\alpha)}(u) \in \mathbb{C}_\alpha[\varkappa_1, \varkappa_2]$, then*

$${}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(u) = \mathcal{G}(\varkappa_2) - \mathcal{G}(\varkappa_1).$$

- (2) *If $\chi(u), \mathcal{G}(u) \in \mathcal{D}_\alpha[\varkappa_1, \varkappa_2]$, and $\chi^{(\alpha)}(u), \mathcal{G}^{(\alpha)}(u) \in \mathbb{C}_\alpha[\varkappa_1, \varkappa_2]$, then*

$${}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi(u)\mathcal{G}^{(\alpha)}(u) = \chi(u)\mathcal{G}(u)|_{\varkappa_1}^{\varkappa_2} - {}_{\varkappa_1}I_{\varkappa_2}^{(\alpha)}\chi^{(\alpha)}(u)\mathcal{G}(u).$$

Lemma 2.5 (See [53]) *The formulas*

$$\begin{aligned} \frac{d^\alpha u^{k\alpha}}{du^\alpha} &= \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k - 1)\alpha)} u^{(k-1)\alpha}, \\ \frac{1}{\Gamma(1 + \alpha)} \int_{\varkappa_1}^{\varkappa_2} u^{k\alpha}(du)^\alpha &= \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k + 1)\alpha)} (\varkappa_2^{(k+1)\alpha} - \varkappa_1^{(k+1)\alpha}), \end{aligned}$$

hold for $k > 0$.

Lemma 2.6 (Generalized Hölder inequality [60]) *Let $s, q > 1$ with $s^{-1} + q^{-1} = 1$ and $\chi, \mathcal{G} \in \mathbb{C}_\alpha[\varkappa_1, \varkappa_2]$. Then*

$$\begin{aligned} &\frac{1}{\Gamma(1 + \alpha)} \int_{\varkappa_1}^{\varkappa_2} |\chi(u)\mathcal{G}(u)|(du)^\alpha \\ &\leq \left(\frac{1}{\Gamma(1 + \alpha)} \int_{\varkappa_1}^{\varkappa_2} |\chi(u)|^s(du)^\alpha \right)^{\frac{1}{s}} \left(\frac{1}{\Gamma(1 + \alpha)} \int_{\varkappa_1}^{\varkappa_2} |\mathcal{G}(u)|^q(du)^\alpha \right)^{\frac{1}{q}}. \end{aligned}$$

3 Main results and discussions

In this section, we first inaugurate a local fractional integral identity for generalized p -convex functions, and then we utilize the said identity to establish certain Simpson-type variants in the context of the fractal domain. We now present the concept of generalized p -convex functions on fractal space as follows.

Definition 3.1 Let $p \in \mathbb{R} \setminus \{0\}$. Then $\chi : \Omega = [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}^{\dot{\alpha}}$ is said to be generalized p -convex if the inequality

$$\chi\left(\left[(1-\xi)x^p + \xi y^p\right]^{\frac{1}{p}}\right) \leq (1-\xi)^{\dot{\alpha}} \chi(x) + \xi^{\dot{\alpha}} \chi(y) \tag{3.1}$$

holds for $x, y \in \Omega$, and $\xi \in [0, 1]$. χ is said to be generalized p -concave if inequality (3.1) is reversed.

Remark 3.2 From Definition 3.1 we clearly see that:

- (1) If we take $\dot{\alpha} = 1$, then we get a definition given in [69].
- (2) If we take $p = 1$, then we get a definition given in [58].
- (3) If we take $p = -1$, then we get a definition given in [66].
- (4) If we take $p = 1$ with $\dot{\alpha} = 1$, then we get the classical convex function.
- (5) If we take $p = -1$ and $\dot{\alpha} = 1$, then we get a definition given in [70].

It is worth mentioning that generalized p -convex functions collapse to generalized convex, generalized harmonically convex functions, harmonically convex functions, and classical convex functions as special cases. This shows that outcomes derived in the present paper continue to hold for these classes of convex functions and their variant forms.

Example 3.3 Let $\chi : \Omega = [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}^{\dot{\alpha}}$ with $\chi_1(u) = u^{\dot{\alpha}p}$ ($p \neq 0$) and $\chi_2 : \Omega \rightarrow \mathbb{R}^{\dot{\alpha}}$ with $\chi_2(u) = c^{\dot{\alpha}}$ ($c^{\dot{\alpha}} \in \mathbb{R}^{\dot{\alpha}}$). Then χ_1 is generalized p -convex and χ_2 is generalized p -concave.

Lemma 3.4 Let $p \in \mathbb{R} \setminus \{0\}$ and $\chi : \Omega^\circ \subset \mathbb{R} \rightarrow \mathbb{R}^{\dot{\alpha}}$ such that $\chi \in \mathcal{D}_{\dot{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω) and $\chi^{(\dot{\alpha})} \in \mathcal{C}_{\dot{\alpha}}[\varkappa_1, \varkappa_2]$. Then the inequality

$$\begin{aligned} & \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + \chi(\varkappa_2) \right] \\ & - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} {}_{\varkappa_1} \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(\xi - \frac{1}{3})^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right] \\ & \times \chi^{(\dot{\alpha})} \left(\left[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ & + \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(\xi - \frac{2}{3})^{\dot{\alpha}}}{[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} \right] \\ & \times \chi^{(\dot{\alpha})} \left(\left[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \end{aligned}$$

holds for all $u \in [\varkappa_1, \varkappa_2]$.

Proof Firstly, let us calculate the following integrals:

$$\left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(\xi - \frac{1}{3})^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right]$$

$$\begin{aligned} & \times \chi^{(\dot{\alpha})} \left(\left[(1-\xi)z_1^p + \xi \frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ & + \left(\frac{z_2^p - z_1^p}{4p} \right)^{\dot{\alpha}} \left[\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(\xi - \frac{2}{3})^{\dot{\alpha}}}{[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p]^{\dot{\alpha}(1-1/p)}} \right] \\ & \times \chi^{(\dot{\alpha})} \left(\left[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} := I_1 + I_2. \end{aligned}$$

Utilizing the local fractional integration by parts and the change of the variable technique for the integrals I_1 and I_2 , we get

$$\begin{aligned} I_1 &= \left[\int_0^1 \frac{(\xi - \frac{1}{3})^{\dot{\alpha}}}{[(1-\xi)z_1^p + \xi \frac{z_1^p + z_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right] \chi^{(\dot{\alpha})} \left(\left[(1-\xi)z_1^p + \xi \frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ &= \left(\xi - \frac{1}{3} \right)^{\dot{\alpha}} \chi \left(\left[(1-\xi)z_1^p + \xi \frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) \Big|_0^1 \\ &\quad - \left(\frac{2p}{z_2^p - z_1^p} \right)^{\dot{\alpha}} \frac{\Gamma(1+\dot{\alpha})}{\Gamma(1+\dot{\alpha})} \int_0^1 \chi \left(\left[(1-\xi)z_1^p + \xi \frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \chi \left(\left[\frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) - \left(-\frac{1}{3} \right)^{\dot{\alpha}} \chi(z_1) \\ &\quad - \left(\frac{2p}{z_2^p - z_1^p} \right)^{\dot{\alpha}} \frac{\Gamma(1+\dot{\alpha})}{\Gamma(1+\dot{\alpha})} \int_{z_1}^{(\frac{z_1^p + z_2^p}{2})^{1/p}} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} (du)^{\dot{\alpha}} \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \chi \left(\left[\frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) - \left(-\frac{1}{3} \right)^{\dot{\alpha}} \chi(z_1) \\ &\quad - \left(\frac{2p}{z_2^p - z_1^p} \right)^{\dot{\alpha}} \Gamma(1+\dot{\alpha}) z_1 \mathcal{I}_{(\frac{z_1^p + z_2^p}{2})^{1/p}}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} I_2 &= \left[\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(\xi - \frac{2}{3})^{\dot{\alpha}}}{[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p]^{\dot{\alpha}(1-1/p)}} \right] \\ &\quad \times \chi^{(\dot{\alpha})} \left(\left[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ &= \left(\xi - \frac{2}{3} \right)^{\dot{\alpha}} \chi^{(\dot{\alpha})} \left(\left[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p \right]^{1/p} \right) \Big|_0^1 \\ &\quad - \left(\frac{2p}{z_2^p - z_1^p} \right)^{\dot{\alpha}} \int_0^1 \chi^{(\dot{\alpha})} \left(\left[(1-\xi) \frac{z_1^p + z_2^p}{2} + \xi z_2^p \right]^{1/p} \right) (d\xi)^{\dot{\alpha}} \\ &= \left(\frac{1}{3} \right)^{\dot{\alpha}} \chi(z_2) - \left(-\frac{2}{3} \right)^{\dot{\alpha}} \chi \left(\left[\frac{z_1^p + z_2^p}{2} \right]^{1/p} \right) \\ &\quad - \left(\frac{2p}{z_2^p - z_1^p} \right)^{\dot{\alpha}} \frac{\Gamma(1+\dot{\alpha})}{\Gamma(1+\dot{\alpha})} \int_{(\frac{z_1^p + z_2^p}{2})^{1/p}}^{z_2} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} (du)^{\dot{\alpha}} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{1}{3}\right)^{\dot{\alpha}} \chi(\varkappa_2) - \left(-\frac{2}{3}\right)^{\dot{\alpha}} \chi\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) \\
 &\quad - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha}) \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}}.
 \end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3) and then multiplying the obtained result by $\frac{1}{2^{\dot{\alpha}}}$, we have

$$\begin{aligned}
 \frac{I_1 + I_2}{2^{\dot{\alpha}}} &= \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + \chi(\varkappa_2) \right] \\
 &\quad - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}}.
 \end{aligned} \tag{3.4}$$

Equation (3.4) gives the desired result. □

Theorem 3.5 *Let $p \in \mathbb{R} \setminus \{0\}$, $s, q > 1$ with $s^{-1} + q^{-1} = 1$ and $\chi : \Omega^\circ \subset \mathbb{R} \rightarrow \mathbb{R}^{\dot{\alpha}}$ such that $\chi \in \mathcal{D}_{\dot{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω), $\chi^{(\dot{\alpha})} \in C_{\dot{\alpha}}[\varkappa_1, \varkappa_2]$ and $|\chi^{(\dot{\alpha})}|^q$ is a generalized p -convex function on Ω . Then one has*

$$\begin{aligned}
 &\left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + \chi(\varkappa_2) \right] \right. \\
 &\quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 &\leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\Lambda_1^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p \right) \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left[\Lambda_2^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p \right) \left| \chi^{(\dot{\alpha})}(\varkappa_1) \right|^q \right. \\
 &\quad \left. + \Lambda_3^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p \right) \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q \right]^{\frac{1}{q}} \\
 &\quad + \left(\Lambda_4^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) \right)^{1-\frac{1}{q}} \\
 &\quad \times \left[\Lambda_5^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q \right. \\
 &\quad \left. + \Lambda_6^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) \left| \chi^{(\dot{\alpha})}(\varkappa_2) \right|^q \right]^{\frac{1}{q}},
 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
 &\Lambda_1^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p \right) \\
 &:= \left(\frac{2}{3}\right)^{\dot{\alpha}} \left(\frac{p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} \left[\frac{(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \left\{ \left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{\frac{1}{p}} - \varkappa_1\right)^{\dot{\alpha}} \right. \right. \\
 &\quad \left. \left. - \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{\frac{1}{p}}\right)^{\dot{\alpha}} \right\} - \frac{6^{\dot{\alpha}} \Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p+1)\dot{\alpha})} \right]
 \end{aligned}$$

$$\times \left\{ \left(2^{\dot{\alpha}} \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(1+\frac{1}{p})} - \mathcal{K}_1^{\dot{\alpha}(p+1)} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} \right) \right\}, \tag{3.6}$$

$$\begin{aligned} & \Lambda_2^{(\dot{\alpha})} \left(\mathcal{K}_1, \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p}; p \right) \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\mathcal{K}_2^p - \mathcal{K}_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{((5\mathcal{K}_1^p + \mathcal{K}_2^p)(\mathcal{K}_1^p + \mathcal{K}_2^p))^{\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \left\{ \left(\left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{1/p} - \mathcal{K}_1 \right)^{\dot{\alpha}} \right. \right. \\ & \quad \left. \left. - \left(\left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p} - \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} - \frac{8^{\dot{\alpha}}(2\mathcal{K}_1^p + \mathcal{K}_2^p)\Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p + 1)\dot{\alpha})} \right. \\ & \quad \times \left\{ 2^{\dot{\alpha}} \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(1+1/p)} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \mathcal{K}_1^{\dot{\alpha}(p+1)} \right\} + \frac{(12^{\dot{\alpha}})\Gamma(1 + 2p\dot{\alpha})}{\Gamma(1 + (2p + 1)\dot{\alpha})} \\ & \quad \left. \times \left\{ 2^{\dot{\alpha}} \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(2+1/p)} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(2+1/p)} - \mathcal{K}_1^{\dot{\alpha}(2p+1)} \right\} \right], \tag{3.7} \end{aligned}$$

$$\begin{aligned} & \Lambda_3^{(\dot{\alpha})} \left(\mathcal{K}_1, \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p}; p \right) \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\mathcal{K}_2^p - \mathcal{K}_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{(11\mathcal{K}_1^p + \mathcal{K}_2^p)\Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(1+1/p)} \right. \right. \\ & \quad \left. \left. - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \mathcal{K}_1^{\dot{\alpha}(p+1)} \right\} - \frac{(6^{\dot{\alpha}})\Gamma(1 + 2p\dot{\alpha})}{\Gamma(1 + (2p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(2+1/p)} \right. \right. \\ & \quad \left. \left. - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(2+1/p)} - \mathcal{K}_1^{\dot{\alpha}(2p+1)} \right\} - \frac{(\mathcal{K}_1^{\dot{\alpha}}(5\mathcal{K}_1^p + \mathcal{K}_2^p))}{\Gamma(1 + \dot{\alpha})} \left\{ \left(\left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{1/p} - \mathcal{K}_1 \right)^{\dot{\alpha}} \right. \right. \\ & \quad \left. \left. - \left(\left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p} - \left(\frac{5\mathcal{K}_1^p + \mathcal{K}_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} \right], \tag{3.8} \end{aligned}$$

$$\begin{aligned} & \Lambda_4^{(\dot{\alpha})} \left(\left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p}, \mathcal{K}_2; p \right) \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\mathcal{K}_2^p - \mathcal{K}_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{(\mathcal{K}_1^p + 5\mathcal{K}_2^p)^{\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \left\{ \left(\left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{\frac{1}{p}} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\frac{1}{p}} \right)^{\dot{\alpha}} \right. \right. \\ & \quad \left. \left. - \left(\mathcal{K}_2 - \left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} - \frac{6^{\dot{\alpha}}\Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p + 1)\dot{\alpha})} \right. \\ & \quad \left. \times \left\{ 2^{\dot{\alpha}} \left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(1+\frac{1}{p})} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(1+\frac{1}{p})} - \mathcal{K}_2^{\dot{\alpha}(p+1)} \right\} \right], \tag{3.9} \end{aligned}$$

$$\begin{aligned} & \Lambda_5^{(\dot{\alpha})} \left(\left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p}, \mathcal{K}_2; p \right) \\ &= \left(\frac{2}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\mathcal{K}_2^p - \mathcal{K}_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{6^{\dot{\alpha}}\Gamma(1 + 2p\dot{\alpha})}{\Gamma(1 + (2p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(2+1/p)} - \mathcal{K}_2^{\dot{\alpha}(2p+1)} \right. \right. \\ & \quad \left. \left. - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(2+1/p)} \right\} - \frac{(\mathcal{K}_1^p + 11\mathcal{K}_2^p)^{\dot{\alpha}}\Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{\dot{\alpha}(1+1/p)} \right. \right. \\ & \quad \left. \left. - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \mathcal{K}_1^{\dot{\alpha}(p+1)} \right\} - \frac{(\mathcal{K}_2^{\dot{\alpha}}(\mathcal{K}_1^p + 5\mathcal{K}_2^p))^{\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \right. \\ & \quad \left. \times \left\{ \left(\left(\frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{1/p} - \left(\frac{\mathcal{K}_1^p + \mathcal{K}_2^p}{2} \right)^{1/p} \right)^{\dot{\alpha}} - \left(\left(\mathcal{K}_2 - \frac{\mathcal{K}_1^p + 5\mathcal{K}_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} \right], \tag{3.10} \end{aligned}$$

and

$$\begin{aligned}
 & \Lambda_6^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; p \right) \\
 & =: \left(\frac{2}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^2} \right)^{\dot{\alpha}} \\
 & \quad \times \left[\frac{(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}} (\varkappa_1^p + 5\varkappa_2^p)^{\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \left\{ \left(\left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{1/p} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} \right)^{\dot{\alpha}} \right. \right. \\
 & \quad \left. \left. - \left(\varkappa_2 - \left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} + \frac{8^{\dot{\alpha}} (\varkappa_1^p + 2\varkappa_2^p)^{\dot{\alpha}} \Gamma(1 + p\dot{\alpha})}{\Gamma(1 + (p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{\dot{\alpha}(1+1/p)} \right. \right. \\
 & \quad \left. \left. - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \varkappa_2^{\dot{\alpha}(p+1)} \right\} - \frac{(12)^{\dot{\alpha}} \Gamma(1 + 2p\dot{\alpha})}{\Gamma(1 + (2p + 1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{\dot{\alpha}(2+1/p)} \right. \right. \\
 & \quad \left. \left. - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}(2+1/p)} - \varkappa_1^{\dot{\alpha}(2p+1)} \right\} \right]. \tag{3.11}
 \end{aligned}$$

Proof It follows from Lemma 3.4 and the generalized power mean inequality that

$$\begin{aligned}
 & \left| \left(\frac{1}{6} \right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\
 & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right. \\
 & \quad \times \left| \chi^{(\dot{\alpha})} \left(\left[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) \right| (d\xi)^{\dot{\alpha}} \\
 & \quad + \frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} \\
 & \quad \times \left| \chi^{(\dot{\alpha})} \left(\left[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p \right]^{1/p} \right) \right| (d\xi)^{\dot{\alpha}} \Bigg] \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{1-\frac{1}{q}} \right. \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right. \\
 & \quad \times \left| \chi^{(\dot{\alpha})} \left(\left[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) \right|^q (d\xi)^{\dot{\alpha}} \Bigg]^{\frac{1}{q}} \\
 & \quad + \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{1-\frac{1}{q}} \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} \right. \\
 & \quad \times \left| \chi^{(\dot{\alpha})} \left(\left[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p \right]^{1/p} \right) \right|^q (d\xi)^{\dot{\alpha}} \Bigg]^{\frac{1}{q}}.
 \end{aligned}$$

Making use of the generalized p -convexity of $|\chi^{(\dot{\alpha})}|^q$ on Ω , we have

$$\begin{aligned}
 & \left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + \chi(\varkappa_2) \right] \right. \\
 & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{1 - \frac{1}{q}} \right. \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \right. \\
 & \quad \times \left. \left[(1 - \xi)^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\varkappa_1)|^q + \xi^{\dot{\alpha}} \left| \chi^{(\dot{\alpha})}\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) \right|^q \right] (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \\
 & \quad + \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{1 - \frac{1}{q}} \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} \right. \\
 & \quad \times \left. \left[(1 - \xi)^{\dot{\alpha}} \left| \chi^{(\dot{\alpha})}\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) \right|^q + \xi^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\varkappa_2)|^q \right] (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \Big] \\
 & = \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\Lambda_1^{(\dot{\alpha})}\left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p\right)\right)^{1 - \frac{1}{q}} \right. \\
 & \quad \times \left[\Lambda_2^{(\dot{\alpha})}\left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p\right) |\chi^{(\dot{\alpha})}(\varkappa_1)|^q \right. \\
 & \quad + \Lambda_3^{(\dot{\alpha})}\left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p\right) \left| \chi^{(\dot{\alpha})}\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) \right|^q \Big]^{\frac{1}{q}} \\
 & \quad + \left(\Lambda_4^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right)\right)^{1 - \frac{1}{q}} \\
 & \quad \times \left[\Lambda_5^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right) \left| \chi^{(\dot{\alpha})}\left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) \right|^q \right. \\
 & \quad \left. + \Lambda_6^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right) |\chi^{(\dot{\alpha})}(\varkappa_2)|^q \right]^{\frac{1}{q}} \Big]. \tag{3.12}
 \end{aligned}$$

We use Lemma 2.5 and the facts that

$$\begin{aligned}
 & \Lambda_1^{(\dot{\alpha})}\left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; p\right) \\
 & =: \frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \\
 & = \frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^{\frac{1}{3}} \frac{(\frac{1}{3} - \xi)^{\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(1+\dot{\alpha})} \int_{\frac{1}{3}}^1 \frac{(\xi - \frac{1}{3})^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \\
 = & \left(\frac{2}{3}\right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^2}\right)^{\dot{\alpha}} \left[\int_{\varkappa_1}^{(\frac{5\varkappa_1^p + \varkappa_2^p}{6})^{1/p}} [(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}} - 6^{\dot{\alpha}} u^{\dot{\alpha}p}] (du)^{\dot{\alpha}} \right. \\
 & \left. + \int_{(\frac{5\varkappa_1^p + \varkappa_2^p}{6})^{1/p}}^{(\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}} [6^{\dot{\alpha}} u^{\dot{\alpha}p} - (\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}}] (du)^{\dot{\alpha}} \right] \\
 = & \left(\frac{2}{3}\right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^2}\right)^{\dot{\alpha}} \left[\frac{(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}}}{\Gamma(1+\dot{\alpha})} \left\{ \left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\frac{1}{p}} - \varkappa_1 \right)^{\dot{\alpha}} \right. \right. \\
 & \left. \left. - \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\frac{1}{p}} \right)^{\dot{\alpha}} \right\} \right. \\
 & \left. - \frac{6^{\dot{\alpha}} \Gamma(1+p\dot{\alpha})}{\Gamma(1+(p+1)\dot{\alpha})} \left\{ \left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\dot{\alpha}(1+\frac{1}{p})} - \varkappa_1^{\dot{\alpha}(p+1)} \right) \right. \right. \\
 & \left. \left. - \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\dot{\alpha}(1+\frac{1}{p})} \right) \right\} \right], \tag{3.13}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_2^{(\dot{\alpha})} & \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; p \right) \\
 = & \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}} (1-\xi)^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \\
 = & \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^{1/3} \frac{(\frac{1}{3} - \xi)^{\dot{\alpha}} (1-\xi)^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \\
 & + \frac{1}{\Gamma(1+\dot{\alpha})} \int_{1/3}^1 \frac{(\xi - \frac{1}{3})^{\dot{\alpha}} (1-\xi)^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \\
 = & \left(\frac{2}{3}\right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^2}\right)^{\dot{\alpha}} \left[\frac{((5\varkappa_1^p + \varkappa_2^p)(\varkappa_1^p + \varkappa_2^p))^{\dot{\alpha}}}{\Gamma(1+\dot{\alpha})} \left\{ \left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{1/p} - \varkappa_1 \right)^{\dot{\alpha}} \right. \right. \\
 & \left. \left. - \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{1/p} \right)^{\dot{\alpha}} \right\} - \frac{8^{\dot{\alpha}} (2\varkappa_1^p + \varkappa_2^p) \Gamma(1+p\dot{\alpha})}{\Gamma(1+(p+1)\dot{\alpha})} \right. \\
 & \left. \times \left\{ 2^{\dot{\alpha}} \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\dot{\alpha}(1+1/p)} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}(1+1/p)} - \varkappa_1^{\dot{\alpha}(p+1)} \right\} \right. \\
 & \left. + \frac{(12^{\dot{\alpha}}) \Gamma(1+2p\dot{\alpha})}{\Gamma(1+(2p+1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{\dot{\alpha}(2+1/p)} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}(2+1/p)} - \varkappa_1^{\dot{\alpha}(2p+1)} \right\} \right], \tag{3.14}
 \end{aligned}$$

and

$$\begin{aligned}
 \Lambda_3^{(\dot{\alpha})} & \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; p \right) \\
 = & \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{\dot{\alpha}} \xi^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^{1/3} \frac{(\frac{1}{3}-\xi)^{\dot{\alpha}} \xi^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \\
 &\quad + \frac{1}{\Gamma(1+\dot{\alpha})} \int_{1/3}^1 \frac{(\xi-\frac{1}{3})^{\dot{\alpha}} \xi^{\dot{\alpha}} (d\xi)^{\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{\dot{\alpha}(1-1/p)}} \\
 &= \left(\frac{2}{3}\right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^2}\right)^{\dot{\alpha}} \left[\frac{(11\varkappa_1^p + \varkappa_2^p)\Gamma(1+p\dot{\alpha})}{\Gamma(1+(p+1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{\dot{\alpha}(1+1/p)} \right. \right. \\
 &\quad \left. \left. - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{\dot{\alpha}(1+1/p)} - \varkappa_1^{\dot{\alpha}(p+1)} \right\} - \frac{(6^{\dot{\alpha}})\Gamma(1+2p\dot{\alpha})}{\Gamma(1+(2p+1)\dot{\alpha})} \left\{ 2^{\dot{\alpha}} \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{\dot{\alpha}(2+1/p)} \right. \right. \\
 &\quad \left. \left. - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{\dot{\alpha}(2+1/p)} - \varkappa_1^{\dot{\alpha}(2p+1)} \right\} - \frac{(\varkappa_1^{\dot{\alpha}}(5\varkappa_1^p + \varkappa_2^p))}{\Gamma(1+\dot{\alpha})} \right. \\
 &\quad \left. \times \left\{ \left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{1/p} - \varkappa_1 \right)^{\dot{\alpha}} - \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6}\right)^{1/p} \right)^{\dot{\alpha}} \right\} \right], \tag{3.15}
 \end{aligned}$$

where we have used the identities

$$\begin{aligned}
 \Lambda_4^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}}}{[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}}, \\
 \Lambda_5^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}} (1-\xi)^{\dot{\alpha}}}{[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}}, \tag{3.16} \\
 \Lambda_6^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\xi - \frac{2}{3}|^{\dot{\alpha}} \xi^{\dot{\alpha}}}{[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}}.
 \end{aligned}$$

Substituting (3.14)–(3.16) in (3.12), we conclude the immediate consequence (3.5). This completes the proof. □

Remark 3.6 Theorem 3.5 leads to the following conclusions:

- (1) Let $\dot{\alpha} = 1$. Then we get Theorem 2.1 of [29].
- (2) Let $\dot{\alpha} = p = 1$. Then we get Corollary 10 of [71] if we choose $s = 1$, which also coincides with the first part of Corollary 2.1 of [29].
- (3) Let $\dot{\alpha} = 1$ and $p = -1$. Then we get the second part of Corollary 2.1 of [29].

Theorem 3.7 Let $p \in \mathbb{R} \setminus \{0\}$, $s, q > 1$ with $s^{-1} + q^{-1} = 1$ and $\chi : \Omega^\circ \subset \mathbb{R} \rightarrow \mathbb{R}^{\dot{\alpha}}$ such that $\chi \in \mathcal{D}_{\dot{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω), $\chi^{(\dot{\alpha})} \in C_{\dot{\alpha}}[\varkappa_1, \varkappa_2]$ and $|\chi^{(\dot{\alpha})}|^q$ is generalized p -convex on Ω . Then

$$\begin{aligned}
 &\left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p} \right) + \chi(\varkappa_2) \right] - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} {}_{\varkappa_1} \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 &\leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; s, p \right) \right)^{\frac{1}{s}} \right. \\
 &\quad \left. \times \left[\frac{2^{\dot{\alpha}} \Gamma(1+\dot{\alpha})}{\Gamma(1+2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})} \left(\left| \chi^{(\dot{\alpha})}(\varkappa_1) \right|^q, \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q \right) \right]^{\frac{1}{q}} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \right)^{\frac{1}{s}} \\
 & \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})} \left(\left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} \right|^q, \left| \chi^{(\dot{\alpha})}(\varkappa_2) \right|^q \right) \right]^{\frac{1}{q}}, \tag{3.17}
 \end{aligned}$$

where

$$\begin{aligned}
 & \Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; s, p \right) \\
 & := 6^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{(2-s)+s/p} - \varkappa_1^{p(2-s)+s} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \left. + \frac{\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-s)+s/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\} \\
 & \quad - (5\varkappa_1^p + \varkappa_2^p)^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{(1-s)+s/p} - \varkappa_1^{p(1-s)+s} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \left. + \frac{\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-s)+s/p} - \left(\frac{5\varkappa_1^p + \varkappa_2^p}{6} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\}, \tag{3.18}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \\
 & := (\varkappa_1^p + 5\varkappa_2^p)^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{(1-s)+s/p} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \left. + \frac{\varkappa_2^{p(1-s)+s} - \left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\} \\
 & \quad - 6^{s\dot{\alpha}} \left\{ \frac{\left[\left(\left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{(2-s)+s/p} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}} \right]}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \left. + \frac{\left(\varkappa_2^{p(2-s)+s} - \left(\frac{\varkappa_1^p + 5\varkappa_2^p}{6} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\}, \tag{3.19}
 \end{aligned}$$

and $\mathbb{A} = \frac{\varkappa_1 + \varkappa_2}{2}$ denotes the arithmetic mean.

Proof It follows from Lemma 3.4 and the generalized Hölder inequality that

$$\begin{aligned}
 & \left| \left(\frac{1}{6} \right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\
 & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{s\dot{\alpha}}}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \right. \\
 & \quad \left. \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \left| \chi^{(\dot{\alpha})} \left(\left[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) \right|^q (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{(\xi - \frac{2}{3})^{s\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \\
 & \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \left| \chi^{(\dot{\alpha})} \left(\left[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p \right]^{1/p} \right) \right|^q (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}}.
 \end{aligned}$$

From the generalized p -convexity of $|\chi^{(\dot{\alpha})}|^q$ on Ω , we have

$$\begin{aligned}
 & \left| \left(\frac{1}{6} \right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\
 & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \mathcal{I}_{\varkappa_1 \varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{s\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \right. \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \left[(1 - \xi)^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\varkappa_1)|^q + \xi^{\dot{\alpha}} \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right) \right|^{1/p} \right]^q (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \\
 & \quad + \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{(\xi - \frac{2}{3})^{s\dot{\alpha}}}{[(1 - \xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \\
 & \quad \times \left(\frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \left[(1 - \xi)^{\dot{\alpha}} \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right) \right|^{1/p} \right]^q + \xi^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\varkappa_2)|^q \right] (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \\
 & = \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; s, p \right) \right)^{\frac{1}{s}} \right. \\
 & \quad \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \left(\frac{|\chi^{(\dot{\alpha})}(\varkappa_1)|^q + |\chi^{(\dot{\alpha})}(\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}|^q}{2^{\dot{\alpha}}} \right) \right]^{\frac{1}{q}} \\
 & \quad + \left(\Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \right)^{\frac{1}{s}} \\
 & \quad \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \left(\frac{|\chi^{(\dot{\alpha})}(\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}|^q + |\chi^{(\dot{\alpha})}(\varkappa_2)|^q}{2^{\dot{\alpha}}} \right) \right]^{\frac{1}{q}} \\
 & = \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; s, p \right) \right)^{\frac{1}{s}} \right. \\
 & \quad \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})} \left(|\chi^{(\dot{\alpha})}(\varkappa_1)|^q, \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right) \right|^{1/p} \right) \right]^{\frac{1}{q}} \\
 & \quad + \left(\Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \right)^{\frac{1}{s}} \\
 & \quad \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})} \left(\left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right) \right|^{1/p}, |\chi^{(\dot{\alpha})}(\varkappa_2)|^q \right) \right]^{\frac{1}{q}} \Big]. \tag{3.20}
 \end{aligned}$$

Making use of Lemma 2.5 and using the change of variable technique, we get

$$\begin{aligned}
 & \Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; s, p \right) \\
 & := \frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^1 \frac{|\xi - \frac{1}{3}|^{s\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \\
 & = \frac{1}{\Gamma(1 + \dot{\alpha})} \int_0^{\frac{1}{3}} \frac{(\frac{1}{3} - \xi)^{s\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \\
 & \quad + \frac{1}{\Gamma(1 + \dot{\alpha})} \int_{\frac{1}{3}}^1 \frac{(\xi - \frac{1}{3})^{s\dot{\alpha}}}{[(1 - \xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{s\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \\
 & = \left(\frac{2^s}{3} \right)^{\dot{\alpha}} \left(\frac{p}{(\varkappa_2^p - \varkappa_1^p)^{2s}} \right)^{\dot{\alpha}} \\
 & \quad \times \left[\frac{1}{\Gamma(1 + \dot{\alpha})} \int_{\varkappa_1}^{(\frac{5\varkappa_1 + \varkappa_2}{6})^{1/p}} \left(\frac{6^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-2p+1)}} - \frac{(5\varkappa_1^p + \varkappa_2^p)^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-p+1)}} \right) (du)^{\dot{\alpha}} \right. \\
 & \quad \left. + \frac{1}{\Gamma(1 + \dot{\alpha})} \int_{(\frac{5\varkappa_1 + \varkappa_2}{6})^{1/p}}^{\varkappa_2} \left(\frac{(5\varkappa_1^p + \varkappa_2^p)^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-p+1)}} - \frac{6^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-2p+1)}} \right) (du)^{\dot{\alpha}} \right]. \tag{3.21}
 \end{aligned}$$

Now, choosing $\frac{1}{u^{ps-s-2p}} = \vartheta$ and using $\frac{(du)^{\dot{\alpha}}}{u^{ps-s-2p-1}} = \frac{(d\vartheta)^{\dot{\alpha}}}{p(2-s)+s}$, we obtain

$$\begin{aligned}
 & \frac{1}{\Gamma(1 + \dot{\alpha})} \int_{\varkappa_1}^{(\frac{5\varkappa_1 + \varkappa_2}{6})^{1/p}} \frac{6^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-2p+1)}} (du)^{\dot{\alpha}} \\
 & = \frac{6^{s\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \int_{\varkappa_1^{p(2-s)+s}}^{(\frac{5\varkappa_1 + \varkappa_2}{6})^{(2-s)+s/p}} (d\vartheta)^{\dot{\alpha}} \\
 & = \frac{6^{s\dot{\alpha}} \left((\frac{5\varkappa_1 + \varkappa_2}{6})^{(2-s)+s/p} - \varkappa_1^{p(2-s)+s} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \tag{3.22}
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{1}{\Gamma(1 + \dot{\alpha})} \int_{(\frac{5\varkappa_1 + \varkappa_2}{6})^{1/p}}^{\varkappa_2} \frac{6^{s\dot{\alpha}}}{u^{\dot{\alpha}(ps-s-2p+1)}} (du)^{\dot{\alpha}} \\
 & = \frac{6^{s\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \int_{(\frac{5\varkappa_1 + \varkappa_2}{6})^{(2-s)+s/p}}^{(\frac{\varkappa_1 + \varkappa_2}{2})^{(2-s)+s/p}} (d\vartheta)^{\dot{\alpha}} \\
 & = \frac{6^{s\dot{\alpha}} \left((\frac{\varkappa_1 + \varkappa_2}{2})^{(2-s)+s/p} - (\frac{5\varkappa_1 + \varkappa_2}{6})^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}. \tag{3.23}
 \end{aligned}$$

Again, choosing $\frac{1}{u^{ps-s-p}} = \vartheta$ and using $\frac{(du)^{\dot{\alpha}}}{u^{ps-s-p+1}} = \frac{(d\vartheta)^{\dot{\alpha}}}{p(1-s)+s}$, we get

$$\begin{aligned}
 & \frac{(5\varkappa_1^p + \varkappa_2^p)^{s\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \int_{\varkappa_1}^{(\frac{5\varkappa_1 + \varkappa_2}{6})^{1/p}} \frac{1}{u^{\dot{\alpha}(ps-s-p+1)}} (du)^{\dot{\alpha}} \\
 & = \frac{(5\varkappa_1^p + \varkappa_2^p)^{s\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \int_{\varkappa_1^{p(1-s)+s}}^{(\frac{5\varkappa_1 + \varkappa_2}{6})^{(1-s)+s/p}} (d\vartheta)^{\dot{\alpha}}
 \end{aligned}$$

$$= \frac{(5\mathcal{I}_1^p + \mathcal{I}^p)^{s\dot{\alpha}} \left(\left(\frac{5\mathcal{I}_1 + \mathcal{I}_2}{6} \right)^{(1-s)+s/p} - \mathcal{I}_1^{p(1-s)+s} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \tag{3.24}$$

and

$$\begin{aligned} & \frac{(5\mathcal{I}_1^p + \mathcal{I}^p)^{s\dot{\alpha}}}{\Gamma(1 + \dot{\alpha})} \int_{\left(\frac{5\mathcal{I}_1 + \mathcal{I}_2}{6} \right)^{1/p}}^{\left(\frac{\mathcal{I}_1 + \mathcal{I}_2}{2} \right)^{1/p}} \frac{1}{u^{\dot{\alpha}(ps-s-p+1)}} (du)^{\dot{\alpha}} \\ &= \frac{(5\mathcal{I}_1^p + \mathcal{I}^p)^{s\dot{\alpha}}}{(p(1-s) + s)^{s\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \int_{\left(\frac{5\mathcal{I}_1 + \mathcal{I}_2}{6} \right)^{(1-s)+s/p}^{\left(\frac{\mathcal{I}_1 + \mathcal{I}_2}{2} \right)^{(1-s)+s/p}} (d\vartheta)^{\dot{\alpha}} \\ &= \frac{(5\mathcal{I}_1^p + \mathcal{I}^p)^{s\dot{\alpha}} \left(\left(\frac{\mathcal{I}_1 + \mathcal{I}_2}{2} \right)^{(1-s)+s/p} - \left(\frac{5\mathcal{I}_1 + \mathcal{I}_2}{6} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}. \end{aligned} \tag{3.25}$$

Substituting (3.22)–(3.25), in (3.21) we get

$$\begin{aligned} & \Lambda_7^{(\dot{\alpha})} \left(\mathcal{I}_1, \left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{1/p}; s, p \right) \\ &:= 6^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{5\mathcal{I}_1^p + \mathcal{I}_2^p}{6} \right)^{(2-s)+s/p} - \mathcal{I}_1^{p(2-s)+s} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\ & \quad \left. + \frac{\left(\left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{(2-s)+s/p} - \left(\frac{5\mathcal{I}_1^p + \mathcal{I}_2^p}{6} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\} \\ & \quad - (5\mathcal{I}_1^p + \mathcal{I}^p)^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{5\mathcal{I}_1^p + \mathcal{I}_2^p}{6} \right)^{(1-s)+s/p} - \mathcal{I}_1^{p(1-s)+s} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\ & \quad \left. + \frac{\left(\left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{(1-s)+s/p} - \left(\frac{5\mathcal{I}_1^p + \mathcal{I}_2^p}{6} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\}. \end{aligned}$$

Analogously,

$$\begin{aligned} & \Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{1/p}, \mathcal{I}_2; s, p \right) \\ &:= (\mathcal{I}_1^p + 5\mathcal{I}^p)^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{\mathcal{I}_1^p + 5\mathcal{I}_2^p}{6} \right)^{(1-s)+s/p} - \left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\ & \quad \left. + \frac{\mathcal{I}_2^{p(1-s)+s} - \left(\frac{\mathcal{I}_1^p + 5\mathcal{I}_2^p}{6} \right)^{(1-s)+s/p} \right)^{\dot{\alpha}}}{(p(1-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\} \\ & \quad - 6^{s\dot{\alpha}} \left\{ \frac{\left(\left(\frac{\mathcal{I}_1^p + 5\mathcal{I}_2^p}{6} \right)^{(2-s)+s/p} - \left(\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\ & \quad \left. - \frac{\left(\mathcal{I}_2^{p(2-s)+s} - \left(\frac{\mathcal{I}_1^p + 5\mathcal{I}_2^p}{6} \right)^{(2-s)+s/p} \right)^{\dot{\alpha}}}{(p(2-s) + s)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right\}. \end{aligned}$$

Putting the values of $\Lambda_7^{(\dot{\alpha})}(\mathcal{I}_1, (\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2})^{1/p}; s, p)$ and $\Lambda_8^{(\dot{\alpha})}((\frac{\mathcal{I}_1^p + \mathcal{I}_2^p}{2})^{1/p}, \mathcal{I}_2; s, p)$ in (3.20), we get the desired inequality (3.17). This completes the proof. \square

Remark 3.8 Theorem 3.7 leads to the following conclusions:

- (1) If $\dot{\alpha} = 1$, then we get Theorem 2.2 of [29].

- (2) If $\dot{\alpha} = p = 1$, then we get Theorem 8 of [28] when we choose $s = 1$, which coincides with the first part of Corollary 2.2 of [29].
- (3) If $\dot{\alpha} = 1$ and $p = -1$, then we get the second part of Corollary 2.2 of [29].

Theorem 3.9 *Let $p \in \mathbb{R} \setminus \{0\}$, $s, q > 1$ with $s^{-1} + q^{-1} = 1$ and $\chi : \Omega^\circ \subset \mathbb{R} \rightarrow \mathbb{R}^{\dot{\alpha}}$ be a function such that $\chi \in \mathcal{D}_{\dot{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω), $\chi^{(\dot{\alpha})} \in \mathcal{C}_{\dot{\alpha}}[\varkappa_1, \varkappa_2]$ and $|\chi^{(\dot{\alpha})}|^q$ is a generalized p -convex function on Ω . Then*

$$\begin{aligned}
 & \left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\
 & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\
 & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left[\frac{\Gamma(1 + \dot{\alpha}s)}{\Gamma(1 + (s+1)\dot{\alpha})} \right]^{\frac{1}{s}} \left[\Theta_1^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \left| \chi^{(\dot{\alpha})}(\varkappa_1) \right|^q \right. \right. \\
 & \quad \left. \left. + \Theta_2^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \left| \chi^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} \right) \right|^q \right]^{\frac{1}{q}} \\
 & \quad + \left[\frac{\Gamma(1 + \dot{\alpha}s)}{\Gamma(1 + (s+1)\dot{\alpha})} \right]^{\frac{1}{s}} \left[\Theta_3^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; q, p \right) \left| \chi^{(\dot{\alpha})} \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} \right|^q \right. \\
 & \quad \left. \left. + \Theta_4^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; q, p \right) \right]^{\frac{1}{q}} \left| \chi^{(\dot{\alpha})}(\varkappa_2) \right|^q \right], \tag{3.26}
 \end{aligned}$$

where

$$\begin{aligned}
 & \Theta_1^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \\
 & := \left(\frac{2p}{(\varkappa_2^p - \varkappa_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}}}{(p(1-q) + q)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \times \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-q)+q/p} - \varkappa_1^{p(1-q)+q} \right)^{\dot{\alpha}} \\
 & \quad \left. - \frac{2^{\dot{\alpha}}}{(p(2-q) + q)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-q)+q/p} - \varkappa_1^{p(2-q)+q} \right)^{\dot{\alpha}} \right], \tag{3.27}
 \end{aligned}$$

$$\begin{aligned}
 & \Theta_2^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \\
 & := \left(\frac{4p}{(\varkappa_2^p - \varkappa_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{1}{(p(1-q) + q)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right. \\
 & \quad \times \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-q)+q/p} - \varkappa_1^{p(1-q)+q} \right)^{\dot{\alpha}} \\
 & \quad \left. - \frac{(\varkappa_1^p)^{\dot{\alpha}}}{(p(2-q) + q)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-q)+q/p} - \varkappa_1^{p(2-q)+q} \right)^{\dot{\alpha}} \right], \tag{3.28}
 \end{aligned}$$

$$\begin{aligned}
 & \Theta_3^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; \varkappa_2, q, p \right) \\
 & := \left(\frac{4p}{(\varkappa_2^p - \varkappa_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{(\varkappa_2^p)^{\dot{\alpha}}}{(p(1-q) + q)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})} \right.
 \end{aligned}$$

$$\begin{aligned} & \times \left(\varkappa_2^{p(1-q)+q} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-q)+q/p} \right)^{\dot{\alpha}} \\ & - \frac{1}{(p(2-q)+q)^{\dot{\alpha}} \Gamma(1+\dot{\alpha})} \left(\varkappa_2^{p(1-q)+q} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-q)+q/p} \right)^{\dot{\alpha}} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} & \Theta_4^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; q, p \right) \\ & := \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \left[\frac{(2)^{\dot{\alpha}}}{(p(1-q)+q)^{\dot{\alpha}} \Gamma(1+\dot{\alpha})} \right. \\ & \quad \times \left(\varkappa_2^{p(1-q)+q} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(1-q)+q/p} \right)^{\dot{\alpha}} \\ & \quad \left. - \frac{(\varkappa_1^p + \varkappa_2^p)^{\dot{\alpha}}}{(p(2-q)+q)^{\dot{\alpha}} \Gamma(1+\dot{\alpha})} \left(\varkappa_2^{p(1-q)+q} - \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{(2-q)+q/p} \right)^{\dot{\alpha}} \right]. \end{aligned} \tag{3.30}$$

Proof It follows from Lemma 3.4 and the generalized Hölder inequality that

$$\begin{aligned} & \left| \left(\frac{1}{6} \right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\ & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_1 \varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \left| \xi - \frac{1}{3} \right|^{s\dot{\alpha}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \right. \\ & \quad \times \left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\chi^{(\dot{\alpha})}([(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{1/p})|^q}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{q\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \left(\xi - \frac{2}{3} \right)^{s\dot{\alpha}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \\ & \quad \left. \times \left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{|\chi^{(\dot{\alpha})}([(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{1/p})|^q}{[(1-\xi)\frac{\varkappa_1^p + \varkappa_2^p}{2} + \xi \varkappa_2^p]^{q\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

By the generalized p -convexity of $|\chi^{(\dot{\alpha})}|^q$ on Ω , we have

$$\begin{aligned} & \left| \left(\frac{1}{6} \right)^{\dot{\alpha}} \left[\chi(\varkappa_1) + (4)^{\dot{\alpha}} \chi \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2} \right]^{1/p} \right) + \chi(\varkappa_2) \right] \right. \\ & \quad \left. - \left(\frac{2p}{\varkappa_2^p - \varkappa_1^p} \right)^{\dot{\alpha}} \varkappa_1 \mathcal{I}_{\varkappa_1 \varkappa_2}^{(\dot{\alpha})} \frac{\chi(u)}{u^{\dot{\alpha}(1-p)}} \right| \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \left| \xi - \frac{1}{3} \right|^{s\dot{\alpha}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{s}} \right. \\ & \quad \times \left(\frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{[(1-\xi)^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\varkappa_1)|^q + \xi^{\dot{\alpha}} |\chi^{(\dot{\alpha})}(\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}|^q]}{[(1-\xi)\varkappa_1^p + \xi \frac{\varkappa_1^p + \varkappa_2^p}{2}]^{q\dot{\alpha}(1-1/p)}} (d\xi)^{\dot{\alpha}} \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\xi - \frac{2}{3} \right)^{s\alpha} (d\xi)^\alpha \right)^{\frac{1}{s}} \\
 & \times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{[(1-\xi)^\alpha |\chi^{(\alpha)}(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2})|^{1/p}]^q + \xi^\alpha |\chi^{(\alpha)}(\mathcal{x}_2)|^q}{[(1-\xi)\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} + \xi \mathcal{x}_2]^{s\alpha(1-1/p)}} (d\xi)^\alpha \right)^{\frac{1}{q}} \\
 & = \left(\frac{\mathcal{x}_2^p - \mathcal{x}_1^p}{4p} \right)^\alpha \left[(\Theta_1^{(\alpha)}(s)^{\frac{1}{s}}) \left[\Theta_2^{(\alpha)} \left(\mathcal{x}_1, \left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{1/p}; q, p \right) |\chi^{(\alpha)}(\mathcal{x}_1)|^q \right. \right. \\
 & \quad \left. \left. + \Theta_3^{(\alpha)} \left(\mathcal{x}_1, \left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{1/p}; q, p \right) \left| \chi^{(\alpha)} \left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right) \right|^q \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + (\Theta_1^{(\alpha)}(s)^{\frac{1}{s}}) \left[\Theta_4^{(\alpha)} \left(\left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{1/p}, \mathcal{x}_2; q, p \right) \left| \chi^{(\alpha)} \left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right) \right|^q \right. \right. \\
 & \quad \left. \left. + \Theta_5^{(\alpha)} \left(\left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{1/p}, \mathcal{x}_2; q, p \right) \right]^{\frac{1}{q}} |\chi^{(\alpha)}(\mathcal{x}_2)|^q \right]. \tag{3.31}
 \end{aligned}$$

From Lemma 2.5 we clearly see that

$$\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| \xi - \frac{1}{3} \right|^{s\alpha} (d\xi)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| \xi - \frac{2}{3} \right|^{s\alpha} (d\xi)^\alpha = \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}. \tag{3.32}$$

Again, using Lemma 2.5 and the change of the variable technique, we have

$$\begin{aligned}
 & \Theta_1^{(\alpha)} \left(\mathcal{x}_1, \left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{1/p}; q, p \right) \\
 & := \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{(1-\xi)^\alpha (d\xi)^\alpha}{[(1-\xi)\mathcal{x}_1^p + \xi \frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2}]^{q\alpha(1-1/p)}} \\
 & = \left(\frac{2p}{(\mathcal{x}_2^p - \mathcal{x}_1^p)^2} \right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_{\mathcal{x}_1}^{(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2})^{\frac{1}{p}}} \left(\frac{(\mathcal{x}_1^p + \mathcal{x}_2^p)^\alpha}{u^{\alpha(pq-q-p+1)}} - \frac{2^\alpha}{u^{\alpha(pq-q-2p+1)}} \right) (du)^\alpha. \tag{3.33}
 \end{aligned}$$

Now, choosing $\frac{1}{u^{pq-q-p}} = \vartheta$ and using $\frac{(du)^\alpha}{u^{pq-q-p+1}} = \frac{(d\vartheta)^\alpha}{(p(1-q)+q)^\alpha}$, we obtain

$$\begin{aligned}
 & \frac{(\mathcal{x}_1^p + \mathcal{x}_2^p)^\alpha}{(p(1-q)+q)^\alpha \Gamma(1+\alpha)} \int_{\mathcal{x}_1}^{(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2})^{\frac{1}{p}}} \frac{1}{u^{\alpha(pq-q-p+1)}} (du)^\alpha \\
 & = \frac{(\mathcal{x}_1^p + \mathcal{x}_2^p)^\alpha}{(p(1-q)+q)^\alpha \Gamma(1+\alpha)} \int_{\mathcal{x}_1^{(1-q)+q}}^{(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2})^{(1-q)+q/p}} (d\vartheta)^\alpha \\
 & = \frac{(\mathcal{x}_1^p + \mathcal{x}_2^p)^\alpha}{(p(1-q)+q)^\alpha \Gamma(1+\alpha)} \left(\left(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2} \right)^{(1-q)+q/p} - \mathcal{x}_1^{p(1-q)+q} \right)^\alpha. \tag{3.34}
 \end{aligned}$$

Again, choosing $\frac{1}{u^{pq-q-2p}} = \vartheta$ and from $\frac{(du)^\alpha}{u^{pq-q-2p+1}} = \frac{(d\vartheta)^\alpha}{(p(2-q)+q)^\alpha}$, we obtain

$$\frac{2^\alpha}{(p(2-q)+q)^\alpha \Gamma(1+\alpha)} \int_{\mathcal{x}_1}^{(\frac{\mathcal{x}_1^p + \mathcal{x}_2^p}{2})^{\frac{1}{p}}} \frac{1}{u^{\alpha(pq-q-p+1)}} (du)^\alpha$$

$$\begin{aligned}
 &= \frac{2^{\dot{\alpha}}}{(p(2-q)+q)^{\dot{\alpha}}\Gamma(1+\dot{\alpha})} \int_{\mathcal{I}_1^{p(2-q)+q}}^{(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2})^{(2-q)+q/p}} (d\vartheta)^{\dot{\alpha}} \\
 &= \frac{2^{\dot{\alpha}}}{(p(2-q)+q)^{\dot{\alpha}}\Gamma(1+\dot{\alpha})} \left(\left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{(2-q)+q/p} - \mathcal{I}_1^{p(2-q)+q} \right)^{\dot{\alpha}}. \tag{3.35}
 \end{aligned}$$

After substituting (3.34) and (3.35) in (3.33), we get

$$\begin{aligned}
 &\Theta_1^{(\dot{\alpha})} \left(\mathcal{I}_1, \left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{1/p}; q, p \right) \\
 &:= \left(\frac{2p}{(\mathcal{I}_2^p-\mathcal{I}_1^p)^2} \right)^{\dot{\alpha}} \left[\frac{(\mathcal{I}_1^p+\mathcal{I}_2^p)^{\dot{\alpha}}}{(p(1-q)+q)^{\dot{\alpha}}\Gamma(1+\dot{\alpha})} \right. \\
 &\quad \times \left(\left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{(1-q)+q/p} - \mathcal{I}_1^{p(1-q)+q} \right)^{\dot{\alpha}} \\
 &\quad \left. - \frac{2^{\dot{\alpha}}}{(p(2-q)+q)^{\dot{\alpha}}\Gamma(1+\dot{\alpha})} \left(\left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{(2-q)+q/p} - \mathcal{I}_1^{p(2-q)+q} \right)^{\dot{\alpha}} \right], \tag{3.36}
 \end{aligned}$$

where we have used the identities

$$\begin{aligned}
 \Theta_2^{(\dot{\alpha})} \left(\mathcal{I}_1, \left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{1/p}; q, p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{\xi^{\dot{\alpha}}(d\xi)^{\dot{\alpha}}}{[(1-\xi)\mathcal{I}_1^p+\xi\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2}]^{q\dot{\alpha}(1-1/p)}}, \\
 \Theta_3^{(\dot{\alpha})} \left(\left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{1/p}; \mathcal{I}_2, q, p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{(1-\xi)^{\dot{\alpha}}(d\xi)^{\dot{\alpha}}}{[(1-\xi)\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2}+\xi\mathcal{I}_2]^{q\dot{\alpha}(1-1/p)}}, \tag{3.37} \\
 \Theta_4^{(\dot{\alpha})} \left(\mathcal{I}_1, \left(\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2} \right)^{1/p}, \mathcal{I}_2; q, p \right) &:= \frac{1}{\Gamma(1+\dot{\alpha})} \int_0^1 \frac{\xi^{\dot{\alpha}}(d\xi)^{\dot{\alpha}}}{[(1-\xi)\frac{\mathcal{I}_1^p+\mathcal{I}_2^p}{2}+\xi\mathcal{I}_2]^{q\dot{\alpha}(1-1/p)}}.
 \end{aligned}$$

Substituting (3.32)–(3.37) in (3.31), we get the desired inequality (3.26). This completes the proof. □

Remark 3.10 Theorem 3.9 leads to the following conclusions:

- (1) Let $\dot{\alpha} = 1$. Then we get Theorem 2.3 of [29].
- (2) Let $\dot{\alpha} = p = 1$. Then we get the first part of Corollary 2.3 of [29].
- (3) Let $\dot{\alpha} = 1$ and $p = -1$. Then we get the second part of Corollary 2.3 of [29].

4 Applications

4.1 Probability density functions

Consider a random variable \mathcal{Z} whose generalized probability density function is $p : [\mathcal{I}_1, \mathcal{I}_2] \rightarrow [0^{\dot{\alpha}}, 1^{\dot{\alpha}}]$, which is generalized convex with the cumulative distribution function $\mathcal{F}_{\dot{\alpha}}$ defined by

$$P_{\dot{\alpha}}(\mathcal{Z} \leq x) = \mathcal{F}_{\dot{\alpha}}(u) := \frac{1}{\Gamma(1+\dot{\alpha})} \int_{\mathcal{I}_1}^u p(\xi)(d\xi)^{\dot{\alpha}}.$$

Moreover, the generalized expectation can be expressed as

$$E_{\dot{\alpha}} = \frac{1}{\Gamma(1+\dot{\alpha})} \int_{\mathcal{I}_1}^{\mathcal{I}_2} \xi^{\dot{\alpha}} p(\xi)(d\xi)^{\dot{\alpha}}.$$

We clearly see that

$$E_{\dot{\alpha}}(u) = \varkappa_2^{\dot{\alpha}} - \frac{1}{\Gamma(1 + \dot{\alpha})} \int_{\varkappa_1}^{\varkappa_2} \mathcal{F}_{\dot{\alpha}}(\xi)(d\xi)^{\dot{\alpha}}.$$

The following two propositions can be obtained from our results obtained in Sect. 3 immediately.

Proposition 4.1 *Let $p \in \mathbb{R} \setminus \{0\}$. Then Theorem 3.5 leads to*

$$\begin{aligned} & \left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[P_{\dot{\alpha}}(\mathcal{Y} \leq \varkappa_1) + (4)^{\dot{\alpha}} P_{\dot{\alpha}}(\mathcal{Y} \leq \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + P_{\dot{\alpha}}(\mathcal{Y} \leq \varkappa_2) \right] \right. \\ & \quad \left. - \left(\frac{2}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} (\varkappa_2 - E_{\dot{\alpha}}(\mathcal{Y})) \right| \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\Lambda_1^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_1; p\right)\right)^{1-\frac{1}{q}} \right. \\ & \quad \times \left[\Lambda_2^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_1; p\right) \left| \mathfrak{p}(\varkappa_1) \right|^q \right. \\ & \quad \left. + \Lambda_3^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_1; p\right) \left| \mathfrak{p}\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q \right]^{\frac{1}{q}} \\ & \quad + \left(\Lambda_4^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right)\right)^{1-\frac{1}{q}} \\ & \quad \times \left[\Lambda_5^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right) \left| \mathfrak{p}\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q \right. \\ & \quad \left. + \Lambda_6^{(\dot{\alpha})}\left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}, \varkappa_2; p\right) \left| \mathfrak{p}(\varkappa_2) \right|^q \right]^{\frac{1}{q}}, \end{aligned} \tag{4.1}$$

where $\Lambda_1^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; p)$, $\Lambda_2^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_1; p)$, $\Lambda_3^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; p)$, $\Lambda_4^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; p)$, $\Lambda_5^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; p)$, and $\Lambda_6^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; p)$ are given in (3.6)–(3.10), respectively.

Proposition 4.2 *Let $p \in \mathbb{R} \setminus \{0\}$. Then Theorem 3.7 leads to the conclusion that*

$$\begin{aligned} & \left| \left(\frac{1}{6}\right)^{\dot{\alpha}} \left[P_{\dot{\alpha}}(\mathcal{Y} \leq \varkappa_1) + (4)^{\dot{\alpha}} P_{\dot{\alpha}}(\mathcal{Y} \leq \left(\left[\frac{\varkappa_1^p + \varkappa_2^p}{2}\right]^{1/p}\right) + P_{\dot{\alpha}}(\mathcal{Y} \leq \varkappa_2) \right] \right. \\ & \quad \left. - \left(\frac{2}{\varkappa_2^p - \varkappa_1^p}\right)^{\dot{\alpha}} (\varkappa_2 - E_{\dot{\alpha}}(\mathcal{Y})) \right| \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p}\right)^{\dot{\alpha}} \left[\left(\Lambda_7^{(\dot{\alpha})}\left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p}; s, p\right)\right)^{\frac{1}{s}} \right. \\ & \quad \times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})}\left(\left| \mathfrak{p}(\varkappa_1) \right|^q, \left| \mathfrak{p}\left(\frac{\varkappa_1^p + \varkappa_2^p}{2}\right)^{1/p} \right|^q\right) \right]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &+ \left(\Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \right)^{\frac{1}{s}} \\
 &\times \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \mathbb{A}^{(\dot{\alpha})} \left(\left| p \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p} \right|^q, |p(\varkappa_2)|^q \right) \right]^{\frac{1}{q}},
 \end{aligned}$$

where $\Lambda_7^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; s, p)$ and $\Lambda_8^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; s, p)$ are given in (3.18) and (3.19), respectively.

4.2 Generalized special means

For $\varkappa_1 < \varkappa_2$ and $\varkappa_1, \varkappa_2 \in \mathbb{R}^{\dot{\alpha}}$, considering the following $\dot{\alpha}$ -type special means [72–80].

I. The generalized arithmetic mean:

$$\mathcal{A}_{\dot{\alpha}}(\varkappa_1, \varkappa_2) := \left(\frac{\varkappa_1 + \varkappa_2}{2} \right)^{\dot{\alpha}} = \frac{\varkappa_1^{\dot{\alpha}} + \varkappa_2^{\dot{\alpha}}}{2^{\dot{\alpha}}}.$$

II. The generalized weighted arithmetic mean:

$$\mathcal{A}_{\dot{\alpha}}(\varkappa_1, \varkappa_2; w_1, w_2) := \frac{w_1^{\dot{\alpha}} \varkappa_1^{\dot{\alpha}} + w_2^{\dot{\alpha}} \varkappa_2^{\dot{\alpha}}}{w_1^{\dot{\alpha}} + w_2^{\dot{\alpha}}} \quad (w_1^{\dot{\alpha}}, w_2^{\dot{\alpha}} \in \mathbb{R}^{\dot{\alpha}}).$$

III. The generalized logarithmic mean:

$$\mathcal{L}_{\dot{\alpha}}(\varkappa_1, \varkappa_2) := \left[\frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n + 1)\dot{\alpha})} \frac{\varkappa_2^{\dot{\alpha}+1} - \varkappa_1^{\dot{\alpha}+1}}{(\varkappa_2 - \varkappa_1)^{\dot{\alpha}}} \right] \quad (n \in \mathbb{Z} \setminus \{-1, 0\}; \varkappa_1, \varkappa_2 \in \mathbb{R}, \varkappa_1 \neq \varkappa_2).$$

Let $\chi(u) = u^{n\dot{\alpha}}$ ($u \in \mathbb{R}; n \in \mathbb{Z}, |n| \geq 2$). Then Theorems 3.7 and 3.9 lead to Propositions 4.3 and 4.4 immediately.

Proposition 4.3 *Let $\varkappa_1, \varkappa_2 \in \mathbb{R}$ with $\varkappa_1 < \varkappa_2$, $0 \notin [\varkappa_1, \varkappa_2]$, $p \in \mathbb{R} \setminus \{0\}$, and $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned}
 &\left| \left[\mathcal{A}_{\dot{\alpha}}^n \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\frac{1}{p}}, \varkappa_2; 1^{\dot{\alpha}}, 4^{\dot{\alpha}}, 1^{\dot{\alpha}} \right) \right] - (2p)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha}) \mathcal{L}_{\dot{\alpha}(p+(n-1))} \right| \\
 &\leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left(\Lambda_7^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; s, p \right) \right)^{\frac{1}{s}} \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \right. \right. \\
 &\quad \times \mathbb{A}^{(\dot{\alpha})} \left(\frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \varkappa_1^{q\dot{\alpha}(n-1)} \right|, \frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\dot{\alpha}q(n-1)/p} \right| \right) \right]^{\frac{1}{q}} \\
 &+ \left(\Lambda_8^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; s, p \right) \right)^{\frac{1}{s}} \left[\frac{2^{\dot{\alpha}} \Gamma(1 + \dot{\alpha})}{\Gamma(1 + 2\dot{\alpha})} \right. \\
 &\quad \times \mathbb{A}^{(\dot{\alpha})} \left(\frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{q\dot{\alpha}(n-1)/p} \right|, \frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \varkappa_2^{q\dot{\alpha}(n-1)} \right| \right) \right]^{\frac{1}{q}},
 \end{aligned}$$

where $\Lambda_7^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; s, p)$ and $\Lambda_8^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; s, p)$ are given in (3.18) and (3.19), respectively.

Proposition 4.4 *Let $\varkappa_1, \varkappa_2 \in \mathbb{R}$ with $\varkappa_1 < \varkappa_2$, $0 \notin [\varkappa_1, \varkappa_2]$, $p \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned} & \left| \left[\mathcal{A}_{\dot{\alpha}}^n \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{\frac{1}{p}}, \varkappa_2; 1^{\dot{\alpha}}, 4^{\dot{\alpha}}, 1^{\dot{\alpha}} \right) \right] - (2p)^{\dot{\alpha}} \Gamma(1 + \dot{\alpha}) \mathcal{L}_{\dot{\alpha}(p+(n-1))} \right| \\ & \leq \left(\frac{\varkappa_2^p - \varkappa_1^p}{4p} \right)^{\dot{\alpha}} \left[\left[\frac{\Gamma(1 + \dot{\alpha}s)}{\Gamma(1 + (s + 1)\dot{\alpha})} \right]^{\frac{1}{s}} \right. \\ & \quad \times \left[\Theta_1^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \varkappa_1^{q\dot{\alpha}(n-1)} \right| \right. \\ & \quad \left. \left. + \Theta_2^{(\dot{\alpha})} \left(\varkappa_1, \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}; q, p \right) \frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{q\dot{\alpha}(n-1)/p} \right| \right]^{\frac{1}{q}} \\ & \quad + \left[\frac{\Gamma(1 + \dot{\alpha}s)}{\Gamma(1 + (s + 1)\dot{\alpha})} \right]^{\frac{1}{s}} \left[\Theta_3^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; q, p \right) \right. \\ & \quad \times \frac{\Gamma(1 + n\dot{\alpha})}{\Gamma(1 + (n - 1)\dot{\alpha})} \left| \left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{q\dot{\alpha}(n-1)/p} \right| \\ & \quad \left. \left. + \Theta_4^{(\dot{\alpha})} \left(\left(\frac{\varkappa_1^p + \varkappa_2^p}{2} \right)^{1/p}, \varkappa_2; q, p \right) \right]^{\frac{1}{q}} \left| \varkappa_2^{q\dot{\alpha}(n-1)} \right| \right], \end{aligned}$$

where $\Theta_1^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; q, p)$, $\Theta_2^{(\dot{\alpha})}(\varkappa_1, (\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}; q, p)$, $\Theta_3^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; q, p)$ and $\Theta_4^{(\dot{\alpha})}((\frac{\varkappa_1^p + \varkappa_2^p}{2})^{1/p}, \varkappa_2; q, p)$ are given in (3.27)–(3.30), respectively.

5 Conclusion

In this article, we have obtained several Simpson-type inequalities for the generalized p -convex functions via the local fractional calculus, our results include a large number of particular cases of the generalized convexity of segmental type and generalized harmonically convex functions. Our ideas and techniques used to obtain the main results in this paper may generate new results within other classes of generalized convexity, which are not included in generalized p -convexity type. These types of results are useful in all the pure and applied domains of science and technique when approximation schemes are involved. It is worth mentioning that many significant real-life problems are generally defined in fractal spaces. Therefore, it is of interest to extend our results to the inequality theory, fuzzy sets and systems and machine learning, which is more general than the current branch of mathematics.

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Authors' contributions

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Author details

¹Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. ²Department of Medical Research, China Medical University, Taichung, Taiwan. ³Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. ⁴Department of Mathematics, Government College University, Faisalabad, Pakistan. ⁵Département de Mathématiques, Faculté des Sciences et Techniques, Université Moulay Ismail, Errachidia, Morocco. ⁶Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey. ⁷Department of Mathematics, Huzhou University, Huzhou, China. ⁸Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China.

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