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Blending type approximation by τ-Baskakov-Durrmeyer type hybrid operators

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Abstract
In this work, we construct a Durrmeyer type modification of the τ-Baskakov operators depends on two parameters $\alpha > 0$ and $\tau \in [0,1]$. We derive the rate of approximation of these operators in a weighted space and also obtain a quantitative Voronovskaja type asymptotic formula as well as a Grüss Voronovskaya type approximation.

MSC: 41A25; 26A15

Keywords: Positive approximation; τ-Baskakov operators; Lipschitz-type space

1 Introduction

Chen et al. [9] recently defined a new kind of Bernstein operators by assuming fixed τ in $\mathbb{R}$ (the set of real numbers) and showed that newly defined τ-Bernstein operators are positive and linear with the choice of $\tau \in [0,1]$. The Kantorovich variant of aforesaid operators was reported by Mohiuddine et al. [22] and investigated several approximation properties, and most recently their Stancu and Schurer types generalization have been constructed and studied by Mohiuddine and Özger [26] and Özger et al. [33].

Inspired from the τ-Bernstein operators, for τ in $[0,1]$ and $m \in \mathbb{N}$ (the set of natural numbers), Aral and Erbay [7] constructed τ-Baskakov as follows:

$$\mathscr{B}_m^{(\tau)}(\zeta;y) = \sum_{j=0}^{\infty} p_{m,j}^{(\tau)}(y)\zeta\left(\frac{j}{m}\right), \quad y \in [0,\infty), \tag{1.1}$$

where

$$\begin{aligned} p_{m,j}^{(\tau)}(y) = \frac{y^{j-1}}{(1+y)^{m+j-1}}\Bigg[&\frac{\tau y}{(1+y)}\binom{m+j-1}{j} \\ &- (1-\tau)(1+y)\binom{m+j-3}{j-2} + (1-\tau)y\binom{m+j-1}{j}\Bigg], \end{aligned}$$

$$\binom{m-3}{-2} = 0 \quad \text{and} \quad \binom{m-2}{-1} = 0.$$

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Setting $\tau = 1$ in (1.1) leads to the Baskakov operators [8]. Later, İlarslan et al. [16] presented a generalization of the above operators (1.1) in Kantorovich sense. Such type of operators are also defined and studied by Nasiruzzaman et al. [31].

In [36], the authors considered an integral modification of a Szász–Mirakjan–Beta type operators and presented several approximation results for their operators. In 2015, Gupta [13] presented a general class of hybrid integral type operators and proved some significant approximation properties of the operators. Kajla and Agrawal [20] obtained an interesting generalization of Szász operators with the help of Charlier polynomials. By taking these operators into account, they studied a Voronovskaya type asymptotic formula and the degree of approximation. Goyal and Kajla [12] constructed an integral type modification of generalized Lupaş operators involving a parameter $\alpha > 0$ and derived the order of approximation for these operators. For further investigation concerning such types of operators as well as statistical approximation, we refer to [1–6, 11, 14, 15, 17–21, 23–25, 27–30, 34, 35, 37–39] and the references therein.

Motivated by the operators constructed in [7, 16, 31], in the next section, we give Durrmeyer type modification of (1.1) and obtain some basic properties for further study in the next sections. Section 3 is devoted to obtain Voronovskaja type results of our new operators. In Sect. 4, we obtain approximation theorems by considering weighted function. In the last section, we considered some terminology defined in [40] and establish a quantitative and Grüss Voronovskaja type approximation.

2 Construction of operators and basic results

It depends on two parameters $\alpha > 0$ and $\tau \in [0,1]$. For $\Lambda > 0$ and $C_\Lambda[0,\infty) := \{\zeta \in C[0,\infty) : \zeta(t) = O(t^\Lambda), t \geq 0\}$, we define the operators

$$\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y) = \sum_{j=1}^{\infty} p_{m,j}^{(\tau)}(y) \int_0^\infty l_{m,j}^{\alpha}(t)\zeta(t)\,dt + p_{m,0}^{(\tau)}(y)\zeta(0), \tag{2.1}$$

where

$$l_{m,j}^{\alpha}(t) = \frac{1}{B(j\alpha, m\alpha+1)} \frac{t^{j\alpha-1}}{(1+t)^{j\alpha+m\alpha+1}}$$

and $p_{m,j}^{(\tau)}(y)$ is defined as above.

Lemma 1 *For the operators $\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)$, we have*

(i) $\mathscr{A}_{m,\alpha}^{(\tau)}(e_0;y) = 1$;

(ii) $\mathscr{A}_{m,\alpha}^{(\tau)}(e_1;y) = y - \frac{2y}{m} + \frac{2y\tau}{m}$;

(iii) $\mathscr{A}_{m,\alpha}^{(\tau)}(e_2;y) = \frac{y^2(-3+m+4\tau)\alpha}{(m\alpha-1)} + \frac{y(-2+m+2\tau+(-4+m+4\tau)\alpha)}{m(m\alpha-1)}$;

(iv) $$\mathscr{A}_{m,\alpha}^{(\tau)}(e_3;y) = \frac{(1+m)y^3(-4+m+6\tau)\alpha^2}{(m\alpha-2)(m\alpha-1)} + \frac{3y^2\alpha(-3+m+4\tau+(-5+m+6\tau)\alpha)}{(m\alpha-2)(m\alpha-1)} + \frac{y(1+\alpha)(m(2+\alpha)+4(-1+\tau)(1+2\alpha))}{m(m\alpha-2)(m\alpha-1)};$$

$$\text{(v)}\quad \mathscr{A}_{m,\alpha}^{(\tau)}(e_4;y) = \frac{(1+m)(2+m)y^4(-5+m+8\tau)\alpha^3}{(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{6(1+m)y^3\alpha^2(-4+m+6\tau+(-6+m+8\tau)\alpha)}{(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{y^2\alpha(1+\alpha)(11(-3+m+4\tau)+(-57+7m+64\tau)\alpha)}{(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{y(1+\alpha)(m(2+\alpha)(3+\alpha)+4(-1+\tau)(3+4\alpha(2+\alpha)))}{m(m\alpha-3)(m\alpha-2)(m\alpha-1)};$$

$$\text{(vi)}\quad \mathscr{A}_{m,\alpha}^{(\tau)}(e_5;y) = \frac{(1+m)(2+m)(3+m)y^5(-6+m+10\tau)\alpha^4}{(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{10(1+m)(2+m)y^4\alpha^3(-5+m+8\tau+(-7+m+10\tau)\alpha)}{(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{5(1+m)y^3\alpha^2(1+\alpha)(7(-4+m+6\tau)+(-44+5m+54\tau)\alpha)}{(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{1}{(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} \times \bigl[5y^2\alpha(1+\alpha)\bigl(m(2+\alpha)(5+3\alpha)+2\tau(4+3\alpha)(5+7\alpha) - 3\bigl(10+\alpha(25+13\alpha)\bigr)\bigr)\bigr] + \frac{y(1+\alpha)(2+\alpha)(m(3+\alpha)(4+\alpha)+8(-1+\tau)(3+4\alpha(2+\alpha)))}{m(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)};$$

$$\text{(vii)}\quad \mathscr{A}_{m,\alpha}^{(\tau)}(e_6;y) = \frac{(1+m)(2+m)(3+m)(4+m)y^6(-7+m+12\tau)\alpha^5}{(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{15(1+m)(2+m)(3+m)y^5\alpha^4(-6+m+10\tau+(-8+m+12\tau)\alpha)}{(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} + \frac{1}{(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} \times \bigl[5(1+m)(2+m)y^4\alpha^3(1+\alpha)\bigl(17(-5+m+8\tau) + (-125+13m+164\tau)\alpha\bigr)\bigr] + \frac{1}{(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} \times \bigl[15(1+m)y^3\alpha^2(1+\alpha)\bigl(15(-4+m+6\tau) + (-144+19m+182\tau)\alpha+2(-38+3m+44\tau)\alpha^2\bigr)\bigr] + \frac{1}{(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)} \times \bigl[y^2\alpha\bigl(274(-3+m+4\tau)+675(-5+m+6\tau)\alpha + 85(-57+7m+64\tau)\alpha^2+225(-13+m+14\tau)\alpha^3 + (-633+31m+664\tau)\alpha^4\bigr)\bigr] + \frac{1}{m(m\alpha-5)(m\alpha-4)(m\alpha-3)(m\alpha-2)(m\alpha-1)}$$

$$\times \big[y(1+\alpha)(2+\alpha)\big(m(3+\alpha)(4+\alpha)(5+\alpha)$$
$$+ 8(-1+\tau)(1+2\alpha)(3+2\alpha)(5+2\alpha)\big)\big].$$

Lemma 2 *From Lemma 1, we obtain*

(i) $\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y);y\big) = \frac{2y(\tau-1)}{m}$;

(ii) $\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big) = \frac{y^2(4(\tau-1)+m(1+\alpha))}{m(m\alpha-1)} + \frac{y(2(\tau-1)+4(\tau-1)\alpha+m(1+\alpha))}{m(m\alpha-1)}$;

(iii)
$$\begin{aligned}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^4;y\big) &= \frac{1}{m(m\alpha-3)(m\alpha-2)(m\alpha-1)}\big[y^4\big(48(\tau-1)+3m^2\alpha(1+\alpha)^2\\ &\quad + 2m(1+\alpha)\big(9+\alpha(-19+28\tau-5\alpha+8\tau\alpha)\big)\big)\big]\\ &\quad + \frac{1}{m(m\alpha-3)(m\alpha-2)(m\alpha-1)}\big[y^3\big(72(\tau-1)+144(\tau-1)\alpha\\ &\quad + 6m^2\alpha(1+\alpha)^2 + 2m(1+\alpha)\big(9+\alpha(-19+28\tau-5\alpha+8\tau\alpha)\big)\\ &\quad + 2m(1+\alpha)\big(9+\alpha\big(-5-13\alpha+2\tau(7+8\alpha)\big)\big)\big)\big]\\ &\quad + \frac{1}{m(m\alpha-3)(m\alpha-2)(m\alpha-1)}\big[y^2\big(48(\tau-1)+144(\tau-1)\alpha\\ &\quad + 96(\tau-1)\alpha^2 + 3m^2\alpha(1+\alpha)^2 + m(1+\alpha)(2+\alpha)(3+\alpha)\\ &\quad + 2m(1+\alpha)\big(9+\alpha\big(-5-13\alpha+2\tau(7+8\alpha)\big)\big)\big)\big]\\ &\quad + \frac{1}{m(m\alpha-3)(m\alpha-2)(m\alpha-1)}\big[y\big(12(-1+\tau)+44(-1+\tau)\alpha\\ &\quad + 48(\tau-1)\alpha^2 + 16(\tau-1)\alpha^3 + m(1+\alpha)(2+\alpha)(3+\alpha)\big)\big].\end{aligned}$$

Remark 1 We have

$$\lim_{m\to\infty} m\mathscr{F}_{m,\alpha}^{\tau,1}(y) = 2y(\tau-1),$$
$$\lim_{m\to\infty} m\mathscr{F}_{m,\alpha}^{\tau,2}(y) = \frac{y(1+y)(1+\alpha)}{\alpha},$$
$$\lim_{m\to\infty} m^2\mathscr{F}_{m,\alpha}^{\tau,4}(y) = \frac{3y^2(1+y)^2(1+\alpha)^2}{\alpha^2},$$
$$\lim_{m\to\infty} m^3\mathscr{F}_{m,\alpha}^{\tau,6}(y) = \frac{15y^3(1+y)^3(1+\alpha)^3}{\alpha^3},$$

where $\mathscr{F}_{m,\alpha}^{\tau,\nu} := \mathscr{A}_{m,\alpha}^{(\tau)}((t-y)^\nu;y)$, $\nu = 1,2,4,6$.

3 Direct results

Theorem 1 *Suppose that* $\zeta \in C_\Lambda[0,\infty)$. *Then* $\lim_{m\to\infty}\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y) = \zeta(y)$, *uniformly in each compact subset of* $[0,\infty)$.

3.1 Voronovskaja type theorem

Theorem 2 *Suppose that* $\zeta \in C_\Lambda[0,\infty)$. *If* ζ'' *exists at a point* $y \in [0,\infty)$, *then*

$$\lim_{m\to\infty} m\big[\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y) - \zeta(y)\big] = 2y(\tau-1)\zeta'(y) + \frac{1}{2}\frac{y(1+y)(1+\alpha)}{\alpha}\zeta''(y).$$

Proof Applying Taylor's expansion, one writes

$$\zeta(t)=\zeta(y)+\zeta'(y)(t-y)+\frac{1}{2}\zeta''(y)(t-y)^2+\varpi(t,y)(t-y)^2, \tag{3.1}$$

where $\lim_{t\to y}\varpi(t,y)=0$. By using the linearity of the operator $\mathscr{A}_{m,\alpha}^{(\tau)}$, we get

$$\begin{aligned}\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)&=\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y);y\big)\zeta'(y)+\frac{1}{2}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\zeta''(y)\\&\quad+\mathscr{A}_{m,\alpha}^{(\tau)}\big(\varpi(t,y)(t-y)^2;y\big).\end{aligned}$$

By using the Cauchy–Schwarz inequality in the last term of the last inequality, we obtain

$$m\mathscr{A}_{m,\alpha}^{(\tau)}\big(\varpi(t,y)(t-y)^2;y\big)\le\sqrt{\mathscr{A}_{m,\alpha}^{(\tau)}\big(\varpi^2(t,y);y\big)}\sqrt{m^2\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^4;y\big)}. \tag{3.2}$$

As $\varpi^2(y,y)=0$ and $\varpi^2(\cdot,y)\in C_\Lambda[0,\infty)$, we have

$$\lim_{m\to\infty}\mathscr{A}_{m,\alpha}^{(\tau)}\big(\varpi^2(t,y);y\big)=\varpi^2(y,y)=0. \tag{3.3}$$

Combining (3.2)–(3.3) and Remark 1, we have

$$\lim_{m\to\infty}m\mathscr{A}_{m,\alpha}^{(\tau)}\big(\varpi(t,y)(t-y)^2;y\big)=0. \tag{3.4}$$

Hence

$$\lim_{m\to\infty}m\big[\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\big]=2y(\tau-1)\zeta'(y)+\frac{1}{2}\frac{y(1+y)(1+\alpha)}{\alpha}\zeta''(y).$$ □

Let $\mu_1\ge 0$, $\mu_2>0$ be fixed. We consider Lipschitz-type space (see [32]) as follows:

$$\mathrm{Lip}_M^{(\mu_1,\mu_2)}(r):=\left\{\zeta\in C[0,\infty):\big|\zeta(t)-\zeta(y)\big|\le M\frac{|t-y|^r}{(t+\mu_1y^2+\mu_2y)^{\frac{r}{2}}};y,t\in(0,\infty)\right\},$$

where $0<r\le 1$.

Theorem 3 *Let $\zeta\in\mathrm{Lip}_M^{(\mu_1,\mu_2)}(r)$ and $r\in(0,1]$. Then, for all $y\in(0,\infty)$, we have*

$$\big|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\big|\le M\left(\frac{\mathscr{F}_{m,\alpha}^{\tau,2}(y)}{\mu_1y^2+\mu_2y}\right)^{\frac{r}{2}}.$$

Proof Using Hölder's inequality with $p=\frac{2}{r}$, $q=\frac{2}{2-r}$, we obtain

$$\begin{aligned}&\big|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\big|\\&\quad=\sum_{j=1}^{\infty}p_{m,j}^{(\tau)}(y)\int_0^\infty l_{m,j}^{\alpha}(t)\big|\zeta(t)-\zeta(y)\big|\,dt+p_{m,0}^{(\tau)}(y)\big|\zeta(0)-\zeta(y)\big|\\&\quad\le\sum_{j=1}^{\infty}p_{m,j}^{(\tau)}(y)\left(\int_0^\infty l_{m,j}^{\alpha}(t)\big|\zeta(t)-\zeta(y)\big|^{\frac{2}{r}}\,dt\right)^{\frac{r}{2}}+p_{m,0}^{(\tau)}(y)\big|\zeta(0)-\zeta(y)\big|\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\sum_{j=1}^{\infty} p_{m,j}^{(\tau)}(y) \int_0^\infty l_{m,j}^{\alpha}(t)\left|\zeta(t)-\zeta(y)\right|^{\frac{2}{r}}\,dt + p_{m,0}^{(\tau)}(y)\left|\zeta(0)-\zeta(y)\right|^{\frac{2}{r}}\right\}^{\frac{r}{2}} \\
&\quad\times \left(\sum_{j=0}^{\infty} p_{m,j}^{(\tau)}(y)\right)^{\frac{2-r}{2}} \\
&= \left\{\sum_{j=1}^{\infty} p_{m,j}^{(\tau)} \int_0^\infty l_{m,j}^{\alpha}(t)\left|\zeta(t)-\zeta(y)\right|^{\frac{2}{r}}\,dt + p_{m,0}^{(\tau)}(y)\left|\zeta(0)-\zeta(y)\right|^{\frac{2}{r}}\right\}^{\frac{r}{2}} \\
&\leq M\left(\sum_{j=1}^{\infty} p_{m,j}^{(\tau)}(y) \int_0^\infty l_{m,j}^{\alpha}(t)\frac{(t-y)^2}{(t+\mu_1 y^2+\mu_2 y)}\,dt + p_{m,0}^{(\tau)}(y)\frac{y^2}{(\mu_1 y^2+\mu_2 y)}\right)^{\frac{r}{2}} \\
&\leq \frac{M}{(\mu_1 y^2+\mu_2 y)^{\frac{r}{2}}}\left(\sum_{j=1}^{\infty} p_{m,j}^{(\tau)}(y) \int_0^\infty l_{m,j}^{\alpha}(t)(t-y)^2\,dt + y^2 p_{m,0}^{(\tau)}(y)\right)^{\frac{r}{2}} \\
&= \frac{M}{(\mu_1 y^2+\mu_2 y)^{\frac{r}{2}}}\left(\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right)^{\frac{r}{2}} = \frac{M}{(\mu_1 y^2+\mu_2 y)^{\frac{r}{2}}}\left(\mathscr{F}_{m,\alpha}^{\tau,2}(y)\right)^{\frac{r}{2}}.
\end{aligned}
$$

Thus, the proof is completed. □

4 Weighted approximation

Suppose $H_\xi[0,\infty)$ is the space of all real valued functions on $[0,\infty)$ satisfies the relation $|\zeta(y)| \leq N_\zeta \xi(y)$, where $\xi(y) = 1 + y^2$ is a weight function and N_ζ is a positive constant depending only on ζ. Let $C_\xi[0,\infty)$ be the space of all continuous functions in $H_\xi[0,\infty)$ endowed with the norm considered by

$$
\|\zeta\|_\xi := \sup_{y\in[0,\infty)} \frac{|\zeta(y)|}{\xi(y)}
$$

and

$$
C_\xi^0[0,\infty) := \left\{\zeta \in C_\xi[0,\infty) : \lim_{y\to\infty} \frac{|\zeta(y)|}{\xi(y)} \text{ exists and is finite}\right\}.
$$

Theorem 4 *For each $\zeta \in C_\xi^0[0,\infty)$ and $r > 0$, we have*

$$
\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} = 0.
$$

Proof Let $y_0 > 0$ be arbitrary but fixed. Then we get

$$
\begin{aligned}
\sup_{y\in[0,\infty)} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} &\leq \sup_{y\leq y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} + \sup_{y>y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} \\
&\leq \sup_{y\leq y_0}\left\{\left|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\right|\right\} + \sup_{y>y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)|}{(1+y^2)^{1+r}} \\
&\quad + \sup_{y>y_0} \frac{|\zeta(y)|}{(1+y^2)^{1+r}}.
\end{aligned}
$$

Since $|\zeta(t)| \leq \|\zeta\|_\xi(1+t^2)$, $\forall t \geq 0$

$$\begin{aligned}\sup_{y\in[0,\infty)} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} &\leq \left\|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\right\|_{C[0,y_0]} + \|\zeta\|_\xi \sup_{y>y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(1+t^2;y)|}{(1+y^2)^{1+r}} \\ &\quad + \sup_{y>y_0} \frac{\|\zeta\|_\xi}{(1+y^2)^r} \\ &= I_1 + I_2 + I_3, \quad \text{say.} \end{aligned} \tag{4.1}$$

Applying Theorem 1, therefore for a given $\epsilon > 0$, $\exists\, m_1 \in \mathbb{N}$, such that

$$I_1 = \left\|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\right\|_{C[0,y_0]} < \frac{\epsilon}{3}, \quad \text{for all } m \geq m_1. \tag{4.2}$$

Since $\lim_{m\to\infty}\sup_{y>y_0} \frac{\mathscr{A}_{m,\alpha}^{(\tau)}(1+t^2;y)}{1+y^2} = 1$, it follows that $\exists\, m_2 \in \mathbb{N}$ such that

$$\sup_{y>y_0} \frac{\mathscr{A}_{m,\alpha}^{(\tau)}(1+t^2;y)}{1+y^2} \leq \frac{(1+y_0^2)^r}{\|\zeta\|_\xi}\cdot\frac{\epsilon}{3}+1, \quad \text{for all } m \geq m_2.$$

Hence,

$$\begin{aligned} I_2 = \|\zeta\|_\xi \sup_{y>y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(1+t^2;y)|}{(1+y^2)^{1+r}} &\leq \frac{\|\zeta\|_\xi}{(1+y_0^2)^r} \sup_{y>y_0} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(1+t^2;y)|}{1+y^2} \\ \leq \frac{\|\zeta\|_\xi}{(1+y_0^2)^r} + \frac{\epsilon}{3}, \quad \text{for all } m \geq m_2. \end{aligned} \tag{4.3}$$

Let us choose y_0 to be so large that

$$\frac{\|\zeta\|_\xi}{(1+y_0^2)^r} < \frac{\epsilon}{6},$$

then

$$I_3 = \sup_{y>y_0} \frac{\|\zeta\|_\xi}{(1+y^2)^r} \leq \frac{\|\zeta\|_\xi}{(1+y_0^2)^r} < \frac{\epsilon}{6}. \tag{4.4}$$

Let $m_0 = \max\{m_1, m_2\}$, then by combining (4.2)–(4.4)

$$\sup_{y\in[0,\infty)} \frac{|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)|}{(1+y^2)^{1+r}} < \epsilon, \quad \text{for all } m \geq m_0.$$

Hence the proof is done. □

Theorem 5 *Let $\zeta \in C_\xi^0[0,\infty)$. Then we have*

$$\lim_{m\to\infty} \left\|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta)-\zeta\right\|_\xi = 0. \tag{4.5}$$

Proof To prove (4.5), by [10], it is sufficient to show the following:

$$\lim_{m\to\infty} \left\|\mathscr{A}_{m,\alpha}^{(\tau)}(t^\nu;y)-e_\nu\right\|_\xi = 0, \quad \nu = 0,1,2. \tag{4.6}$$

Since $\mathscr{A}_{m,\alpha}^{(\tau)}(1;y) = 1$, so (4.6) holds true for $\nu = 0$.

From Lemma 1, we obtain

$$\left\|\mathscr{A}_{m,\alpha}^{(\tau)}(t;y)-y\right\|_{\xi}=\sup_{y\geq 0}\frac{1}{1+y^2}\left|y+\frac{2y(\tau-1)}{m}-y\right|\leq\sup_{y\geq 0}\left(\frac{y}{1+y^2}\right)\frac{2|\tau-1|}{m}. \tag{4.7}$$

Thus, $\lim_{m\to\infty}\|\mathscr{A}_{m,\alpha}^{(\tau)}(t;y)-y\|_{\xi}=0$.

Finally, we obtain

$$\begin{aligned}
&\left\|\mathscr{A}_{m,\alpha}^{(\tau)}(t^2;y)-y^2\right\|_{\xi}\\
&\quad=\sup_{y\geq 0}\frac{1}{1+y^2}\left|\frac{y^2(-3+m+4\tau)\alpha}{(m\alpha-1)}+\frac{y(-2+m+2\tau+(-4+m+4\tau)\alpha)}{m(m\alpha-1)}-y^2\right|\\
&\quad\leq\sup_{y\geq 0}\frac{y^2}{1+y^2}\left|\frac{(m+m(-3+4\alpha)\rho)}{m(m\alpha-1)}\right|\\
&\qquad+\sup_{y\geq 0}\frac{y}{1+y^2}\left|\frac{(m+m\alpha+2(\tau-1)(1+2\alpha))}{m(m\alpha-1)}\right|,
\end{aligned} \tag{4.8}$$

which implies that $\lim_{m\to\infty}\|\mathscr{A}_{m,\alpha}^{(\tau)}(t^2;y)-y^2\|_{\xi}=0$. □

5 Some Voronoskaja type approximation theorem

To examine the degree of approximation of functions in $C_{\xi}[0,\infty)$, Yüksel and Ispir [40] presented the weighted modulus of smoothness $\Omega(\zeta;\sigma)$ as follows:

$$\Omega(\zeta;\sigma)=\sup_{0\leq h<\sigma,y\in[0,\infty)}\frac{|\zeta(y+h)-\zeta(y)|}{(1+h^2)(1+y^2)} \tag{5.1}$$

for $\zeta\in C_{\xi}[0,\infty)$. It was proved in [40] that, if $\zeta\in C_{\xi}^{0}[0,\infty)$, then $\Omega(\cdot;\sigma)$ has the properties

$$\lim_{\sigma\to 0}\Omega(\zeta;\sigma)=0$$

and

$$\Omega(\zeta;\lambda\sigma)\leq 2(1+\lambda)\left(1+\sigma^2\right)\Omega(\zeta;\sigma),\quad \lambda>0. \tag{5.2}$$

For $\zeta\in C_{\xi}^{0}[0,\infty)$, it follows from (5.1) and (5.2) that

$$\begin{aligned}
\left|\zeta(t)-\zeta(y)\right|&\leq\left(1+(t-y)^2\right)\left(1+y^2\right)\Omega\left(\zeta;|t-y|\right)\\
&\leq 2\left(1+\frac{|t-y|}{\sigma}\right)\left(1+\sigma^2\right)\Omega(\zeta;\sigma)\left(1+(t-y)^2\right)\left(1+y^2\right).
\end{aligned} \tag{5.3}$$

In the next theorem, we compute the degree of approximation of ζ by the operator $\mathscr{A}_{m,\alpha}^{(\tau)}$ in the weighted space of continuous functions $C_{\xi}^{0}[0,\infty)$ in terms of the weighted modulus of smoothness $\Omega(\cdot;\sigma)$, $\sigma>0$.

5.1 Quantitative Voronovskaya type theorem

Theorem 6 *Suppose that* $\zeta \in C_{\xi}^{0}[0,\infty)$ *such that* $\zeta'(y), \zeta''(y) \in C_{\xi}^{0}[0,\infty)$. *Then, for sufficiently large m and each* $y \in [0,\infty)$,

$$\begin{aligned}&\left| m\left\{ \mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y) - \zeta(y) - \zeta'(y)\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y);y\big) - \frac{\zeta''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right\}\right| \\ &\quad = O(1)\Omega\big(\zeta'';\sqrt{1/m}\big).\end{aligned}$$

Proof By Taylor's formula

$$\begin{aligned}\zeta(t) &= \zeta(y) + \zeta'(y)(t-y) + \frac{\zeta''(\eta)}{2!}(t-y)^2 \\ &= \zeta(y) + \zeta'(y)(t-y) + \frac{\zeta''(y)}{2!}(t-y)^2 + h_2(t,y),\end{aligned} \tag{5.4}$$

where $\eta \in (y,t)$ and hence

$$h_2(t,y) = \frac{\zeta''(\eta) - \zeta''(y)}{2!}(t-y)^2. \tag{5.5}$$

In view of the inequality (5.3) of the weighted modulus of continuity, we obtain

$$\begin{aligned}\left|\zeta''(\eta) - \zeta''(y)\right| &\le \big(1+(\eta-y)^2\big)\big(1+y^2\big)\Omega\big(\zeta'';|\eta-y|\big) \\ &\le \big(1+(t-y)^2\big)\big(1+y^2\big)\Omega\big(\zeta'';|t-y|\big) \\ &\le 2\big(1+(t-y)^2\big)\big(1+y^2\big)\left(1+\frac{|t-y|}{\sigma}\right)\big(1+\sigma^2\big)\Omega\big(\zeta'';\sigma\big),\end{aligned} \tag{5.6}$$

but

$$\left(1+\frac{|t-y|}{\sigma}\right)\big(1+(t-y)^2\big) \le \begin{cases} 2(1+\sigma^2), & |t-y| < \sigma, \\ 2\frac{(t-y)^4}{\sigma^4}(1+\sigma^2), & |t-y| \ge \sigma, \end{cases}$$

that is,

$$\left(1+\frac{|t-y|}{\sigma}\right)\big(1+(t-y)^2\big) \le 2\left(1+\frac{(t-y)^4}{\sigma^4}\right)\big(1+\sigma^2\big). \tag{5.7}$$

Combining (5.5)–(5.7) and choosing $0 < \sigma < 1$, we obtain

$$\left|h_2(t,y)\right| \le 2\big(1+\sigma^2\big)^2\big(1+y^2\big)\Omega\big(\zeta'';\sigma\big)\left(1+\frac{(t-y)^4}{\sigma^4}\right)(t-y)^2. \tag{5.8}$$

Operating $\mathscr{A}_{m,\alpha}^{(\tau)}$ and Lemma 2 on both sides of (5.4), we get

$$\begin{aligned}&\left|\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y) - \zeta(y) - \zeta'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y) - \frac{\zeta''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right| \\ &\quad \le \mathscr{A}_{m,\alpha}^{(\tau)}\big(\left|h_2(t,y)\right|;y\big).\end{aligned} \tag{5.9}$$

Applying Remark 1 and using Eq. (5.8), we get

$$\begin{aligned}
&\mathscr{A}_{m,\alpha}^{(\tau)}\big(\big|h_2(t,y)\big|;y\big)\\
&\quad\le 2\big(1+\sigma^2\big)^2\big(1+y^2\big)\Omega\big(\zeta'';\sigma\big)\mathscr{A}_{m,\alpha}^{(\tau)}\left(\left((t-y)^2+\frac{(t-y)^6}{\sigma^4}\right);y\right)\\
&\quad=2\big(1+\sigma^2\big)^2\big(1+y^2\big)\Omega\big(\zeta'';\sigma\big)\left(\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)+\frac{1}{\sigma^4}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^6;y\big)\right)\\
&\quad=2\big(1+\sigma^2\big)^2\big(1+y^2\big)\Omega\big(\zeta'';\sigma\big)\left(O(1/m)+\frac{1}{\sigma^4}O\big(1/m^3\big)\right).
\end{aligned}$$

By choosing $\sigma=\sqrt{1/m}$, we get

$$m\mathscr{A}_{m,\alpha}^{(\tau)}\big(\big|h_2(t,y)\big|;y\big)=O(1)\Omega\big(\zeta'';\sqrt{1/m}\big).\tag{5.10}$$

Hence, from (5.9) and (5.10), we get

$$\begin{aligned}
&\left|m\left\{\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)\zeta'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y)-\frac{\zeta''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right\}\right|\\
&\quad=O(1)\Omega\big(\zeta'';\sqrt{1/m}\big),\quad \text{as } m\to\infty.
\end{aligned}$$

This completes the proof. □

5.2 Grüss Voronovskaya type theorem

Theorem 7 *Suppose that ζ, g and $\zeta g\in C_\xi^0[0,\infty)$ such that ζ', g', $(\zeta g)'$, ζ'', g'' and $(\zeta g)''\in C_\xi^0[0,\infty)$. Then, for each $y\in[0,\infty)$,*

$$\lim_{m\to\infty}m\big\{\mathscr{A}_{m,\alpha}^{(\tau)}\big((\zeta g);y\big)-\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\mathscr{A}_{m,\alpha}^{(\tau)}(g;y)\big\}=\zeta'(y)g'(y)\frac{y(1+y)(1+\alpha)}{\alpha}.$$

Proof Since $(\zeta g)(y)=\zeta(y)g(y)$, $(\zeta g)'(y)=\zeta'(y)g(y)+\zeta(y)g'(y)$ and $(\zeta g)''(y)=\zeta''(y)g(y)+2\zeta'(y)g'(y)+\zeta(y)g''(y)$, we may write

$$\begin{aligned}
&\mathscr{A}_{m,\alpha}^{(\tau)}\big((\zeta g);y\big)-\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\mathscr{A}_{m,\alpha}^{(\tau)}(g;y)\\
&\quad=\left\{\mathscr{A}_{m,\alpha}^{(\tau)}\big((\zeta g);y\big)-\zeta(y).g(y)-(\zeta g)'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y)-\frac{(\zeta g)''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right\}\\
&\qquad-g(y)\left\{\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)-\zeta(y)-\zeta'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y)-\frac{\zeta''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right\}\\
&\qquad-\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\left\{\mathscr{A}_{m,\alpha}^{(\tau)}(g;y)-g(y)-g'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y)-\frac{g''(y)}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\right\}\\
&\qquad+\frac{1}{2!}\mathscr{A}_{m,\alpha}^{(\tau)}\big((t-y)^2;y\big)\big\{\zeta(y)g''(y)+2\zeta'(y)g'(y)-g''(y)\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\big\}\\
&\qquad+\mathscr{A}_{m,\alpha}^{(\tau)}(t-y;y)\big\{\zeta(y)g'(y)-g'(y)\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\big\}.
\end{aligned}$$

Now, by using Lemma 2 and Theorems 1 and 6, we get

$$\lim_{m\to\infty}m\big\{\mathscr{A}_{m,\alpha}^{(\tau)}\big((\zeta g);y\big)-\mathscr{A}_{m,\alpha}^{(\tau)}(\zeta;y)\mathscr{A}_{m,\alpha}^{(\tau)}(g;y)\big\}=\zeta'(y)g'(y)\frac{y(1+y)(1+\alpha)}{\alpha},$$

which proves our theorem. □

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