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Solutions for impulsive fractional pantograph differential equation via generalized anti-periodic boundary condition

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Abstract

This study investigates the solutions of an impulsive fractional differential equation incorporated with a pantograph. This work extends and improves some results of the impulsive fractional differential equation. A differential equation of an impulsive fractional pantograph with a more general anti-periodic boundary condition is proposed. By employing the well-known fixed point theorems of Banach and Krasnoselskii, the existence and uniqueness of the solution of the proposed problem are established. Furthermore, two examples are presented to support our theoretical analysis.

Keywords: Pantograph differential equation; Impulsive; Anti-periodic condition; Fixed point theorems

1 Introduction

Fractional differential equations are generally applicable in many fields such as chemistry, mechanics, fluid systems, electronics, electromagnetic and other fields; for an overview, the reader should see the literature on fractional differential equations, e.g., [3, 4, 12, 17, 21, 24, 26, 33–36, 39] and the references therein. Fractional and impulsive differential equations were used as a powerful method to gain insight into certain emerging problems from various science and engineering fields [32, 40, 46]. In particular, much attention has been given to the theoretical studies such as existence, uniqueness, and stability of analytical solutions, in recent years (we refer, for example, to [1, 2, 6, 14–16, 38, 43]).

Several works on boundary value problems for an impulsive differential equation with anti-periodic boundary conditions were conducted, and results on the existence of solutions for mixed-type fractional integro-differential equation were established (see, e.g., [5, 19, 30, 42, 51, 52, 54, 55]). More recently, the theory of existence, uniqueness, and stability analysis for impulsive fractional differential equations with different kinds of fractional operators and initial/boundary conditions has attracted the attention of many researchers; for an overview of the literature, we refer the reader to [8–11, 49, 50]. For example, in [47] Wang and Lin investigated the impulsive fractional anti-periodic boundary value problem with constant coefficients. Recently, motivated by [47], Zuo et al. [56] investigated the existence results for an equation with impulsive and anti-periodic boundary conditions described by:

$$\begin{cases} {}^cD^q_{0+}u(t) + \lambda u(t) = f(t, u(t), Tu(t), Su(t)), \quad t \in J' = J\backslash\{t_1, \dots, t_m\}, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = -u(1), \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, $\lambda > 0$, $I_k \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, $J = [0, 1]$, $\mathbb{R}$ is the set of real numbers, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, S and T are linear operators. Under Lipschitz and nonlinear growth conditions, they established sufficient conditions for the existence and uniqueness of a solution to (1.1) using Banach mapping principle and Krasnoselskii's fixed point theorem.

On the other hand, in the deterministic situation there is a very special delay differential equation known as the pantograph equation given by

$$z'(\tau) = \alpha z(\tau) + \beta z(\lambda \tau), \quad 0 \le \tau \le T,$$

where $0 < \lambda < 1$. It is used in various fields of applied and pure mathematics, such as number theory, probability, dynamic system, and quantum mechanics. In particular, an important studies were conducted on the properties of both the analytical and numerical solutions of this equation (see [18, 22, 23, 31]), also recently multi-pantograph and generalized nonlinear multi-pantograph equations were studied in [29, 37, 53]. Owing to the increasing interest and importance of this equation, Balachandran et al. [13] established the solution of abstract fractional pantograph equation via fractional calculus techniques and fixed point method. They consider the following equation:

$$\begin{cases} {}^cD^p_{0+}y(t) = f(t, y(t), y(\lambda t)), \quad t \in J, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $0 < p < 1$, $0 < \lambda < 1$, and $f : J \times X \times X \to X$ is a continuous function.

In this paper, motivated by [13, 47, 56], we consider the following impulsive fractional pantograph differential equation:

$$\begin{cases} {}^cD^\alpha_{0+}x(t) + \lambda x(t) = f(t, x(t), x(\gamma t)), \\ \quad t \in J^* = J\backslash\{t_1, \dots, t_k\}, 0 < \alpha < 1, 0 < \gamma < 1, \\ \Delta x|_{t=t_m}(0) = I_m(x(t_m)), \quad m = 1, 2, \dots, k, \\ ax(0) + bx(1) = 0, \quad a \ge b > 0, \end{cases} \tag{1.3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $J = [0, 1]$, $\Delta x|_{t=t_m} = x(t_m^+) - x(t_m^-)$, with $x(t_m^+)$ and $x(t_m^-)$ representing the right and left limits of $x(t)$ at $t = t_m$.

The main aim of this paper is to establish the existence and uniqueness of solutions for the boundary value problem (1.3), by using the contraction principle of Banach and the fixed point theorem of Krasnoselskii. Presently, different techniques have been extensively applied in obtaining solutions to the impulsive fractional differential equations (see, e.g., [20, 41, 44]). However, in this article, we adopt the solution approach used in [47] to solve the impulsive fractional equation (1.3).

We highlight the main contributions of this paper as follows:

- We consider the impulsive pantograph fractional differential equation.
- We consider more general anti-periodic boundary value problems with constant coefficients.

In general, this paper contributes toward the development of qualitative analysis of fractional differential equations.

This paper is organized as follows: the statement of the problem, preliminaries, and some useful lemmas that will be required for the later sections are presented in Sect. 2. In Sect. 3, we prove the existence and uniqueness of solutions for problem (1.3) via Banach and Krasnoselskii's fixed point theorems with some illustrative examples. Conclusions on our findings are presented in the last section.

2 Preliminaries and lemmas

Let $J_0 = (0,t_1], J_1 = (t_1,t_2],\ldots,J_{k-1}, J_k = (t_k,1]$, and $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x \in C(J_m,\mathbb{R})\}$, where $m = 0,1,2,\ldots,k$, $x(t_m^+)$ and $x(t_m^-)$ exist, $m = 1,2,\ldots,k$, is a space of continuous real-valued functions on the interval J, and $x(t_m^-) = x(t_m)$.

Then, clearly, $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$, and let the norm of a measurable function $\varphi : J \to \mathbb{R}$ be defined by:

$$\|\varphi\|_{L^p(J)} = \begin{cases} (\int_J |\varphi(t)|^p \, dt)^{1/p}, & 1 \leq p < \infty, \\ \inf_{\operatorname{mes}(\bar{J})=0}\{\sup_{t\in J\setminus\bar{J}} |x(t)|\}, & p = \infty. \end{cases}$$

Then $L^p(J,\mathbb{R})$ is a Banach space of Lebesque-measurable functions with $\|\varphi\|_{L^p(J)} < \infty$.

Definition 2.1 (see [26]) The fractional integral of order ρ with the lower limit zero for a function g is defined as

$$I_{0+}^{\rho} g(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} f(\tau)\, d\tau, \quad \rho > 0, n \in \mathbb{N},$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 (see [26]) The Riemann–Liouville derivative of order ρ with the lower limit zero for a function g is defined as

$${}^{L}D_{0+}^{\rho} g(t) = \frac{1}{\Gamma(n-\rho)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\rho-1} g(\tau)\, d\tau, \quad \rho > 0, n-1 < \rho < n,$$

provided the function g is absolutely continuous up to order $(n-1)$ derivatives, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.3 (see [26]) The Caputo derivative of order $\rho > 0$ with the lower limit zero for a function g is defined as

$$^{c}D^{\rho}_{0+}g(t) = \frac{1}{\Gamma(n-\rho)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\rho-1}\left(g(\tau) - \sum_{l=0}^{n-1}\frac{s^l}{l!}g^{(l)}(0)\right)d\tau, \quad n = [\rho]+1, n \in \mathbb{N},$$

provided the function $g : [0,\infty) \to \mathbb{R}$, where $\Gamma(\cdot)$ denotes the Gamma function.

Remark 2.4 (see [28]) If $g \in C^n[0,+\infty)$, then

$$^{c}D^{\rho}_{0+}g(t) = \frac{1}{\Gamma(n-\rho)}\int_0^t (t-\tau)^{n-\rho-1}g^{(n)}(\tau)\,d\tau = I^{n-\rho}g^{(n)}(t), \quad t > 0, n = [\rho]+1.$$

Since in this paper we deal with an impulsive problem, Definition 2.3 is appropriate.

Definition 2.5 A function $x \in PC(J,\mathbb{R})$ is said to be a solution of problem (1.3) if it satisfies the equation $^{c}D^{\alpha}_{0+}x(t) + \lambda x(t) = f(t,x(t),x(\gamma t))$ a.e. on J^* and the conditions $\Delta x|_{t=t_m}(0) = I_m(x(t_m)), m = 1,2,\ldots,k$, and $ax(0) + bx(1) = 0$, with $a \geq b > 0$.

Lemma 2.6 (see [45]) *The nonnegative functions* E_α *and* $\mathrm{E}_{\alpha,\alpha}$ *given by*

$$\mathrm{E}_\alpha(u) = \sum_{m=0}^{\infty}\frac{u^m}{\Gamma(\alpha m+1)}, \qquad \mathrm{E}_{\alpha,\alpha}(u) = \sum_{m=0}^{\infty}\frac{u^m}{\Gamma(\alpha m+\alpha)},$$

have the following properties:

(1) *For any* $\lambda > 0$ *and* $t \in J$,

$$\mathrm{E}_\alpha\left(-t^\alpha\lambda\right) \leq 1, \qquad \mathrm{E}_{\alpha,\alpha}\left(-t^\alpha\lambda\right) \leq \frac{1}{\Gamma(\alpha)}.$$

Moreover, $\mathrm{E}_\alpha(0) = 1, \mathrm{E}_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$.

(2) *For any* $\lambda > 0$ *and* $t_1, t_2 \in J$,

$$\mathrm{E}_\alpha\left(-t_2^\alpha\lambda\right) \to \mathrm{E}_\alpha\left(-t_1^\alpha\lambda\right) \quad \text{as } t_2 \to t_1,$$
$$\mathrm{E}_{\alpha,\alpha}\left(-t_2^\alpha\lambda\right) \to \mathrm{E}_{\alpha,\alpha}\left(-t_1^\alpha\lambda\right) \quad \text{as } t_2 \to t_1.$$

(3) *For any* $\lambda > 0$ *and* $t_1, t_2 \in J$ *such that* $t_1 \leq t_2$,

$$\mathrm{E}_\alpha\left(-t_2^\alpha\lambda\right) \leq \mathrm{E}_\alpha\left(-t_1^\alpha\lambda\right), \qquad \mathrm{E}_{\alpha,\alpha}\left(-t_2^\alpha\lambda\right) \leq \mathrm{E}_{\alpha,\alpha}\left(-t_1^\alpha\lambda\right).$$

Lemma 2.7 (see [27]) *Let* P *be a closed, convex, and nonempty subset of a Banach space* X, *and let* F_1, F_2 *be operators such that*:

(1) $F_1x + F_2y \in M$ *whenever* $x, y \in P$,

(2) F_1 *is compact and continuous*,

(3) F_2 *is a contraction mapping*.

Then there exists $z \in P$ *such that* $z = F_1z + F_2z$.

Lemma 2.8 (see [48]) *Let X be a Banach space, and let $J=[0,T]$. Suppose that $\mathcal{W}\subset PC(J,X)$ satisfies the following conditions*:

(1) *$\mathcal{W}$ is a uniformly bounded subset of $PC(J,X)$,*

(2) *$\mathcal{W}$ is equicontinuous in $(t_m,t_{m+1}), m=0,1,\dots,k$, where $t_0=0, t_{k+1}=T$,*

(3) *Its t-sections $\mathcal{W}(t)=\{x(t):x\in\mathcal{W}, t\in J\backslash\{t_1,\dots,t_k\}\}, \mathcal{W}(t_m^+)=\{x(t_m^+):x(t_m^+):x\in W\}$, and $\mathcal{W}(t_m^-)=\{x(t_m^-):x(t_m^-):x\in W\}$ are relatively compact subsets of X.*

Then $\mathcal{W}$ is a relatively compact subset of $PC(J,X)$.

Lemma 2.9 (see [47]) *Let $g:J\to\mathbb{R}$ be a continuous function. The function u given by*

$$u(t)=\begin{cases}\frac{-E_q(-\lambda)E_q(-t^q\lambda)}{1+E_q(-\lambda)}\sum_{i=1}^m\frac{y_i}{E_q(-t_i^q\lambda)}+\int_0^t(t-s)^{q-1}E_{q,q}(-(t-s)^q\lambda)g(s)\,ds\\ \quad-\frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\int_0^1(1-s)^{q-1}E_{q,q}(-(1-s)^q\lambda)g(s)\,ds,\quad t\in J_0,\\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\{\sum_{i=1}^m\frac{y_i}{E_q(-t_i^q\lambda)}-\int_0^1(1-s)^{q-1}E_{q,q}(-(1-s)^q\lambda)g(s)\,ds\}\\ \quad-E_q(-t^q\lambda)\sum_{j=k+1}^m\frac{y_j}{E_q(-t_j^q\lambda)}\\ \quad+\int_0^t(t-s)^{q-1}E_{q,q}(-(t-s)^q\lambda)g(s)\,ds,\quad t\in J_m, m=1,2,\dots,k-1,\\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\{\sum_{i=1}^m\frac{y_i}{E_q(-t_i^q\lambda)}-\int_0^1(1-s)^{q-1}E_{q,q}(-(1-s)^q\lambda)g(s)\,ds\}\\ \quad+\int_0^t(t-s)^{q-1}E_{q,q}(-(t-s)^q\lambda)g(s)\,ds,\quad t\in J_k,\end{cases}\tag{2.1}$$

is a solution of the impulsive problem

$$\begin{cases}{}^cD_{0+}^q u(t)+\lambda u(t)=g(t),\quad t\in J^*,\\ \Delta u|_{t=t_m}(0)=y_m,\quad m=1,2,\dots,k,\\ u(0)+u(1)=0.\end{cases}\tag{2.2}$$

It follows from Lemma 2.9 and by using the boundary condition $ax(0)+bx(1)=0$ that the solution of (1.3) can be expressed by

$$u(t)=\begin{cases}\frac{-E_\alpha(-\lambda)E_\alpha(-t^\alpha\lambda)}{1+E_\alpha(-\lambda)}\sum_{i=1}^k\frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)}\\ \quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\lambda)f(s,x(s),x(\gamma s)\,ds\\ \quad-\frac{E_\alpha(-t^\alpha\lambda)}{1+\kappa E_\alpha(-\lambda)}\kappa\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(-(1-s)^\alpha\lambda)f(s,x(s),x(\gamma s)\,ds,\quad t\in J_0,\\ \frac{E_\alpha(-t^\alpha\lambda)}{1+\kappa E_\alpha(-\lambda)}\{\sum_{i=1}^k\frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)}\\ \quad-\kappa\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(-(1-s)^\alpha\lambda)f(s,x(s),x(\gamma s))\,ds\}\\ \quad-E_\alpha(-t^\alpha\lambda)\sum_{j=m+1}^k\frac{I_j(x(t_j))}{E_\alpha(-t_j^\alpha\lambda)}\\ \quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\lambda)f(s,x(s),x(\gamma s))\,ds,\\ \quad t\in J_m, m=1,2,\dots,k-1,\\ \frac{E_\alpha(-t^\alpha\lambda)}{1+\kappa E_\alpha(-\lambda)}\{\sum_{i=1}^k\frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)}\\ \quad-\kappa\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(-(1-s)^\alpha\lambda)f(s,x(s),x(\gamma s))\,ds\}\\ \quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\lambda)f(s,x(s),x(\gamma s))\,ds,\quad t\in J_k,\end{cases}\tag{2.3}$$

where $\kappa=\frac{b}{a}$.

3 Main results

Theorem 3.1 *Consider the following hypotheses*:

(C_1) *Function* $f \times J \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exists a constant* $L_1 > 0$ *such that*

$$\left|f(t,x,y) - f(t,u,v)\right| \le L_1\left(|x-u| + |y-v|\right),$$

for all $t \in J, x, y, u, v \in \mathbb{R}$.

(C_2) *There exists a positive constant* L_2 *such that*

$$\left|I_m(x) - I_m(y)\right| \le L_2|x-y|, \quad \text{for all } x, y \in \mathbb{R}, m = 1, 2, \dots, k.$$

(C_3)

$$\eta = \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|}\left(\sum_{i=1}^{k} \frac{L_2}{|E_\alpha(-t_i^\alpha\lambda)|} + \frac{(4\kappa+2)L_1}{(2+\kappa)\Gamma(\alpha+1)}\right) < 1.$$

Then the boundary value problem (1.3) *has a unique solution*.

Proof Define a mapping $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}(Tx)(t) = {} & \frac{E_\alpha(-t^\alpha\lambda)}{1+\kappa E_\alpha(-\lambda)}\left\{\sum_{i=1}^{k}\frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)}\right. \\ & \left. - \kappa\int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(1-s)^\alpha\lambda\right)f\left(s,x(s),x(\gamma s)\right)ds\right\} \\ & - E_\alpha\left(-t^\alpha\lambda\right)\sum_{j=m+1}^{k}\frac{I_j(x(t_j))}{E_\alpha(-t_j^\alpha\lambda)} \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-s)^\alpha\lambda\right)f\left(s,x(s),x(\gamma s)\right)ds, \\ & t \in [t_m, t_{m+1}), m = 0, 1, 2, \dots, k, \end{aligned} \tag{3.1}$$

then we show that T has a fixed point, which is a solution of problem (1.3). Letting $G = \sup_{t\in J}|f(t,0,0)|$, $G^* = \max\{|I_i(0)| : i = 1, 2, \dots, k\}$, we choose

$$r \ge \frac{\sum_{i=1}^{k}\frac{G^*}{|E_\alpha(-t_i^\alpha\lambda)|} + \frac{G}{\Gamma(\alpha+1)}}{\frac{|1+\kappa E_\alpha(-\lambda)|}{(2+\kappa)} - \left[\sum_{i=1}^{k}\frac{L_2}{|E_\alpha(-t_i^\alpha\lambda)|} + \frac{(4\kappa+2)L_1}{(2+\kappa)\Gamma(\alpha+1)}\right]}. \tag{3.2}$$

Firstly, we show that $TH_r \subset H_r$, where $H_r = \{x \in PC(J,\mathbb{R}) : \|x\|_{PC} \le r\}$. It follows from the hypotheses above and Lemma 2.6 that

$$\begin{aligned}&\left|(Tx)(t)\right| \\ &\quad \le \left|E_\alpha\left(-t^\alpha\lambda\right)\right|\left|\frac{1}{1+\kappa E_\alpha(-\lambda)}\left\{\sum_{i=1}^{k}\frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)}\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \kappa \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(1-s)^\alpha \lambda\bigr) f\bigl(s, x(s), x(\gamma s)\bigr)\, ds \Biggr\} - \sum_{j=m+1}^{k} \frac{I_j(x(t_j))}{E_\alpha(-t_j^\alpha \lambda)} \Biggr| \\
&\quad + \biggl| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(t-s)^\alpha \lambda\bigr) f\bigl(s, x(s), x(\gamma s)\bigr)\, ds \biggr| \\
&\leq \frac{1}{|1+\kappa E_\alpha(-\lambda)|} \Biggl\{ \sum_{i=1}^{k} \frac{|I_i(x(t_i))|}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{\kappa}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \bigl| f(s, x(s), x(\gamma s)) \bigr|\, ds \Biggr\} \\
&\quad + \sum_{i=1}^{k} \frac{|I_i(x(t_i))|}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl| f(s, x(s), x(\gamma s)) \bigr|\, ds \\
&\leq \frac{1+|1+\kappa E_\alpha(-\lambda)|}{|1+\kappa E_\alpha(-\lambda)|} \Biggl\{ \sum_{i=1}^{k} \frac{|I_i(x(t_i)) - I_i(0)| + G^*}{|E_\alpha(-t_i^\alpha \lambda)|} \Biggr\} \\
&\quad + \frac{\kappa}{\Gamma(\alpha)|1+\kappa E_\alpha(-\lambda)|} \int_0^1 (1-s)^{\alpha-1} \bigl[\bigl| f\bigl(s, x(s), x(\gamma s)\bigr) - f(s,0,0) \bigr| + \bigl| f(s,0,0) \bigr| \bigr]\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl[\bigl| f\bigl(s, x(s), x(\gamma s)\bigr) - f(s,0,0) \bigr| + \bigl| f(s,0,0) \bigr| \bigr]\, ds \\
&\leq \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \sum_{i=1}^{k} \frac{L_2 r + G^*}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{G}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|} + \frac{G}{\Gamma(\alpha+1)} \\
&\quad + \frac{\kappa}{\Gamma(\alpha)|1+\kappa E_\alpha(-\lambda)|} \int_0^1 (1-s)^{\alpha-1} L_1\bigl(\bigl|x(s) + x(\gamma s)\bigr|\bigr)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_1\bigl(\bigl|x(s) + x(\gamma s)\bigr|\bigr)\, ds \\
&\leq \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \Biggl\{ \sum_{i=1}^{k} \frac{L_2 r + G^*}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{G}{\Gamma(\alpha+1)} \Biggr\} \\
&\quad + \frac{2\kappa L_1 r}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|} + \frac{2L_1 r}{\Gamma(\alpha+1)} \\
&\leq \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \Biggl\{ \sum_{i=1}^{k} \frac{G^*}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{G}{\Gamma(\alpha+1)} \\
&\quad + \Biggl[\sum_{i=1}^{k} \frac{L_2}{|E_\alpha(-t_i^\alpha \lambda)|} + \frac{(4\kappa+2)L_1}{(2+\kappa)\Gamma(\alpha+1)} \Biggr] r \Biggr\} \\
&\leq r.
\end{aligned}
$$

Secondly, we show that the mapping T is a contraction. Indeed, given any $x, y \in H_r$ and each $t \in J$, we obtain

$$
\begin{aligned}
&\bigl|(Tx)(t) - (Ty)(t)\bigr| \\
&\quad = \Biggl| \frac{E_\alpha(-t^\alpha \lambda)}{1+\kappa E_\alpha(-\lambda)} \Biggl\{ \sum_{i=1}^{k} \frac{I_i(x(t_i)) - I_i(y(t_i))}{E_\alpha(-t_i^\alpha \lambda)} \\
&\qquad - \kappa \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(1-s)^\alpha \lambda\bigr) \bigl(f\bigl(s, x(s), x(\gamma s)\bigr) - f\bigl(s, y(s), y(\gamma s)\bigr) \bigr)\, ds \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=m+1}^{k} \frac{I_j(x(t_j)) - I_j(y(t_j))}{E_\alpha(-t_j^\alpha \lambda)} \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(t-s)^\alpha \lambda\bigr)\bigl(f\bigl(s,x(s),x(\gamma s)\bigr) - f\bigl(s,y(s),y(\gamma s)\bigr)\bigr)\,ds \Bigg| \\
&\leq \left(\frac{1}{|1+\kappa E_\alpha(-\lambda)|} + 1\right) \sum_{i=1}^{k} \frac{L_2|x(t_i)) - (y(t_i))|}{E_\alpha(-t_i^\alpha \lambda)} \\
&+ \frac{\kappa}{\Gamma(\alpha)|1+\kappa E_\alpha(-\lambda)|} \int_0^1 (1-s)^{\alpha-1} L_1\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|x(\gamma s) - y(\gamma s)\bigr|\bigr)\,ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_1\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|x(\gamma s) - y(\gamma s)\bigr|\bigr)\,ds \\
&\leq \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \sum_{i=1}^{k} \frac{X_2}{E_\alpha(-t_i^\alpha \lambda)} \|x-y\|_{PC} \\
&+ \frac{2\kappa L_1}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|} \|x-y\|_{PC} + \frac{2L_1}{\Gamma(\alpha+1)} \|x-y\|_{PC} \\
&\leq \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \left(\sum_{i=1}^{k} \frac{L_2}{E_\alpha(-t_i^\alpha \lambda)} + \frac{(4\kappa+2)L_1}{(2+\kappa)\Gamma(\alpha+1)} \right) \|x-y\|_{PC} \\
&= \eta \|x-y\|_{PC}.
\end{aligned}
$$

This implies that $\|Tx - Ty\| \leq \eta \|x-y\|_{PC}$. Thus, T is a contraction, Hence we conclude the proof by applying Banach contraction principle. □

Theorem 3.2 *Assume that condition* (C_2) *and the following additional conditions are satisfied*:

(C_4) *A function* $\varphi \in L^{(1/\rho)}(J,(0,+\infty))(0<\rho<\alpha<1)$ *exists, and* $\bar{\omega} \in C([0,+\infty])$ *is a nondecreasing function satisfying the following inequality*:

$$
\bigl|f\bigl(t,x(s),x(\gamma s)\bigr)\bigr| \leq \varphi(t)\bar{\omega}\bigl(\|x\|_{PC}\bigr), \quad x \in PC(J,\mathbb{R}), t \in J.
$$

(C_5)

$$
\frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \left(\frac{(1+2\kappa)\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{(2+\kappa)\Gamma(\alpha)(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \liminf_{r\to+\infty} \frac{\bar{\omega}(r)}{r} + \sum_{i=1}^{k} \frac{L_2}{|E_\alpha(-t_i^\alpha \lambda)|} \right) < 1.
$$

Then problem (1.3) *has at least one solution.*

Proof It is easily to see that the set $H_r = \{x \in PC(J,\mathbb{R}) : \|x\|_{PC} \leq r\}$ is a closed, bounded, and convex set in $PC(J,\mathbb{R})$ for all $r>0$. Let M and N be two operators on H_r defined by

$$
\begin{aligned}
(Mx)(t) = {}& \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(t-s)^\alpha \lambda\bigr) f(s,x(s),x(\gamma s))\,ds \\
& - \frac{\kappa E_\alpha(-t^\alpha \lambda)}{1+\kappa E_\alpha(-\lambda)} \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(1-s)^\alpha \lambda\bigr) f(s,x(s),x(\gamma s))\,ds,
\end{aligned} \tag{3.3}
$$

$$(Nx)(t) = \frac{E_\alpha(-t^\alpha\lambda)}{1+\kappa E_\alpha(-\lambda)} \sum_{i=1}^{k} \frac{I_i(x(t_i))}{E_\alpha(-t_i^\alpha\lambda)} - E_\alpha\left(-t^\alpha\lambda\right) \sum_{j=m+1}^{k} \frac{I_j(x(t_j))}{E_\alpha(-t_j^\alpha\lambda)}. \tag{3.4}$$

It follows from condition (C_4) and Hölder inequality that for any $x \in H_r$ and each $t \in J$,

$$\begin{aligned}
&\int_0^t \left|(t-s)^{\alpha-1} f\left(s, x(s), x(\gamma s)\right)\right| ds \\
&\quad\le \int_0^t \left|(t-s)^{\alpha-1}\varphi(s)\bar{\omega}(r)\right| ds \\
&\quad\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\rho}} ds\right)^{1-\rho} \left(\int_0^t \left(\bar{\omega}(r)\varphi(s)\right)^{\frac{1}{\rho}} ds\right)^{\rho} \\
&\quad\le \frac{\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \bar{\omega}(r).
\end{aligned}$$

Repeating the same procedure as above, we obtain

$$\int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}\left(-(1-s)^\alpha\lambda\right) f\left(s, x(s), x(\gamma s)\right) ds \le \frac{\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \bar{\omega}(r).$$

Next, we show that there exist $r_0 > 0$ with $Mx + Ny \in H_{r_0}$ for $x, y \in H_{r_0}$. Suppose by contradiction that for each $r > 0$ there exist $x_r, y_r \in H_{r_0}$ and $t_r \in J$ such that $|(Mx_r)(t_r) + (Ny_r)(t_r)| > r$. Assumption (C_2) implies that $|I_i(x(t_i)| \le |I_i(x(t_i) - I_i(0) + I_i(0)| \le L_2 r + G^*$.

Thus,

$$\begin{aligned}
r &< \left|(Mx_r)(t_r) + (Ny_r)(t_r)\right| \\
&\le \frac{\kappa\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{\Gamma(\alpha)|1+\kappa E_\alpha(-\lambda)|(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \bar{\omega}(r) \\
&\quad + \frac{\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{\Gamma(\alpha)(\frac{\alpha-\rho}{1-\sigma})^{1-\rho}} \bar{\omega}(r) + \frac{1}{|1+\kappa E_\alpha(-\lambda)|} \sum_{i=1}^{k} \frac{L_2 r + G^*}{|E_\alpha(-t_i^\alpha\lambda)|} + \sum_{i=1}^{k} \frac{L_2 r + G^*}{|E_\alpha(-t_i^\alpha\lambda)|} \\
&\le \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \left(\frac{(1+2\kappa)\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{(2+\kappa)\Gamma(\alpha)(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \bar{\omega}(r) + \sum_{i=1}^{k} \frac{L_2 r + G^*}{|E_\alpha(-t_i^\alpha\lambda)|}\right).
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, yields

$$1 \le \frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|} \left(\frac{(1+2\kappa)\|\varphi\|_{L^{\frac{1}{\rho}}(J)}}{(2+\kappa)\Gamma(\alpha)(\frac{\alpha-\rho}{1-\rho})^{1-\rho}} \liminf_{r\to+\infty} \frac{\bar{\omega}}{r} + \sum_{i=1}^{k} \frac{L_2}{|E_\alpha(-t_i^\alpha\lambda)|}\right),$$

which contradicts condition (C_5). Hence, there exist r_0 such that $Mx + Ny \in H_{r_0}$, for all $x, y \in H_{r_0}$.

Thus, for all $t \in J$ and $x, y \in H_r$, one gets

$$\left|(Nx)(t) - (Ny)(t)\right| \le \frac{|E_\alpha(-t^\alpha\lambda)|}{|1+\kappa E_\alpha(-\lambda)|} \sum_{i=1}^{k} \frac{I_i(x(t_i)) - I_i(y(t_i))}{|E_\alpha(-t_i^\alpha\lambda)|}$$

$$+\left|E_\alpha\left(-t_i^\alpha\lambda\right)\right|\sum_{i=1}^{k}\frac{I_i(x(t_i))-I_i(y(t_i))}{|E_\alpha(-t_i^\alpha\lambda)|}$$
$$\le\sum_{i=1}^{k}\frac{I_i(x(t_i))-I_i(y(t_i))}{|E_\alpha(-t_i^\alpha\lambda)|}\left(1+\frac{1}{|1+\kappa E_\alpha(-\lambda)|}\right)$$
$$\le\left(\frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|}\sum_{i=1}^{k}\frac{L_2}{|E_\alpha(-t_i^\alpha\lambda)|}\right)\|x-y\|_{PC}.$$

Denoting $\eta^*=\frac{(2+\kappa)}{|1+\kappa E_\alpha(-\lambda)|}\sum_{i=1}^{k}\frac{L_2}{|E_\alpha(-t_I^\alpha\lambda)|}$, it follows from (C_5) that $0<\eta^*<1$ and $\|Nx+Ny\|_{PC}\le\eta^*\|x-y\|_{PC}$. Thus N is a contraction mapping.

Since f is continuous, this implies that operator M is also continuous. Now to show M is compact, we apply the same procedure as in Theorem 3.1. One can easily see that $M(H_r)$ is uniformly bounded on $PC(J,\mathbb{R})$. We now show that $M(H_r)$ is equicontinuous on $J_m(m=1,\dots,k)$. Let $\Phi=J\times H_r\times H_r$ and $f^*=\sup_{(t,x(t),x(\gamma s))\in\Phi}|f(t,x(t),x(\gamma s))|$, then, for any $t_m<\xi_2<\xi_1<t_{m+1}$, we have

$$\begin{aligned}
&\left|(Mx)(\xi_2)-(Mx)(\xi_1)\right|\\
&\quad\le\left|\int_0^{\xi_2}(\xi_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\right.\\
&\qquad\left.-\int_0^{\xi_1}(\xi_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\right|\\
&\qquad+\left|\frac{\kappa(E_\alpha(-\tau_2^\alpha\lambda)-E_\alpha(-\tau_1^\alpha\lambda))}{1+\kappa E_\alpha(-\lambda)}\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(1-s)^\alpha\lambda\right)f\left(s,x(s),x(\gamma s)\right)ds\right|\\
&\quad\le\left|\int_0^{\xi_2}(\xi_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\right.\\
&\qquad-\int_0^{\xi_2}(\xi_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\\
&\qquad+\int_0^{\xi_2}(\xi_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\\
&\qquad-\int_0^{\xi_2}(\xi_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\tau_1-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\\
&\qquad\left.-\int_{\xi_2}^{\xi_1}(\xi_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\left(f\left(s,x(s),x(\gamma s)\right)ds\right|\\
&\qquad+\frac{|\kappa(E_\alpha(-\xi_2^\alpha\lambda)-E_\alpha(-\xi_1^\alpha\lambda))|}{\Gamma(\alpha)|1+\kappa E_\alpha(-\lambda)|}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),x(\gamma s)\right)\right|ds\\
&\quad\le\int_0^{\xi_2}\left|(\xi_2-s)^{\alpha-1}-(\xi_1-s)^{\alpha-1}\right|\left|E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)\right|f^*\,ds\\
&\qquad+\int_0^{\xi_2}\left|(\xi_1-s)^{\alpha-1}\right|\left|E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\right|f^*\,ds\\
&\qquad+\frac{f^*}{\Gamma(\alpha)}\left|\int_{\xi_2}^{\xi_1}(\xi_1-s)^{\alpha-1}\right|+\frac{|\kappa(E_\alpha(-\xi_2^\alpha\lambda)-E_\alpha(-\xi_1^\alpha\lambda))|}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|}f^*\\
&\quad\le\frac{f^*}{\Gamma(\alpha)}\left|\int_0^{\xi_2}\left|(\xi_2-s)^{\alpha-1}-(\xi_1-s)^{\alpha-1}\right|ds\right|+\frac{(\xi_1-\xi_2)^\alpha f^*}{\Gamma(\alpha+1)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\kappa(E_\alpha(-\xi_2^\alpha\lambda)-E_\alpha(-\xi_1^\alpha\lambda))|}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|}f^* \\
&+f^*\int_0^{\xi_2}\left|(\xi_1-s)^{\alpha-1}\right|\left|E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\right|ds \\
&\leq\frac{(\xi_1-\xi_2)^\alpha+\xi_1^\alpha-\xi_2^\alpha}{\Gamma(\alpha+1)}f^*+\frac{(\xi_1-\xi_2)^\alpha f^*}{\Gamma(\alpha+1)}+\frac{|\kappa(E_\alpha(-\xi_2^\alpha\lambda)-E_\alpha(-\xi_1^\alpha\lambda))|}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|}f^* \\
&+f^*\int_0^{\xi_2}(\xi_1-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\right|ds.
\end{aligned}
$$

By (2) of Lemma 2.6, it follows that $E_{\alpha,\alpha}(-t^\alpha\lambda)$ is continuous on $t\in J$, and thus $E_{\alpha,\alpha}(-t^\alpha\lambda)$ is uniformly continuous on $t\in J$, hence, there is a sufficiently small $\delta>0$ such that, for $t_1,t_2\in J$ with $|t_1-t_2|\leq\delta$, we have

$$
\left|E_{\alpha,\alpha}\left(-t_1^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-t_2^\alpha\lambda\right)\right|<\frac{\epsilon}{\xi_2^{\frac{\alpha}{2-\alpha}}}.
$$

Let $\rho_1=\frac{2-\alpha}{2(1-\alpha)}$ and $\rho_2=\frac{2-\alpha}{\alpha}$. Then $\rho_1>1,\rho_2>1$, and $\frac{1}{\rho_1}+\frac{1}{\rho_2}=1$. Applying Hölder inequality yields

$$
\begin{aligned}
&\int_0^{\xi_2}(\xi_1-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\right|ds \\
&\leq\left[\int_0^{\xi_2}(\xi_1-s)^{(\alpha-1)\frac{2-\alpha}{2(1-\alpha)}}\,ds\right]^{\frac{2(1-\alpha)}{(2-\alpha)}} \\
&\quad\times\left[\int_0^{\xi_2}\left(E_{\alpha,\alpha}\left(-(\xi_2-s)^\alpha\lambda\right)-E_{\alpha,\alpha}\left(-(\xi_1-s)^\alpha\lambda\right)\right)^{\frac{2-\alpha}{\alpha}}ds\right]^{\frac{\alpha}{2-\alpha}} \\
&\leq\left[\frac{2\xi_1^{\frac{\alpha}{2}}-2(\xi_1-\xi_2)^{\frac{\alpha}{2}}}{\alpha}\right]^{\frac{2(1-\alpha)}{2-\alpha}}\epsilon.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|(Mx)(\xi_2)-(Mx)(\xi_1)\right| \\
&\leq\frac{(\xi_1-\xi_2)^\alpha+\xi_1^\alpha-\xi_2^\alpha}{\Gamma(\alpha+1)}f^*+\frac{(\xi_1-\xi_2)^\alpha f^*}{\Gamma(\alpha+1)}+\frac{|\kappa(E_\alpha(-\xi_2^\alpha\lambda)-E_\alpha(-\xi_1^\alpha\lambda))|}{\Gamma(\alpha+1)|1+\kappa E_\alpha(-\lambda)|}f^* \\
&\quad+\left[\frac{2\xi_1^{\frac{\alpha}{2}}-2(\xi_1-\xi_2)^{\frac{\alpha}{2}}}{\alpha}\right]^{\frac{2(1-\alpha)}{2-\alpha}}\epsilon \\
&\to 0,
\end{aligned}
$$

as $\xi_2\to\xi_1$, which implies that M is equicontinuous on the interval J_m. Hence, we have shown that $M(H_r)$ is relatively compact on J. It now follows by Arzela–Ascoli's theorem that M is compact. Therefore, we conclude from Lemma 2.7 that problem (1.3) has at least one solution. □

We now present some examples to illustrate our result.

Example 1 Consider the following fractional pantograph-differential equation with impulsive and anti-periodic boundary condition described by:

$$\begin{cases} {}^cD_{0^+}^{\frac{1}{2}}x(t) + x(t) = \frac{\sin(t)}{(t+5)^2}\frac{x^2(t)}{(1+x^2(t))} + \frac{\cos(t)}{(e^t+4)^2}x(\frac{1}{3}t), & t \in J \in [0,1]\backslash\{\frac{1}{2}\}, \\ \Delta x|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2})|}{17+|x(\frac{1}{2})|}, \\ x(0) = -x(1). \end{cases} \tag{3.5}$$

By comparing with problem (1.3), we get:
$f(t,x(t),x(\gamma t)) = \frac{\sin(t)}{(t+5)^2}\frac{x^2(t)}{(1+x^2(t))} + \frac{\cos(t)}{(e^t+4)^2}x(\frac{1}{3}t), t \in J \in [0,1]\backslash\{\frac{1}{2}\}$, $a = b = \kappa = 1$, $\alpha = \frac{1}{2}$, $I_k(x) = \frac{|x(\frac{1}{2})|}{17+|x(\frac{1}{2})|}$.

Then for any $x, y, u, v \in \mathbb{R}$ and $t \in J$, we obtain

$$\begin{aligned} \bigl|f(t,x,y) - f(t,u,v)\bigr| &\le \frac{2}{25}\|x-y\|_{PC}, \\ \bigl|I_k(x) - I_k(y)\bigr| &\le \frac{1}{17}\|x-y\|_{PC}. \end{aligned}$$

Then by a simple calculation we can easily see that $L_1 = \frac{2}{25}, L_2 = \frac{1}{17}, E_{\frac{1}{2}}(-1) \approx 0.42$, $E_{\frac{1}{2}}(-(\frac{1}{2})^{\frac{1}{2}}) \approx 0.52, \Gamma(\frac{3}{2}) \approx 0.89$, and

$$\eta = \frac{3}{|1 + E_{\frac{1}{2}}(-1)|}\left(\frac{L_2}{|E_{\frac{1}{2}}(-(\frac{1}{2})^{\frac{1}{2}})|} + \frac{6 \times L_1}{3 \times \Gamma(\frac{3}{2})}\right) \approx \frac{3}{1.42}\left(\frac{\frac{1}{17}}{0.52} + \frac{(6)\frac{2}{25}}{3 \times (0.89)}\right) < 1.$$

Therefore, all the hypotheses of Theorem 3.1 are satisfied. Hence problem (1.3) has a unique solution on $[0,1]$.

Example 2 Consider the following impulsive fractional pantograph differential equation:

$$\begin{cases} {}^cD_{0^+}^{\frac{1}{2}}x(t) + 2x(t) = \frac{\sqrt[3]{t+1}\sin(2t)}{14}(\frac{|x(t)|}{|1+|x(t)|}) + \frac{\sqrt[3]{t+1}}{(7e^t)^2}x(\frac{3}{2}t), & t \in J \in [0,1]\backslash\{\frac{1}{2}\}, \\ \Delta x|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2})|}{15+|x(\frac{1}{2})|}, \\ 2x(0) + 4x(1) = 0. \end{cases} \tag{3.6}$$

Denote

$$f\bigl(t,x(t),x(\gamma t)\bigr) = \frac{\sqrt[3]{t+1}\sin(2t)}{14}\left(\frac{|x(t)|}{|1+|x(t)|}\right) + \frac{\sqrt[3]{t+1}}{(7e^t)^2}x\left(\frac{3}{2}t\right). \tag{3.7}$$

This implies that

$$\bigl|f\bigl(t,x(t),x(\gamma t)\bigr)\bigr| \le \left(\frac{\sqrt[3]{t+1}}{14}\right)\bigl(\|x(t)\| + 1\bigr). \tag{3.8}$$

Thus, applying the same procedure as above yields $\alpha = \frac{1}{2}, \lambda = 2, \rho = \frac{1}{3}, \kappa = 2, \bar{\omega} = r + 1, E_{\frac{1}{2}}(-2) \approx 0.25, E_{\frac{1}{2}}(-2(\frac{1}{2})^{\frac{1}{2}}) \approx 0.40, \Gamma(\frac{1}{2}) \approx 3.14$ and $\varphi(t) = \frac{\sqrt[3]{t+1}}{14}$. This implies that

$\liminf_{r\to+\infty}\frac{\bar{\omega}}{r}=1, L_2=\frac{1}{15}$ and $\|\varphi(t)\|_{L^{\frac{1}{\rho}}}=(\int_0^1(\frac{\sqrt[3]{t+1}}{14})^3\,dt)^{\frac{1}{3}}\approx 0.08$. Therefore

$$\frac{4}{|1+E_{\frac{1}{2}}(-2)|}\left(\frac{5(\int_0^1(\frac{\sqrt[3]{t+1}}{14})^3\,dt)^{\frac{1}{3}}}{4\Gamma(\frac{1}{2})(\frac{\frac{1}{2}-\frac{1}{3}}{1-\frac{1}{3}})^{1-\frac{1}{3}}}+\frac{\frac{1}{15}}{|E_{\frac{1}{2}}(-2(\frac{1}{2})^{\frac{1}{2}})|}\right)\approx 0.58<1.$$

Thus, according to Theorem 3.2, problem (1.3) has at least one solution.

4 Conclusions

Using the Banach and Krasnoselskii's fixed point theorems, we have established the existence and uniqueness of the solution for fractional pantograph differential equation with impulsive and generalized anti-periodic boundary conditions. We note that it would be interesting to study this kind of problem for a certain kind of generalized fractional derivatives and integrals [7, 25]. In addition, this is the first paper, to the best of our knowledge, dealing with a fractional pantograph differential equation with an impulsive and generalized anti-periodic boundary conditions. Therefore, our result improves and generalizes the results in [13] and can be considered as a contribution to the development of the qualitative analysis of fractional differential equations. Lastly, we offered two examples to illustrate the obtained results.

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