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# Dynamics of a stochastic eutrophication-chemostat model with impulsive dredging and pulse inputting on environmental toxicant

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## Abstract

In this paper, we present a stochastic eutrophication-chemostat model with impulsive dredging and pulse inputting on environmental toxicant. The sufficient condition for the extinction of microorganisms is obtained. The sufficient condition for the investigated system with unique ergodic stationary distribution is also obtained. The results show that the stochastic noise, impulsive dredging, and pulse input on the environmental toxicant play important roles in the extinction of microorganisms. The results also indicate the effective and reliable controlling strategy for water resource management. Finally, numerical simulations are employed to illustrate our results.

**Keywords:** Stochastic eutrophication-chemostat model; Impulsive dredging; Pulse input; Extinction

## 1 Introduction

The chemostat is a device for continuous and impulsive cultures of microorganisms in laboratory [1–3]. Impulsive differential equations are found in almost every domain of applied science and have been studied in many investigations [4, 5]. With the development of society, the increasing amount of toxicants and contaminants have entered ecological systems. Environmental pollution has become one of the most important society-ecological problems. Therefore, it is very important to study the effects of toxicants on a population or community. Specially, the toxicant and abundant microorganisms in the water pollution environment are also a threat to the water resource management. Consequently, it is important to discuss chemostat models in a polluted environment [6, 7]. Zhou et al. [8] considered that reservoir dredging is the main and effective way to improve water quality by using a physical method. However, it is well known that many real-world systems may be disturbed by stochastic factors. Population systems are often subjected to various types of environmental noise. In ecology, it is critical to discover whether the presence of this noise has significant effects on population systems. Mao [9, 10] investigated stochastic differential equations and their applications. Lv et al. [11] presented an impulsive stochastic chemostat model with nonlinear perturbation.

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### 2 The model

Inspired by the above discussion, we consider a stochastic eutrophication-chemostat model with impulsive dredging and pulse inputting on environmental toxicant:

$$\left. \begin{aligned}
 dx(t) &= [D(x_0 - x(t)) - \frac{\beta x(t)y(t)}{k(A+x(t)+By(t))}] dt \\
 &\quad + x(t)(\sigma_{11} + \sigma_{12}x(t)) dB_1(t), \\
 dy(t) &= [\frac{\beta x(t)y(t)}{A+x(t)+By(t)} - Dy(t) - rc_o(t)y(t)] dt \\
 &\quad + y(t)(\sigma_{21} + \sigma_{22}y(t)) dB_2(t), \\
 dc_o(t) &= (fc_e(t) - (g + m)c_o(t)) dt, \\
 dc_e(t) &= (-hc_e(t)) dt,
 \end{aligned} \right\} t \neq (n + l)\tau, t \neq (n + 1)\tau,$$

$$\left. \begin{aligned}
 \Delta x(t) &= 0, \\
 \Delta y(t) &= -h_1y(t), \\
 \Delta c_o(t) &= 0, \\
 \Delta c_e(t) &= -h_2c_e(t),
 \end{aligned} \right\} t = (n + l)\tau, n \in Z^+,$$

$$\left. \begin{aligned}
 \Delta x(t) &= 0, \\
 \Delta y(t) &= 0, \\
 \Delta c_o(t) &= 0, \\
 \Delta c_e(t) &= \mu,
 \end{aligned} \right\} t = (n + 1)\tau, n \in Z^+,$$
(2.1)

where  $x(t)$  is the concentration of the nutrient in a lake at time  $t$ .  $y(t)$  is the concentration of the microorganism in a lake at time  $t$ .  $c_o(t)$  is the concentration of the toxicant in the organism of the microorganism in a lake at time  $t$ .  $c_e(t)$  is the concentration of the toxicant in a lake at time  $t$ .  $D$  denotes the input rate from the lakes containing the nutrient and the wash-out rate of nutrients and microorganisms from the lake.  $\beta > 0$  is the uptake constant of the nutrient.  $\frac{x(t)}{A+x(t)+By(t)}$  is a functional response of the Beddington–DeAngelis type.  $k > 0$  is the yield of the microorganism  $y$  per unit mass of the nutrient.  $A > 0$  and  $B > 0$  are the saturating parameters of the Beddington–DeAngelis functional response.  $r > 0$  is the depletion rate coefficient of the microorganism  $y$  due to the microorganism organismal toxicant.  $f > 0$  is the coefficient of the population organism’s net uptake of toxicant from the environment in a lake.  $-g < 0$  and  $-m < 0$ , respectively, represent coefficients of the elimination and depuration rates of the toxicant in the organism in a lake.  $-h < 0$  is the coefficient of the totality of toxicant losses from the system environment in a lake, including processes such as biological transformation, chemical hydrolysis, volatilization, microbial degradation, and photosynthetic degradation.  $\tau$  is the period of impulsive dredging or the pulse input environmental toxin.  $0 < h_1 < 1$  is the effect of impulsive dredging microorganism at time  $t = (n + l)$  ( $0 < l < 1$ ).  $0 < h_2 < 1$  is the effect of impulsive dredging environmental toxicant at time  $t = (n + l)$  ( $0 < l < 1$ ).  $\mu \geq 0$  is the amount of pulse input of environmental toxin concentration in a lake at  $t = (n + 1)\tau$ ,  $n \in Z^+$ , and  $Z^+ = \{1, 2, \dots\}$ .

### 3 The lemmas

In this paper,  $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P)$  stands for a complete probability space with filtration  $\mathcal{F}_{t \geq 0}$  satisfying the usual conditions. Define  $f^l = \inf_{t \in R^+} f(t)$ ,  $f(t)$  is a bounded function on  $[0, +\infty)$ ,  $\langle f(t) \rangle = \frac{1}{t} \int_0^t f(s) ds$ , where  $f(t)$  is an integrable function on  $[0, +\infty)$ .

Consider the subsystem of system (2.1) as follows:

$$\left\{ \begin{array}{l} \left. \begin{array}{l} dc_o(t) = (fc_e(t) - (g + m)c_o(t)) dt, \\ dc_e(t) = (-hc_e(t)) dt, \end{array} \right\} \quad t \neq (n + l)\tau, t \neq (n + 1)\tau, \\ \left. \begin{array}{l} \Delta c_o(t) = 0, \\ \Delta c_e(t) = -h_2c_e(t), \end{array} \right\} \quad t = (n + l)\tau, n \in Z^+, \\ \left. \begin{array}{l} \Delta c_o(t) = 0, \\ \Delta c_e(t) = \mu, \end{array} \right\} \quad t = (n + 1)\tau, n \in Z^+. \end{array} \right. \tag{3.1}$$

With regard to system (3.1), we have the following equations with integrating and solving the first two equations of system (3.1) between pulses:

$$\left\{ \begin{array}{l} c_o(t) = \begin{cases} c_o(n\tau^+)e^{-(g+m)(t-n\tau)} + \frac{fc_e(n\tau^+)(1-e^{-(h-g-m)(t-n\tau)})}{(h-g-m)}, \\ \quad t \in (n\tau, (n + l)\tau], \\ c_o((n + l)\tau^+)e^{-(g+m)(t-(n+l)\tau)} \\ \quad + \frac{fc_e((n+l)\tau^+)(1-e^{-(h-g-m)(t-(n+l)\tau)})}{(h-g-m)}, \\ \quad t \in ((n + l)\tau, (n + 1)\tau], \end{cases} \\ c_e(t) = \begin{cases} c_e(n\tau^+)e^{-h(t-n\tau)}, & t \in (n\tau, (n + l)\tau], \\ c_e((n + l)\tau^+)e^{-h(t-(n+l)\tau)}, & t \in ((n + l)\tau, (n + 1)\tau]. \end{cases} \end{array} \right. \tag{3.2}$$

The stroboscopic map of system (3.1) is obtained by the last two equations of system (3.1):

$$\left\{ \begin{array}{l} c_o((n + 1)\tau^+) = c_o(n\tau^+)e^{-(g+m)\tau} \\ \quad + \frac{c_e(n\tau^+)f(e^{-(g+m)(1-l)\tau} - e^{-(h-g-m)l\tau - (1-l)\tau})}{(h-1)} \\ \quad + \frac{(1-h_2)c_e(n\tau^+)f(e^{-hl\tau} - e^{-(h-g-m)\tau})}{(h-g-m)}, \\ c_e((n + 1)\tau^+) = (1 - h_2)e^{-h\tau}c_e(n\tau^+) + \mu. \end{array} \right. \tag{3.3}$$

We can easily have a unique fixed point  $(c_o^*, c_e^*)$  of system (3.3) as follows:

$$\left\{ \begin{array}{l} c_o^* = \frac{\mu f}{1-e^{-(g+m)\tau}} \times \left[ \frac{(e^{-(g+m)(1-l)\tau} - e^{-(h-g-m)l\tau - (1-l)\tau})}{(h-g-m)(1-(1-h_2)e^{-h\tau})} \right. \\ \quad \left. + \frac{(1-h_2)(e^{-hl\tau} - e^{-(h-g-m)\tau})}{(h-g-m)(1-(1-h_2)e^{-h\tau})} \right], \\ c_e^* = \frac{\mu}{1-(1-h_2)e^{-h\tau}}. \end{array} \right. \tag{3.4}$$

The unique fixed point  $(c_o^*, c_e^*)$  of system (3.3) is globally asymptotically stable for the eigenvalues of the coefficient matrix of system (3.3)

$$\begin{pmatrix} e^{-\tau} & \star \\ 0 & (1 - h_2)e^{-h\tau} \end{pmatrix}$$

are less 1, there is no need for calculating  $\star$ .

Similar to Lemma 3.3 in reference [6], we can obtain the following lemma.

**Lemma 3.1** *System (3.1) has a unique positive  $\tau$ -periodic solution  $(\widetilde{c}_o(t), \widetilde{c}_e(t))$ , which is also globally asymptotically stable,  $\widetilde{c}_o(t)$  and  $\widetilde{c}_e(t)$  are defined as follows:*

$$\begin{cases} \widetilde{c}_o(t) = \begin{cases} c_o^* e^{-(g+m)(t-n\tau)} + \frac{fc_e^*(1-e^{-(h-g-m)(t-n\tau)})}{(h-g-m)}, & t \in (n\tau, (n+l)\tau], \\ c_o^{**} e^{-(g+m)(t-(n+l)\tau)} + \frac{fc_e^{**}(1-e^{-(h-g-m)(t-(n+l)\tau)})}{(h-g-m)}, & t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ \widetilde{c}_e(t) = \begin{cases} c_e^* e^{-h(t-n\tau)}, & t \in (n\tau, (n+l)\tau], \\ c_e^{**} e^{-h(t-(n+l)\tau)}, & t \in ((n+l)\tau, (n+1)\tau], \end{cases} \end{cases} \tag{3.5}$$

where  $c_o^*, c_e^*$  are defined as (3.4), and

$$\begin{cases} c_o^{**} = c_o^* e^{-(g+m)l\tau} + \frac{fc_e^*(1-e^{-(h-g-m)l\tau})}{(h-g-m)}, \\ c_e^{**} = (1-h_2)e^{-hl\tau} c_e^*. \end{cases} \tag{3.6}$$

**Remark 3.2** For any positive solution  $(c_o(t), c_e(t))$  of system (3.1) with the initial value  $(c_o(0), c_e(0)) \in R_2^+$ , we can obtain

$$\begin{aligned} & \lim_{t \rightarrow +\infty} (c_o(t)) \\ &= \frac{c_o^*(1-e^{-(g+m)l\tau})}{(g+m)\tau} \\ &+ \frac{fc_e^*}{h-g-m} - \frac{fc_e^*(1-e^{-(h-g-m)l\tau})}{(h-g-m)^2\tau} \\ &+ \frac{c_o^{**}(e^{-(g+m)l\tau} - e^{-(g+m)\tau})}{\tau} \\ &+ \frac{f(1-l)c_e^{**}}{h-g-m} - \frac{fc_e^{**}(e^{-(h-g-m)l\tau} - e^{-(h-g-m)\tau})}{(h-g-m)^2\tau} \triangleq \widetilde{c}_0, \end{aligned} \tag{3.7}$$

where  $c_o^*$  and  $c_e^*$  are defined as (3.4), and  $c_o^{**}$  and  $c_e^{**}$  are defined as (3.6).

For convenience, we consider the following notation:

$$\tau_n = n\tau, \quad \tau_{n+l} = (n+l)\tau, \quad h_{n+l} = h_1.$$

Define  $(x(t), y(t))$  and  $(w(t), z(t))$  are the solutions of the subsystem of system (2.1), respectively:

$$\begin{cases} \left. \begin{aligned} dx(t) &= [D(x_0 - x(t)) \\ &\quad - \frac{\beta x(t)y(t)}{k(A+x(t)+By(t))}] dt \\ &\quad + x(t)(\sigma_{11} + \sigma_{12}x(t)) dB_1(t), \\ dy(t) &= [\frac{\beta x(t)y(t)}{A+x(t)+By(t)} \\ &\quad - Dy(t) - rc_o(t)y(t)] dt \\ &\quad + y(t)(\sigma_{21} + \sigma_{22}y(t)) dB_2(t), \end{aligned} \right\} t \neq (n+l)\tau, \\ \left. \begin{aligned} \Delta x(t) &= 0, \\ \Delta y(t) &= -h_1 y(t), \end{aligned} \right\} t = (n+l)\tau, n \in Z^+, \end{cases} \tag{3.8}$$

and the following SDE without impulsive perturbations:

$$\begin{cases} dw(t) = [D(x_0 - w(t)) - \frac{\beta w(t) \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)}{k(A+w(t)+B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t))}] dt \\ \quad + w(t)(\sigma_{11} + \sigma_{12} w(t)) dB_1(t), \\ dz(t) = [\frac{\beta w(t) z(t)}{A+w(t)+B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)} - Dz(t) - rc_o(t)z(t)] dt \\ \quad + z(t)[\sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)] dB_2(t), \end{cases} \tag{3.9}$$

with the initial value  $w(0) = x(0)$  and  $z(0) = y(0)$ .

**Lemma 3.3** *The solutions  $(x(t), y(t))$  of the subsystem of system (2.1) can also be expressed as follows:*

$$\begin{cases} x(t) = w(t), \\ y(t) = \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t), \end{cases} \tag{3.10}$$

where  $(w(t), z(t))$  is the solution of (3.10).

*Proof* One can find that  $(x(t), y(t))$  is continuous on the interval  $(\tau_n, \tau_{n+l})$ , and for  $t \neq \tau_{n+l}$ ,

$$\begin{cases} dx(t) = dw(t) \\ \quad = [D(x_0 - w(t)) - \frac{\beta w(t) \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)}{k(A+w(t)+B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t))}] dt \\ \quad \quad + w(t)(\sigma_{11} + \sigma_{12} w(t)) dB_1(t) \\ \quad = [D(x_0 - w(t)) - \frac{\beta w(t) y(t)}{k(A+w(t)+B y(t))}] dt \\ \quad \quad + w(t)(\sigma_{11} + \sigma_{12} w(t)) dB_1(t), \\ dy(t) = \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) dz(t) \\ \quad = \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \{ [\frac{\beta w(t) z(t)}{A+w(t)+B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)} - Dz(t) - rc_o(t)z(t)] dt \\ \quad \quad + z(t)[\sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)] dB_2(t) \} \\ \quad = [\frac{\beta w(t) \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)}{A+w(t)+B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)} - D \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) - rc_o(t) \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)] dt \\ \quad \quad + \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) [\sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)] dB_2(t) \\ \quad = [\frac{\beta w(t) y(t)}{A+w(t)+B y(t)} - Dy(t) - rc_o(t)y(t)] dt \\ \quad \quad + y(t)[\sigma_{21} + \sigma_{22} y(t)] dB_2(t). \end{cases} \tag{3.11}$$

For every  $n \in N$ , and  $\tau_{n+l} \in [0, +\infty)$ ,

$$\begin{aligned} y(\tau_{n+l}^+) &= \lim_{t \rightarrow \tau_{n+l}^+} \prod_{0 < \tau_j < t} (1 - h_j) z(t) \\ &= \prod_{0 < \tau_j \leq \tau_{n+l}} (1 - h_{\tau_j}) z(\tau_{n+l}^+) \\ &= (1 - h_{\tau_{n+l}}) \prod_{0 < \tau_j < \tau_{n+l}} (1 - h_{\tau_j}) z(\tau_{n+l}) \\ &= (1 - h_{\tau_{n+l}}) y(\tau_{n+l}), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
 y(\tau_{n+1}^-) &= \lim_{t \rightarrow \tau_{n+1}^-} \prod_{0 < \tau_j < t} (1 - h_j) z(t) \\
 &= \prod_{0 < \tau_j < \tau_{n+1}} (1 - h_{\tau_j}) z(\tau_{n+1}^-) \\
 &= \prod_{0 < \tau_j < \tau_{n+1}} (1 - h_{\tau_j}) z(\tau_{n+1}) = y(\tau_{n+1}).
 \end{aligned} \tag{3.13}$$

□

**Assumption 3.4** ([12]) There exists a bounded domain  $U \subset E_d$  with regular boundary, then

- (A<sub>1</sub>) In the open domain  $U$  and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix  $A(x)$  is bounded away from zero;
- (A<sub>2</sub>) If  $x \in E_d \setminus U$ , the mean time  $\tau$  at which a path issuing from  $x$  reaches the set  $U$  is finite, and  $\sup_{x \in K} E_x \tau < \infty$  for every compact subset  $K \subset U_d$ .

Assumption 3.4 is a general assumption which is the condition for Lemma 3.6.

**Lemma 3.5** ([12]) *If Assumption 3.4 holds, the Markov process  $X(t)$  has a stationary distribution  $\mu(\cdot)$ , and*

$$\mathbb{P} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \int_{E_d} f(x) \mu(dx) \right\} = 1,$$

where  $f$  is an integrable function with respect to the measure  $\mu$ .

#### 4 The dynamics

In the following theorem, we devote ourselves to investigating system (3.10).

**Theorem 4.1** *If  $\frac{\beta}{A} \int_0^\infty w \phi(w) dw < D + r \tilde{c}_0 + \frac{\sigma_{21}^2}{2}$  holds, then*

$$\lim_{t \rightarrow +\infty} z(t) = 0 \quad a.s., \tag{4.1}$$

where for  $x \in (0, +\infty)$

$$\begin{aligned}
 \phi(x) &= Cx^{-\frac{2(2Dx_0\sigma_{12}+D\sigma_{11})}{\sigma_1^3}} \times (\sigma_{11} + \sigma_{12}x)^{-\frac{2(2Dx_0\sigma_{12}+D\sigma_{11})}{\sigma_{11}^3}} \\
 &\times e^{-\frac{2}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}(\frac{Dx_0}{x} + \frac{2Dx_0\sigma_{12}+D\sigma_{11}}{\sigma_{11}})},
 \end{aligned} \tag{4.2}$$

and constant  $C$  satisfies that  $\int_0^\infty \phi(x) dx = 1$ .

*Proof* Constructing the following auxiliary differential equation:

$$dW(t) = [D(x_0 - W(t))] dt + W(t)(\sigma_{11} + \sigma_{12}W(t)) dB_1(t), \tag{4.3}$$

with the initial value  $W(0) = x(0) > 0$ , we assume that  $W(t)$  is the solution of (4.3). Obviously, the following inequality can be obtained by the comparison theorem for stochastic

differential equations:

$$w(t) \leq W(t) \quad \text{a.s.} \tag{4.4}$$

We set

$$a(w) = D(x_0 - w(t)), \quad \sigma(w) = w(\sigma_{11} + \sigma_{12}w), \quad w \in (0, +\infty), \tag{4.5}$$

and compute the following indefinite integral:

$$\begin{aligned} \int \frac{a(t)}{\sigma^2(t)} dt &= \int \frac{D(x_0 - t)}{t^2(\sigma_{11} + \sigma_{12}t)^2} dt \\ &= \frac{2Dx_0\sigma_{12} + D\sigma_{11}}{\sigma_{11}^3} \ln \frac{\sigma_{11} + \sigma_{12}t}{t} \\ &\quad - \frac{Dx_0}{\sigma_{11}t(\sigma_{11} + \sigma_{12}t)} - \frac{2Dx_0\sigma_{12} + D\sigma_{11}}{\sigma_{11}^2t(\sigma_{11} + \sigma_{12}t)} + C. \end{aligned} \tag{4.6}$$

Then

$$e^{\int \frac{a(t)}{\sigma^2(t)} dt} = e^C \left( \frac{\sigma_{11} + \sigma_{12}t}{t} \right)^{\frac{2Dx_0\sigma_{12} + D\sigma_{11}}{\sigma_{11}^3}} e^{-\frac{1}{\sigma_{11}(\sigma_{11} + \sigma_{12}t)} \left( \frac{Dx_0}{t} + \frac{2Dx_0\sigma_{12} + D\sigma_{11}}{\sigma_{11}} \right)}. \tag{4.7}$$

Hence,

$$\begin{aligned} \int_0^\infty \frac{1}{\sigma^2(w)} e^{\int_0^w \frac{2a(s)}{\sigma^2(s)} ds} dw &= \int_0^\infty w^{-2} (\sigma_{11} + \sigma_{12}w)^2 \left( \frac{\sigma_{11} + \sigma_{12}w}{w} \right)^{\frac{2(2Dx_0\sigma_{12} + D\sigma_{11})}{\sigma_{11}^3}} \\ &\quad \times e^{-\frac{2}{\sigma_{11}(\sigma_{11} + \sigma_{12}w)} \left( \frac{2(2Dx_0\sigma_{12} + D\sigma_{11})}{\sigma_{11}^3} \right)} dw < \infty. \end{aligned} \tag{4.8}$$

This indicates that SDE (4.3) has the ergodic property. By the ergodic theorem, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t w(s) ds = \int_0^\infty w\phi(w) dw \quad \text{a.s.} \tag{4.9}$$

Applying Itô's formula, we have

$$\begin{aligned} d \ln z(t) &= \left[ \frac{\beta w(t)}{A + w(t) + B \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t)} - D - rc_0(t) \right. \\ &\quad \left. - \frac{1}{2} \left( \sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) \right)^2 \right] dt \\ &\quad + \left( \sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) \right) dB_2(t) \\ &\leq \left[ \frac{\beta w(t)}{A} - D - rc_0(t) - \frac{\sigma_{21}^2}{2} \right. \\ &\quad \left. - \sigma_{21}\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) - \frac{(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \sigma_{22})^2}{2} z^2(t) \right] dt \\ &\quad + \left( \sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z(t) \right) dB_2(t). \end{aligned} \tag{4.10}$$

Integrating with respect to  $t$  from 0 to  $t$  on both sides of (4.10), we have

$$\begin{aligned} \ln z(t) \leq & \frac{\beta}{A} \int_0^t w(s) ds - \left( D + \frac{\sigma_{21}^2}{2} \right) t - \int_0^t rc_o(s) ds \\ & - \frac{(\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2}{2} \int_0^t z^2(s) ds + \sigma_{21} B_2(t) + M(t) + \ln y(0), \end{aligned} \tag{4.11}$$

where  $M(t) = \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \int_0^t z(s) dB_2(t)$  and its quadratic variation is given by

$$\langle M, M \rangle(t) = \left( \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right)^2 \int_0^t z^2(s) ds. \tag{4.12}$$

According to the exponential martingales inequality, for any positive  $\tau, \alpha, \beta$ ,

$$P \left\{ \sup_{0 \leq t \leq T} \left[ M(t) - \frac{a}{2 \langle M, M \rangle(t)} \right] > b \right\} \leq e^{-ab}. \tag{4.13}$$

Let  $T = k_1, a = 1, b = \ln k_1$ , then

$$P \left\{ \sup_{0 \leq t \leq T} \left[ M(t) - \frac{1}{2 \langle M, M \rangle(t)} \right] > \ln k_1 \right\} \leq \frac{1}{k_1}. \tag{4.14}$$

There exists random  $k_1^0 \in k_1(\omega)$  such that  $k_1 > k_1^0$  for almost all  $\omega \in \Omega$ . We can obtain the following by the Borel–Cantelli lemma:

$$\sup_{0 \leq t \leq T} \left[ M(t) - \frac{1}{2 \langle M, M \rangle(t)} \right] \leq \ln k_1. \tag{4.15}$$

Therefore

$$\begin{aligned} M(t) & \leq \ln k_1 + \frac{1}{2} \langle M, M \rangle(t) \\ & = \ln k_1 + \frac{(\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2}{2} \int_0^t z^2(s) ds, \\ & \text{for all } 0 < t < k_1, k_1 > k_1^0 \text{ a.s.} \end{aligned} \tag{4.16}$$

Considering (4.11) and (4.16), we have

$$\ln z(t) \leq \frac{\beta}{A} \int_0^t w(s) ds - \left( D + \frac{\sigma_{21}^2}{2} \right) t - \int_0^t rc_o(s) ds + \sigma_{21} B_2(t) + \ln k_1 + \ln y(0). \tag{4.17}$$

Then, for  $0 \leq k_1 - 1 \leq t \leq k_1$ , we have

$$\begin{aligned} \frac{\ln z(t)}{t} & \leq \frac{\beta}{At} \int_0^t w(s) ds - \left( D + \frac{\sigma_{21}^2}{2} \right) \\ & \quad - \frac{1}{t} \int_0^t rc_o(s) ds + \frac{\sigma_{21} B_2(t)}{t} + \frac{\ln k_1}{k_1 - 1} + \frac{\ln y(0)}{t}. \end{aligned} \tag{4.18}$$



Taking the superior limit on both sides of (4.18), note that

$$\lim_{t \rightarrow +\infty} B_2(t) = 0 \quad \text{a.s.} \tag{4.19}$$

and  $t \rightarrow +\infty \Rightarrow k_1 \rightarrow +\infty$ , we have

$$\lim_{k_1 \rightarrow +\infty} \frac{\ln k_1}{k_1} = \lim_{k_1 \rightarrow +\infty} \frac{1}{k_1} = 0. \tag{4.20}$$

Then we obtain

$$\limsup_{t \rightarrow +\infty} \frac{\ln z(t)}{t} \leq \frac{\beta}{A} \int_0^{+\infty} w\phi(w) dw - \left( D + r\tilde{c}_o + \frac{\sigma_{21}^2}{2} \right). \tag{4.21}$$

This implies that if  $\frac{\beta}{A} \int_0^{+\infty} w\phi(w) dw < D + r\tilde{c}_o + \frac{\sigma_{21}^2}{2}$  holds, then

$$\lim_{t \rightarrow +\infty} z(t) = 0 \quad \text{a.s.}$$

This completes the proof. □

*Remark 4.2* According to Lemma 3.3 and Theorem 4.1, one can easily obtain

$$\lim_{t \rightarrow +\infty} y(t) = 0 \quad \text{a.s.}$$

**Theorem 4.3** *If  $\frac{\beta}{A} \int_0^{+\infty} w\phi(w) dw < D + r\tilde{c}_o + \frac{\sigma_{21}^2}{2}$  holds, the distribution of  $x(t)$  converges weakly to the measure which has the density  $\pi(x)$ .*

*Proof* For any small  $\varepsilon > 0$ , there exist  $t_0$  and a set  $\Omega_\varepsilon \subset \Omega$  such that  $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$  and  $\frac{xy}{k(A+x+By)} \leq \varepsilon x$  for  $t \geq t_0$  and  $\omega \in \Omega_\varepsilon$ . Then

$$\begin{aligned} & \left[ (x_0 - x(t)) - \varepsilon x(t) \right] dt + x(t)(\sigma_{11} + \sigma_{12}x(t)) dB_1(t) \\ & \leq dx(t) \leq \left[ (x_0 - x(t)) \right] dt + x(t)(\sigma_{11} + \sigma_{12}x(t)) dB_1(t). \end{aligned}$$

This shows that the distribution of the process  $x(t)$  converges weakly to the measure with density  $\pi(x)$ . □

**Theorem 4.4** *If  $Dx_0\beta > (D + \sigma_{11}^2 + \frac{2\sigma_{12}Dx_0}{\sigma_{11}})(D + rc_o^u + \frac{1}{2}\sigma_{21}^2)$  holds, system (3.10) admits a unique stationary distribution and it has ergodic property for initial  $(w(0), z(0)) \in \mathbb{R}_+^2$ .*

*Proof* Define

$$\begin{aligned} & V^*(w, z) \\ & = M \left[ -c_1 \ln w - c_2 l \ln z + \frac{2c_1}{\theta(1-\theta)\sigma_{11}^\theta} (\sigma_{11} + \sigma_{12}w)^\theta \right] + \frac{1}{p} w^p + \frac{1}{p} z^p \\ & \triangleq MV_1^* + V_2^*, \end{aligned}$$

where  $\theta, p \in (0, 1)$ , and  $M$  is a sufficiently large constant satisfying the following condition:

$$M \left[ -2Dw_0 \left( \sqrt{\frac{Dx_0\beta}{(D + \sigma_{11}^2 + \frac{2\sigma_{12}Dx_0}{\sigma_{11}})(D + rc_o^u + \frac{1}{2}\sigma_{21}^2)}} - 1 \right) \right] + f^u + g^u \leq -2,$$

where

$$\begin{aligned} c_1 &= \frac{Dw_0}{1 + \sigma_{11}^2 + \frac{2\sigma_{12}Dw_0}{(D-\theta)\sigma_{11}}}, \\ c_2 &= \frac{Dw_0}{D + rc_o^u + \frac{1}{2}\sigma_{21}^2}, \\ f^u &= \sup_{w \in \mathbb{R}_+} \left\{ Dw_0w^{p-1} - Dw^p - \frac{1-p}{2}\sigma_{12}^2w^{p+2} \right\}, \\ g^u &= \sup_{w \in \mathbb{R}_+} \left\{ -Dz^p - rc_o^l z^p - \frac{1-p}{2} \left[ \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right]^2 y^{p+2} \right\}. \end{aligned}$$

It shows that

$$\liminf_{\varepsilon \rightarrow 0, (w,z) \in \mathbb{R}_+ \setminus D} V^*(w, z) = +\infty,$$

where  $D = (\varepsilon, \frac{1}{\varepsilon}) \times (\varepsilon, \frac{1}{\varepsilon})$  and  $V^*(w, z)$  is a continuous function. Therefore,  $V^*(w, z)$  has a minimum point  $(w_0, z_0)$  in the interior of  $\mathbb{R}_+^2$ . Thus, we can define a nonnegative  $C^2$ -function  $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$

$$V(w, z) = V^*(w, z) - V^*(w_0, z_0).$$

We can obtain the following equation by Itô's formula:

$$LV = LV^* = MLV_1^* + LV_2^*.$$

Therefore,

$$\begin{aligned} LV_1^* &= -\frac{c_1Dw_0}{w} + c_1D + \frac{c_1\beta(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z)}{K[A + w + B(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z)]} \\ &\quad + \frac{2c_1\sigma_{12}Dw_0}{(1-\theta)\sigma_{11}^\theta}(\sigma_{11} + \sigma_{12}w)^{\theta-1} - \frac{2c_1\sigma_{12}w}{(1-\theta)\sigma_{11}^\theta}(\sigma_{11} + \sigma_{12}w)^{\theta-1} \\ &\quad - \frac{2c_1\sigma_{12}\beta w(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z)}{(1-\theta)\sigma_{11}^\theta k[A + w + B(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z)]}(\sigma_{11} + \sigma_{12}w)^{\theta-1} \\ &\quad + \frac{c_1}{2}(\sigma_{11} + \sigma_{12}w)^2 - \frac{c_1\sigma_{12}^2}{\sigma_{11}^\theta}(\sigma_{11} + \sigma_{12}w)^\theta w^2 \\ &\quad - \frac{c_2\beta w}{A + w + B(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z)} + c_2D + c_2rc_o(t) \\ &\quad + \frac{c_2}{2} \left[ \sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z \right]^2 \end{aligned}$$

$$\begin{aligned}
 &\leq -\frac{c_1 Dw_0}{w} - c_1 \beta w + c_2 \beta w + c_1 D + \frac{c_1 \beta w}{kA} + \frac{2c_1 \sigma_{12} Dw_0}{(1-\theta)\sigma_{11}} \\
 &\quad + c_1 \sigma_{11}^2 - \frac{c_1}{2}(\sigma_{11} - \sigma_{12} w)^2 + c_2 D + c_2 r c_o^u + \frac{c_2}{2} \left( \sigma_{21} + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z \right)^2 \\
 &\leq -2\sqrt{c_1 c_2 Dw_0 \beta} + c_1 \left[ D + \sigma_{11}^2 + \frac{2\sigma_{12} Dw_0}{(1-\theta)\sigma_{11}} \right] + c_2 \left[ D + r c_o^u + \frac{\sigma_{21}^2}{2} \right] \\
 &\quad + \left[ \frac{c_1 \beta}{kA} + c_2 \sigma_{21} \left( \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right) \right] w \\
 &\quad + \frac{c_2 (\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2 z^2}{2} + c_2 \beta w \\
 &= -2w_0 \left( \sqrt{\frac{w_0}{(1 + \sigma_{11}^2 + \frac{2\sigma_{12} w_0}{\sigma_{11}})(1 + r c_o^u + \frac{1}{2} \sigma_{21}^2)}} - 1 \right) \\
 &\quad + \left[ \frac{c_1 \beta}{kA} + c_2 \sigma_{21} \left( \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right) \right] z \\
 &\quad + \frac{c_2 (\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2 z^2}{2} + c_2 \beta w,
 \end{aligned}$$

where  $c_1$  and  $c_2$  are such that

$$c_1 \left[ D + \sigma_{11}^2 + \frac{2\sigma_{12} Dw_0}{(1-\theta)\sigma_{11}} \right] = c_2 \left[ D + r c_o^u + \frac{1}{2} \sigma_{21}^2 \right] = Dw_0.$$

The function  $\frac{Dw_0 \beta}{(D + \sigma_{11}^2 + \frac{2\sigma_{12} Dw_0}{\sigma_{11}})(D + r c_o^u + \frac{1}{2} \sigma_{21}^2)} > 1$  is continuous. Choose  $\varepsilon > 0$  sufficiently small such that

$$\begin{aligned}
 LV_1^* &\leq -2Dw_0 \left( \sqrt{\frac{Dw_0 \beta}{(D + \sigma_{11}^2 + \frac{2\sigma_{12} Dw_0}{\sigma_{11}})(D + r c_o^u + \frac{1}{2} \sigma_{21}^2)}} - 1 \right) \\
 &\quad + \left[ \frac{c_1 \beta}{kA} + c_2 \sigma_{21} \left( \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right) \right] z \\
 &\quad + \frac{c_2 (\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2 z^2}{2} + c_2 \beta w.
 \end{aligned}$$

We can also have

$$\begin{aligned}
 LV_2^* &\leq Dw_0 w^{p-1} - Dw^p - \frac{\beta \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z w^p}{k(A + w + B(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z))} \\
 &\quad - \frac{1-p}{2} w^{p-2} w^2 (\sigma_{11} + \sigma_{12} w)^2 + \frac{\beta w [\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z]^p}{A + w + B(\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z)} \\
 &\quad - Dw^p - r c_o(t) w^p - \frac{1-p}{2} \left[ \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z \right]^p \left( \sigma_{21} + \sigma_{22} \left[ \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z \right]^2 \right) \\
 &\leq Dw_0 w^{p-1} - Dw^p - \frac{1-p}{2} \sigma_{12}^2 w^{p+2} - Dw^p - r c_o^l z^p \\
 &\quad - \frac{1-p}{2} \sigma_{22}^2 \left[ \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z \right]^{p+2} + \frac{\beta w (\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z)^p}{A}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 LV \leq & M \left\{ -2Dw_0 \left( \sqrt{\frac{Dw_0\beta}{(D + \sigma_{11}^2 + \frac{2\sigma_{12}Dw_0}{\sigma_{11}})(D + rc_o^u + \frac{1}{2}\sigma_{21}^2)}} - 1 \right) \right. \\
 & + \left[ \frac{c_1\beta}{kA} + c_2\sigma_{21}\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right] z + \frac{c_2(\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2}{2} z^2 + c_2\beta w \Big\} \\
 & + Dw_0x^{p-1} - Dx^p - rc_o^l z^p \\
 & - \frac{1-p}{2} \sigma_{22}^2 \left[ \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z \right]^{p+2} + \frac{\beta x (\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z)^p}{A}.
 \end{aligned}$$

Let

$$\begin{aligned}
 H(w, z) = & M \left\{ -2Dw_0 \left( \sqrt{\frac{Dw_0\beta}{(D + \sigma_{11}^2 + \frac{2\sigma_{12}Dw_0}{\sigma_{11}})(D + rc_o^u + \frac{1}{2}\sigma_{21}^2)}} - 1 \right) \right. \\
 & + \left[ \frac{c_1\beta}{kA} + c_2\sigma_{21}\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) \right] z \\
 & + \frac{c_2(\sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}))^2}{2} z^2 + c_2\beta w \Big\} \\
 & + f(x) + g(z) + \frac{\beta x (\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z)^p}{A},
 \end{aligned}$$

where  $f(x) = Dw_0w^{p-1} - Dx^p - \frac{1-p}{2}\sigma_{12}^2x^{p+2}$  and  $g(z) = -Dx^p - rc_o^l z^p - \frac{1-p}{2}\sigma_{22}^2[\prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z]^{p+2}$ . Then we can get

$$H(w, z) \leq \begin{cases} H(+\infty, z) \rightarrow -\infty, & \text{as } w \rightarrow +\infty, \\ H(w, +\infty) \rightarrow -\infty, & \text{as } z \rightarrow +\infty, \\ M[-2Dw_0(\sqrt{\frac{Dw_0\beta}{(D+\sigma_{11}^2 + \frac{2\sigma_{12}Dw_0}{\sigma_{11}})(D+rc_o^u + \frac{1}{2}\sigma_{21}^2)}} - 1)] \\ \quad + f^u + g^u \leq -2, & \text{as } w \rightarrow 0^+, z \rightarrow 0^+. \end{cases} \tag{4.22}$$

Therefore, there exists sufficiently small  $\varepsilon > 0$  such that

$$LV \leq -1, \quad \text{for any } (w, z) \in \mathbb{R}_+^2 \setminus \mathbb{D},$$

where  $\mathbb{D} = (\varepsilon, \frac{1}{\varepsilon}) \times (\varepsilon, \frac{1}{\varepsilon})$ .

On the other hand, the diffusion matrix of system (3.10) is given by

$$\begin{aligned}
 \sum_{i,j=1}^2 a_{ij}(w, z) \xi_i \xi_j = & \left( (\sigma_{11}w + \sigma_{12}w^2) \xi_1, \left( \sigma_{21}z + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z^2 \right) \xi_2 \right) \\
 & \times \begin{pmatrix} (\sigma_{11}w + \sigma_{12}w^2) \xi_1 \\ (\sigma_{21}z + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l}) z^2) \xi_2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 &= (\sigma_{11}w + \sigma_{12}w^2)\xi_1^2 + \left( \sigma_{21}z + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z^2 \right) \xi_2^2 \\
 &\geq G \|\xi\|^2 \quad \text{for any } (w, z) \in D_e \subset \mathbb{D}_+^2,
 \end{aligned}$$

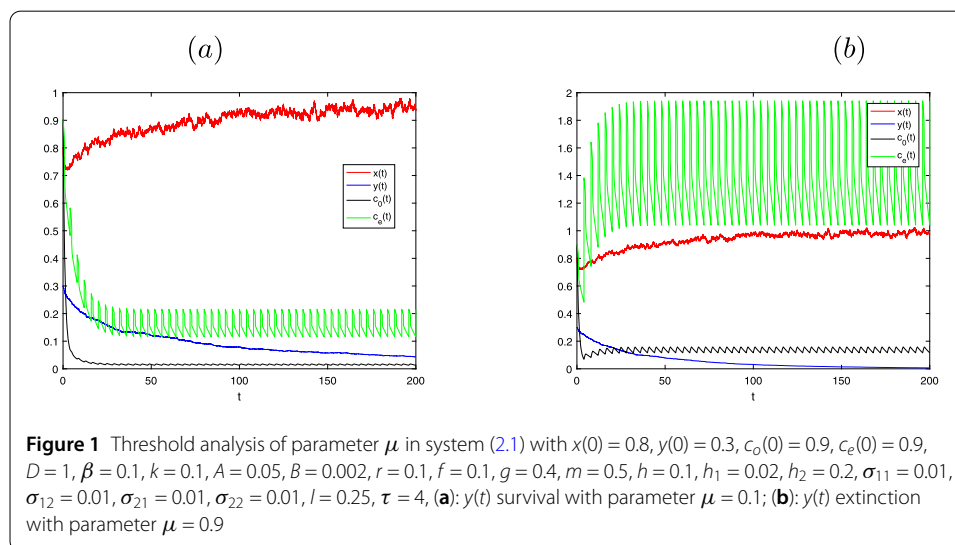
where  $\xi = (\xi_1, \xi_2) \in \mathbb{D}_+^2$ ,  $G = \min_{(w,z) \in D_e} \{(\sigma_{11}w + \sigma_{12}w^2)\xi_1^2 + (\sigma_{21}z + \sigma_{22} \prod_{0 < \tau_{n+l} < t} (1 - h_{n+l})z^2)\xi_2^2\}$ , and  $D_e = [\frac{1}{e}, e] \times [\frac{1}{e}, e]$ .

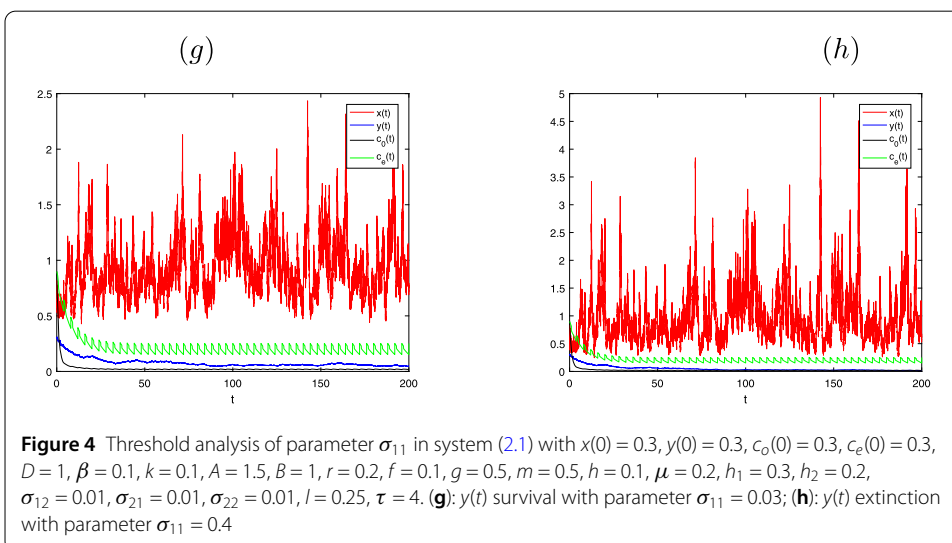
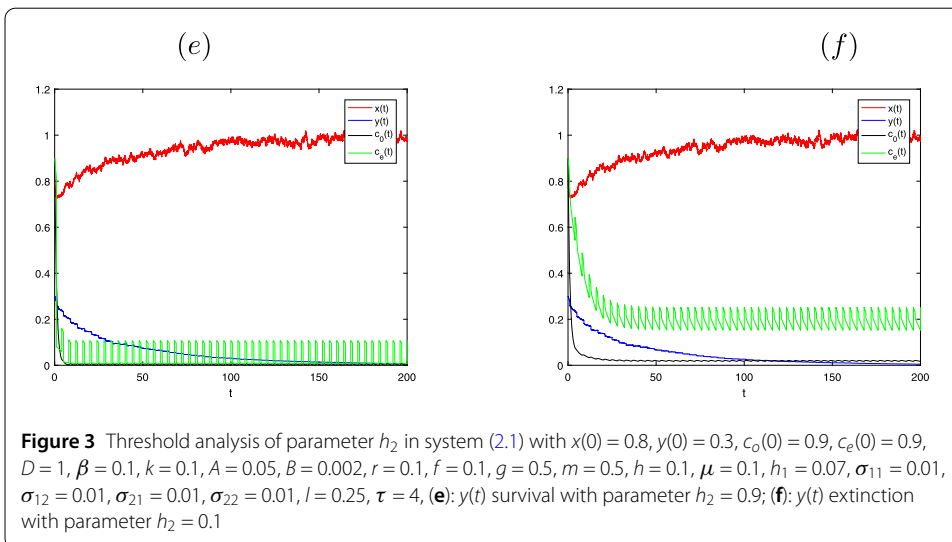
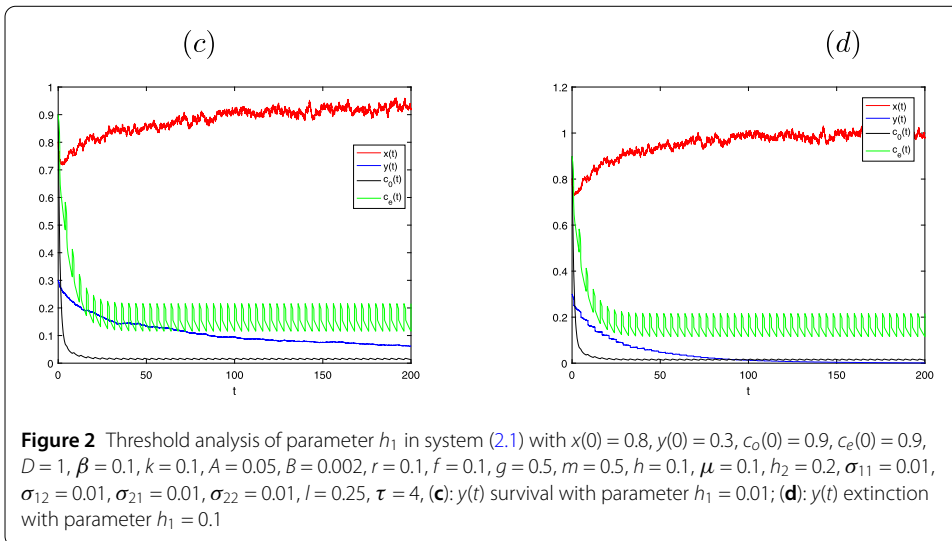
From Theorem 4.7 in reference [9], it can be known that system (3.11) is ergodic and has a unique stationary distribution, and the distribution of the process converges weakly to the measure with density. □

*Remark 4.5* From Lemma 3.3 and Theorem 4.4, we can easily know that if  $Dx_0\beta > (D + \sigma_{11}^2 + \frac{2\sigma_{12}Dx_0}{\sigma_{11}})(D + rc_o^\mu + \frac{1}{2}\sigma_{21}^2)$  holds, system (2.1) admits a unique stationary distribution and it has ergodic property for initial  $(x(0), y(0)) \in \mathbb{R}_+^2$ .

### 5 Numerical simulations

If it is assumed that  $x(0) = 0.8, y(0) = 0.3, c_o(0) = 0.9, c_e(0) = 0.9, D = 1, \beta = 0.1, k = 0.1A = 0.05, B = 0.002, r = 0.1, f = 0.1, g = 0.5, m = 0.5, h = 0.1, h_1 = 0.02, h_2 = 0.2, \sigma_{11} = 0.01, \sigma_{12} = 0.01, \sigma_{21} = 0.01, \sigma_{22} = 0.01, l = 0.25, \tau = 4$ , when  $\mu = 0.1$ , the microorganism  $y(t)$  will survive (it can be seen in (a) of Fig. 1), when  $\mu = 0.9$ , the microorganism  $y(t)$  will be extinct(it can be seen in (b) of Fig. 1). From Theorem 4.1, Remark 4.5, and the computer simulations in Fig. 1, we conjecture that there must exist a threshold  $\mu^*$ , if  $\mu > \mu^*$ , the microorganism  $y(t)$  will be extinct, if  $\mu < \mu^*$ , the microorganism  $y(t)$  will survive. If it is assumed that  $x(0) = 0.8, y(0) = 0.3, c_o(0) = 0.9, c_e(0) = 0.9, D = 1, \beta = 0.1, k = 0.1, A = 0.05, B = 0.002, r = 0.1, f = 0.1, g = 0.5, m = 0.5, h = 0.1, \mu = 0.1, h_2 = 0.2, \sigma_{11} = 0.01, \sigma_{12} = 0.01, \sigma_{21} = 0.01, \sigma_{22} = 0.01, l = 0.25, \tau = 4$ , when  $h_1 = 0.01$ , the microorganism  $y(t)$  will survive (it can be seen in (c) of Fig. 2), when  $h_1 = 0.1$  the microorganism  $y(t)$  will be extinct(it can be seen in (d) of Fig. 2). From Theorem 4.1, Remark 4.5, and the computer simulations in Fig. 2, we conjecture that there must exist a threshold  $h_2^*$ , if  $h_2 < h_2^*$ , the microorganism  $y(t)$  will be extinct, if  $h_2 > h_2^*$ , the microorganism  $y(t)$  will survive. If it is assumed that  $x(0) = 0.8, y(0) = 0.3, c_o(0) = 0.9, c_e(0) = 0.9, D = 1, \beta = 0.1, k = 0.1, A = 0.05, B = 0.002,$





$r = 0.1, f = 0.1, G = 0.5, M = 0.5, h = 0.1, \mu = 0.1, h_1 = 0.07, \sigma_{11} = 0.01, \sigma_{12} = 0.01, \sigma_{21} = 0.01, \sigma_{22} = 0.01, l = 0.25, \tau = 4$ , when  $h_2 = 0.9$ , the microorganism  $y(t)$  will survive (it can be seen in (e) of Fig. 3), when  $h_2 = 0.1$ , the microorganism  $y(t)$  will be extinct (it can be seen in (f) of Fig. 3). From Theorem 4.1, Remark 4.5, and the computer simulations in Fig. 3, we conjecture that there must exist a threshold  $h_1^*$ , if  $h_1 < h_1^*$ , the microorganism  $y(t)$  will be extinct, if  $h_1 > h_1^*$ , the microorganism  $y(t)$  will survive. If it is assumed that  $x(0) = 0.8, y(0) = 0.3, c_o(0) = 0.9, c_e(0) = 0.9, D = 1, \beta = 0.1, k = 0.1, A = 0.05, B = 0.002, r = 0.1, f = 0.1, g = 0.5, m = 0.5, h = 0.1, \mu = 0.1, h_1 = 0.02, h_2 = 0.1, \sigma_{12} = 0.1, \sigma_{21} = 0.05, \sigma_{22} = 0.05, l = 0.25, \tau = 4$ , and when  $\sigma_{11} = 0.3$ , the microorganism  $y(t)$  will survive (it can be seen in (g) of Fig. 4), when  $\sigma_{11} = 0.6$ , the microorganism  $y(t)$  will be extinct (it can be seen in (h) of Fig. 4). From Theorem 4.1, Remark 4.5, and the computer simulations in Fig. 4, we conjecture that there must exist a threshold  $\sigma_{11}^*$ , if  $\sigma_{11} < \sigma_{11}^*$ , the microorganism  $y(t)$  will be extinct, if  $\sigma_{11} > \sigma_{11}^*$ , the microorganism  $y(t)$  will survive.

## 6 Discussion

In this work, we consider a stochastic eutrophication-chemostat model with impulsive dredging and pulse inputting on environmental toxicant. The sufficient condition for the extinction of microorganisms is obtained. The sufficient condition for the investigated system with unique ergodic stationary distribution is also obtained by the Lyapunov functions method. The results of mathematical analysis and numerical analysis show that the stochastic noise, impulsive diffusion, and pulse input on environmental toxicant play important roles in the extinction and survival of the microorganisms. These results indicate the effective and reliable controlling strategy for water resource management with eutrophication.

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### Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

### Ethics approval and consent to participate

Not applicable.

### Competing interests

The authors declare that there is no conflict of interests.

### Consent for publication

Not applicable.

### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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