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RESEARCH

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Degenerate poly-Bernoulli polynomials arising from degenerate polylogarithm

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Abstract

Recently, degenerate polylogarithm functions were introduced by Kim and Kim. In this paper, we introduce degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm functions and investigate some their properties. In more detail, we find certain explicit expressions for those polynomials in terms of the Carlitz degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind. Furthermore, we obtain some expressions for differences of the degenerate poly-Bernoulli polynomials.

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Keywords: Degenerate poly-Bernoulli polynomials; Degenerate polylogarithm

1 Introduction

Carlitz [5, 6] initiated a study of degenerate versions of some special numbers and polynomials, the degenerate Bernoulli and Euler polynomials. In recent years, the idea of studying degenerate versions of many special polynomials and numbers regained interests of some mathematicians, and many interesting results were found (see [10–13, 16, 18, 19]). They have been explored by employing several different tools such as combinatorial methods, generating functions, p-adic analysis, umbral calculus techniques, differential equations, and probability theory.

The aim of this paper is to introduce the degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm functions and to study their properties including their explicit expressions and differences. Here we note that those polynomials are slight modifications of the previously studied ones under the same name.

The outline of this paper is as follows. In Sect. 1, as a preparation to the next section, we recall the Carlitz degenerate Bernoulli polynomials, the degenerate exponential functions, the degenerate polylogarithms, and the degenerate Stirling numbers of the second kind. In Sect. 2, we introduce the degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm functions; note that they reduce to the Carlitz degenerate Bernoulli polynomials when $k = 1$. We express the generating function of the degenerate poly-Bernoulli polynomials as an iterated integral, from which we find an explicit expression for these polynomials when $k = 2$. Also, we find explicit expressions for the degenerate poly-Bernoulli polynomials in terms of the Carlitz degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind. Finally, we find certain expressions for certain differences of the degenerate poly-Bernoulli polynomials.

For $0 \neq \lambda \in \mathbb{R}$, Carlitz introduced the degenerate Bernoulli polynomials given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [5, 6])}. \tag{1}$$

In the case $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda\to 0} \beta_{n,\lambda}(x) = B_n(x)$ $(n \geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [14, 15, 20, 23, 24])}. \tag{2}$$

For $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \qquad e_\lambda(t) = e_\lambda^1(t) \quad \text{(see [10–13, 16, 18, 19])}, \tag{3}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$. Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = e^{xt}$.

For $k \in \mathbb{Z}$, the polylogarithm is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad \text{(see [1–4, 9, 10, 12, 21])}. \tag{4}$$

Note that $\mathrm{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$.

Kaneko [7] considered the poly-Bernoulli numbers arising from the polylogarithm and defined by

$$\frac{1}{1-e^{-t}}\mathrm{Li}_k\bigl(1-e^{-t}\bigr) = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!} \quad \text{(see [1–4, 8, 17, 22])}. \tag{5}$$

Note that $B_n^{(1)} = B_n(1)$ $(n \geq 0)$.

Recently, Kim and Kim introduced the degenerate polylogarithm functions defined by

$$\mathrm{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n \quad (k \in \mathbb{Z}) \text{ (see [10, 19])}. \tag{6}$$

Note that $\lim_{\lambda\to 0} \mathrm{Li}_{k,\lambda}(x) = \mathrm{Li}_k(x)$.

Let $\log_\lambda(t)$ be the inverse function of $e_\lambda(t)$ such that $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n!}t^n = \frac{1}{\lambda}\bigl((1+t)^\lambda - 1\bigr) \quad \text{(see [10, 13, 16, 18])}. \tag{7}$$

Note that from (6) we have $\mathrm{Li}_{1,\lambda}(x) = -\log_\lambda(1-x)$.

In [10] the degenerate poly-Bernoulli polynomials are defined by means of the degenerate polylogarithm function as

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)}e_\lambda^x(-t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{8}$$

Note that $\beta_{n,\lambda}^{(1)}(x)=(-1)^n\beta_{n,\lambda}(x)$, where $\beta_{n,\lambda}(x)$ are the Carlitz degenerate Bernoulli polynomials defined by (1).

In [10, 11, 16] the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l \quad (n\geq 0). \tag{9}$$

From (9) we can easily show that the generating function of the degenerate Stirling numbers of the second kind is given by

$$\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad \text{(see [13, 16, 18])}. \tag{10}$$

2 Degenerate poly-Bernoulli polynomials

We slightly modify the definition of degenerate poly-Bernoulli polynomials in (8) by

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}, \tag{11}$$

which are again called the degenerate poly-Bernoulli polynomials. Note different definitions in (8) and (11). In the case $x=0$, $B_{n,\lambda}^{(k)}=B_{n,\lambda}^{(k)}(0)$ $(n\geq 0)$ are called the degenerate poly-Bernoulli numbers.

Note that by (11)

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(1)}(x)\frac{t^n}{n!}&=\frac{\mathrm{Li}_{1,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)\\&=\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

Thus by (12) we get $B_{n,\lambda}^{(1)}(x)=\beta_{n,\lambda}(x)$ $(n\geq 0)$.

By (11) we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}&=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)\\&=\sum_{l=0}^{\infty}B_{l,\lambda}^{(k)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(k)}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{13}$$

Thus we have

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{l,\lambda}^{(k)}(x)_{n-l,\lambda} \quad (n \geq 0).$$

It is not difficult to show that

$$\begin{aligned}
&\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1} e_\lambda^x(t) \\
&= \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \\
&= \frac{e_\lambda^x(t)}{e_\lambda(t)-1} \underbrace{\int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)} \int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)} \cdots \int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}}_{(k-1)\text{-times}} t\,dt \cdots dt.
\end{aligned} \tag{14}$$

By (14) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^{(2)}(x) \frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)-1} \int_0^t \frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)} t\,dt \\
&= \frac{te_\lambda^x(t)}{e_\lambda(t)-1} \sum_{l=0}^{\infty} \frac{\beta_{l,\lambda}(1-\lambda)}{l+1} (-1)^l \frac{t^l}{l!} \\
&= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{t^m}{m!} \sum_{l=0}^{\infty} (-1)^l \frac{\beta_{l,\lambda}(1-\lambda)}{l+1} \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{\beta_{l,\lambda}(1-\lambda)}{l+1} (-1)^l \beta_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Comparing the coefficients on both sides of (15), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(2)}(x) = \beta_{n,\lambda}(x) - \frac{n}{4}(1-2\lambda)\beta_{n-1,\lambda}(x) + \sum_{l=2}^{n} \binom{n}{l} (-1)^l \frac{\beta_{l,\lambda}(1-\lambda)}{l+1} \beta_{n-l,\lambda}(x).$$

In general,

$$B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{l,\lambda}^{(k)}(x)_{n-l,\lambda}.$$

Note that by (11)

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1} e_\lambda^x(t) \\
&= \frac{t}{e_\lambda(t)-1} e_\lambda^x(t) \cdot \frac{1}{t} \mathrm{Li}_{k,\lambda}\bigl(1-e_\lambda(-t)\bigr).
\end{aligned} \tag{16}$$

On the other hand,

$$
\begin{aligned}
\frac{1}{t}\operatorname{Li}_{k,\lambda}\bigl(1-e_\lambda(-t)\bigr) &= \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}\bigl(1-e_\lambda(-t)\bigr)^n \\
&= \frac{1}{t}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\sum_{l=n}^{\infty}S_{2,\lambda}(l,n)(-1)^{l-1}\frac{t^l}{l!} \\
&= \frac{1}{t}\sum_{l=1}^{\infty}\sum_{n=1}^{l}\frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}(-1)^{l-1}S_{2,\lambda}(l,n)\frac{t^l}{l!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=1}^{l+1}\frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\cdot\frac{S_{2,\lambda}(l+1,n)}{l+1}(-1)^l\right)\frac{t^l}{l!}.
\end{aligned}
\tag{17}
$$

By (16) and (17) we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{p=1}^{l+1}\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}(-1)^l\frac{S_{2,\lambda}(l+1,p)}{l+1}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^l\binom{n}{l}\sum_{p=1}^{l+1}\frac{\lambda^{p-1}}{p^{k-1}}(1)_{p,1/\lambda}\frac{S_{2,\lambda}(l+1,p)}{l+1}\beta_{n-l,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{18}
$$

Therefore by (18) we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$
B_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\left(\sum_{p=1}^{l+1}\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}\cdot\frac{S_{2,\lambda}(l+1,p)}{l+1}\right)\beta_{n-l,\lambda}(x).
$$

Now we observe that

$$
\begin{aligned}
&\frac{\operatorname{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{x+1}(t) - \frac{\operatorname{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{x}(t) \\
&= e_\lambda^x(t)\operatorname{Li}_{k,\lambda}\bigl(1-e_\lambda(-t)\bigr) \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}\bigl(1-e_\lambda(-t)\bigr)^m \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{m=1}^{\infty}\frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\sum_{p=m}^{\infty}S_{2,\lambda}(p,m)(-1)^{p-1}\frac{t^p}{p!} \\
&= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{p=1}^{\infty}\sum_{m=1}^{p}\frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}(-1)^{p-1}S_{2,\lambda}(p,m)\frac{t^p}{p!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{p=1}^{n}\binom{n}{p}(-1)^{p-1}\sum_{m=1}^{p}\frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}S_{2,\lambda}(p,m)(x)_{n-p,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{19}
$$

On the other hand,

$$\begin{aligned}
&\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{x+1}(t)-\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{x}(t)\\
&=\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x+1)\frac{t^n}{n!}-\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\\
&=\sum_{n=1}^{\infty}\bigl(B_{n,\lambda}^{(k)}(x+1)-B_{n,\lambda}^{(k)}(x)\bigr)\frac{t^n}{n!}.
\end{aligned}\tag{20}$$

Therefore by (19) and (20) we obtain the following theorem.

Theorem 3 *For $n\in\mathbb{N}$, we have*

$$B_{n,\lambda}^{(k)}(x+1)-B_{n,\lambda}^{(k)}(x)=\sum_{p=1}^{n}\binom{n}{p}(-1)^{p-1}\Biggl(\sum_{m=1}^{p}\frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}S_{2,\lambda}(p,m)\Biggr)(x)_{n-p,\lambda}.$$

Note that by (11)

$$\begin{aligned}
&\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{x}(t)\\
&=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda^{d}(t)-1}\sum_{a=0}^{d-1}e_\lambda^{a+x}(t)\\
&=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{t}\frac{1}{d}\sum_{a=0}^{d-1}\frac{dt}{e_\lambda^{d}(t)-1}e_\lambda^{a+x}(t)\\
&=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{t}\frac{1}{d}\sum_{a=0}^{d-1}\frac{dt}{e_{\frac{\lambda}{d}}(dt)-1}e_{\frac{\lambda}{d}}^{\frac{a+x}{d}}(dt)\\
&=\frac{1}{t}\sum_{p=1}^{\infty}\frac{(-\lambda)^{p-1}(1)_{p,1/\lambda}}{(p-1)!p^k}\bigl(1-e_\lambda(-t)\bigr)^p\sum_{a=0}^{d-1}\sum_{m=0}^{\infty}\beta_{m,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr)d^{m-1}\frac{t^m}{m!}\\
&=\frac{1}{t}\sum_{p=1}^{\infty}\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}\sum_{l=p}^{\infty}S_{2,\lambda}(l,p)(-1)^{l-1}\frac{t^l}{l!}\sum_{m=0}^{\infty}d^{m-1}\sum_{a=0}^{d-1}\beta_{m,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr)\frac{t^m}{m!}\\
&=\frac{1}{t}\sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{(-1)^{l-1}\lambda^{p-1}}{p^{k-1}}(1)_{p,1/\lambda}S_{2,\lambda}(l,p)\frac{t^l}{l!}\sum_{m=0}^{\infty}d^{m-1}\sum_{a=0}^{d-1}\beta_{m,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr)\frac{t^m}{m!}\\
&=\sum_{l=0}^{\infty}\sum_{p=1}^{l+1}(-1)^l\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}\frac{S_{2,\lambda}(l+1,p)}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}d^{m-1}\sum_{a=0}^{d-1}\beta_{m,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr)\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\sum_{p=1}^{l+1}\frac{\lambda^{p-1}(1)_{p,1.\lambda}}{p^{k-1}}\frac{S_{2,\lambda}(l+1,p)}{l+1}d^{n-l-1}\sum_{a=0}^{d-1}\beta_{n-l,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr))\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\Biggl(\sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}(-1)^l d^{n-l-1}\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}\frac{S_{2,\lambda}(l+1,p)}{l+1}\beta_{n-l,\lambda/d}\Bigl(\frac{a+x}{d}\Bigr)\Biggr)\frac{t^n}{n!}.
\end{aligned}\tag{21}$$

On the other hand,

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{22}$$

Therefore by (21) and (22) we obtain the following theorem.

Theorem 4 *For $n,k\in\mathbb{Z}$ with $n\geq 0$ and $d\in\mathbb{N}$, we have*

$$B_{n,\lambda}^{(k)}(x)=\sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}(-1)^l d^{n-l-1}\frac{\lambda^{p-1}(1)_{p,1/\lambda}}{p^{k-1}}\frac{S_{2,\lambda}(l+1,p)}{l+1}\beta_{n-l,\lambda/d}\left(\frac{a+x}{d}\right).$$

From the definition of the degenerate poly-Bernoulli polynomials we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x+y)\frac{t^n}{n!}&=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^{(x+y)}(t)=\left(\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t)\right)e_\lambda^y(t)\\&=\sum_{l=0}^{\infty}B_{l,\lambda}^{(k)}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(k)}(x)(y)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

Comparing the coefficients on both sides of (23), we have

$$B_{n,\lambda}^{(k)}(x+y)=\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(k)}(x)(y)_{n-l,\lambda}\quad(n\geq 0). \tag{24}$$

Consider the following expression:

$$\frac{1}{\lambda}\sum_{l=0}^{n}\binom{n}{l}B_{n-l,\lambda}^{(k)}\left((x+\lambda)_{l,\lambda}-(x)_{l,\lambda}\right). \tag{25}$$

From (13) we see that (25) is equal to

$$\frac{1}{\lambda}\left(B_{n,\lambda}^{(k)}(x+\lambda)-B_{n,\lambda}^{(k)}(x)\right). \tag{26}$$

On the other hand, we see that (25) is also equal to

$$\begin{aligned}&\frac{1}{\lambda}\sum_{l=1}^{n}\binom{n}{l}B_{n-l,\lambda}^{(k)}\lambda l(x)_{l-1,\lambda}\\&\quad=n\sum_{l=0}^{n-1}\binom{n-1}{l}B_{n-1-l,\lambda}^{(k)}(x)_{l,\lambda}\\&\quad=nB_{n-1,\lambda}^{(k)}(x).\end{aligned} \tag{27}$$

Thus from (24), combining (26) with (27), we obtain the following result.

Theorem 5 *For $n\geq 1$, we have*

$$nB_{n-1,\lambda}^{(k)}(x)=\frac{1}{\lambda}\left(B_{n,\lambda}^{(k)}(x+\lambda)-B_{n,\lambda}^{(k)}(x)\right)$$

and

$$B_{n,\lambda}^{(k)}(x+y)=\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(k)}(x)(y)_{n-l,\lambda}\quad (n\geq 0).$$

3 Conclusion

In this paper, we defined the degenerate poly-Bernoulli polynomials, which are slight modifications of the previous ones. We represented the generating function of those polynomials as an iterated integral, from which we obtained an explicit expression of those polynomials for $k=2$ in terms of the Carlitz degenerate Bernoulli polynomials. We also found two explicit expressions of the degenerate poly-Bernoulli polynomials involving the Carlitz degenerate Bernoulli polynomials and the degenerate Stirling numbers of the second kind in Theorems 2 and 4. In addition, we were able to find certain expressions for differences of the degenerate poly-Bernoulli polynomials in Theorems 3 and 5.

We refer the reader to [18] and the references therein for three possible immediate applications of our results to probability, differential equations, and symmetry.

As one of our future projects, we would like to continue studying degenerate versions of many special polynomials and numbers and their applications to physics, science, and engineering, and mathematics.

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; LCJ, HL, and HYK checked the results of the paper; DSK and TK completed the revision of the paper. All authors have read and approved the final version of the manuscript.

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References

1. Abouzahra, M., Lewin, L.: The polylogarithm in algebraic number fields. J. Number Theory **21**(2), 214–244 (1985)
2. Adelberg, A.: Kummer congruences for universal Bernoulli numbers and related congruences for poly-Bernoulli numbers. Int. Math. J. **1**(1), 53–63 (2002)
3. Bayad, A., Hamahata, Y.: Polylogarithms and poly-Bernoulli polynomials. Kyushu J. Math. **65**(1), 15–24 (2011)
4. Bényi, B., Hajnal, P.: Combinatorics of poly-Bernoulli numbers. Studia Sci. Math. Hung. **52**(4), 537–558 (2015)
5. Carlitz, L.: A degenerate Staudt–Clausen theorem. Arch. Math. (Basel) **7**, 28–33 (1956)
6. Carlitz, L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Util. Math. **15**, 51–88 (1979)
7. Kaneko, M.: Poly-Bernoulli numbers. J. Théor. Nr. Bordx. **9**(1), 221–228 (1997)
8. Khan, W.A., Ahmad, M.: Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **28**(3), 487–496 (2018)

9. Kim, D.S., Kim, T.: A note on polyexponential and unipoly functions. Russ. J. Math. Phys. **26**(1), 40–49 (2019)
10. Kim, D.S., Kim, T.: A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. **27**(2), 227–235 (2020)
11. Kim, D.S., Kim, T., Kwon, J., Lee, H.: A note on λ-Bernoulli numbers of the second kind. Adv. Stud. Contemp. Math. (Kyungshang) **30**(2), 187–195 (2020)
12. Kim, T., Kim, D.S.: Degenerate polyexponential functions and degenerate Bell polynomials. J. Math. Anal. Appl. **487**(2), 124017 (2020)
13. Kim, T., Kim, D.S.: Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials. Symmetry **12**(6), 905 (2020)
14. Kim, T., Kim, D.S.: A note on central Bell numbers and polynomials. Russ. J. Math. Phys. **27**(1), 76–81 (2020)
15. Kim, T., Kim, D.S.: Note on the degenerate gamma function. Russ. J. Math. Phys. **27**(3), 352–358 (2020)
16. Kim, T., Kim, D.S., Jang, L.-C., Lee, H.: Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae–Stirling numbers. Adv. Differ. Equ. **2020**, 245 (2020)
17. Kim, T., Kim, D.S., Kim, H.Y., Kwon, J.: A note on degenerate multi-poly-Genocchi polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **30**(3), 447–454 (2020)
18. Kim, T., Kim, D.S., Kwon, J., Kim, H.Y.: A note on degenerate Genocchi and poly-Genocchi numbers and polynomials. J. Inequal. Appl. **2020**, 110 (2020)
19. Kim, T., Kim, D.S., Kwon, J., Lee, H.: Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials. Adv. Differ. Equ. **2020**, 168 (2020)
20. Kruchinin, D.V., Kruchinin, V.V.: Explicit formula for reciprocal generating function and its application. Adv. Stud. Contemp. Math. (Kyungshang) **29**(3), 365–372 (2019)
21. Lewin, L.: Polylogarithms and Associated Functions. North-Holland, Amsterdam (1981). With a foreword by A.J. Van der Poorten
22. Lim, D., Kwon, J.: A note on poly-Daehee numbers and polynomials. Proc. Jangjeon Math. Soc. **19**(2), 219–224 (2016)
23. Pyo, S.-S.: Degenerate Cauchy numbers and polynomials of the fourth kind. Adv. Stud. Contemp. Math. (Kyungshang) **28**(1), 127–138 (2018)
24. Simsek, Y.: Identities on the Changhee numbers and Apostol-type Daehee polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **27**(2), 199–212 (2017)