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Global behavior and the periodic character of some biological models

Osama Moaaz[1], Jan Awrejcewicz[2*] and Hamida Mahjoub[1,3]

*Correspondence:
jan.awrejcewicz@p.lodz.pl
[2]Department of Automation,
Biomechanics and Mechatronics,
Lodz University of Technology, 1/15
Stefanowski St., 90-924 Lodz, Poland
Full list of author information is
available at the end of the article

Abstract

Biological models are usually described using difference equations. As a result, we are - in this work - interested in studying a general difference model which includes two biological models as special cases. In detail, we study the qualitative behaviors (local and global stability, boundedness and periodicity character) of a general difference model. Furthermore, we apply our general results to the population model with two age classes and the flour beetle model.

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1 Introduction

It should be emphasized that the majority of the mathematical models of real-world processes refer to evolutionary differential equations such as linear/nonlinear ODEs (ordinary differential equations) and PDEs (partial differential equations). On the other hand, owing to the difficulties the occurred while analyzing nonlinear PDEs and finding their solutions, various reduction procedures are employed to reduce problems of infinite dimensions (PDEs) to that of finite dimensions (ODEs). In particular, in mechanical engineering, the problems of reduction procedures are based on FEM (finite element method), FDM (finite difference method), the Bubnov–Galerkin methods of higher-order approximation, etc. (see, for example, [4, 5]).

The so far mentioned approaches allow one to reduce nonlinear PDEs to a finite set of nonlinear ODEs or sometimes to ODEs and AEs (algebraic equations). Thus, the problem is finally reduced to study nonlinear ODEs or ODEs/AEs which cannot be solved, in general, analytically. Therefore, a wide palette of numerical algorithms have been developed including implicit and explicit iterative methods that are based on the temporal discretization. They include, for instance, the classic Runge–Kutta method (RKM), second-order Runge–Kutta methods with two stages, adaptive Runge–Kutta methods with estimations of the local truncation error, implicit Runge–Kutta methods, Euler methods, Dormand–Prince methods, seventh-, sixth- and fifth-order Runge–Kutta–Nyström algorithms modified by Fehlberg, and many other [8]. However, all of them are based on the introduced step-size and hence in fact the problem is discretely governed by difference equations.

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Difference equations are known as a description of the observed evolution of a phenomenon, where the majority of measurements of a time-evolving variable are discrete. Thus, these equations gain their importance in arithmetical models. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogs and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics.

For varied reasons, rational difference equations have interest in the debate of many researchers. First, they afford many examples of nonlinear equations which are treatable, in many cases. But whose dynamics offer some strong features with regard to the linear case. In fact, the importance of studying difference equations comes from their appearance in many biological models which have many applications. One of the interesting models is describe by the Riccati difference equation

$$\omega_{n+1} = \frac{a + b\omega_n}{c + d\omega_n},$$

where a, b, c, d and ω_0 are real numbers. The richness of the dynamics of Riccati equations is very well known [11], and a specific case of these equations provides the classical Beverton–Holt model on the dynamics of exploited fish populations [6]. Another example refers to Kuruklis et al. [23] who investigated the behavior of the Pielou's discrete logistic model

$$\omega_{n+1} = \frac{\alpha\omega_n}{1 + \omega_{n-1}}, \tag{1.1}$$

where $\alpha \leq 1$. This equation was proposed by Pielou in [31] as a discrete analog of the delay logistic differential equation. The case $\alpha > 1$ in Pielou's equation was considered in [19]. As an example of a map generated by a simple model for frequency-dependent natural selection, May [25] introduced the difference equation

$$\omega_{n+1} = \frac{\omega_n \mathbf{e}^{\mu(1-2\omega_n)}}{1 - \omega_n + \omega_n \mathbf{e}^{\mu(1-2\omega_n)}}, \tag{1.2}$$

where $\mu \in (0, \infty)$. May studied the local stability of the positive equilibrium point $\omega^* = 1/2$. Moreover, Kocic et al. [19] investigated the oscillation and the global asymptotic stability of Eq. (1.2). Cooke et al. [10] studied the stability of the discrete epidemic models

$$\omega_{n+1} = (1 - \omega_n - \omega_{n-1})\left(1 - \mathbf{e}^{-A\omega_n}\right),$$

where $A \in (0, \infty)$. Also, Kuang et al. [20] established the global stability of a model of flour beetle population growth

$$\omega_{n+1} = a\omega_n + b\omega_{n-2}\mathbf{e}^{-c\omega_n - d\omega_{n-2}}, \tag{1.3}$$

where $a \in (0, 1)$, $b, c, d \in [0, \infty)$, $b \neq 0$ and $c + d > 0$. As a model that describes the dynamics of baleen whales, the equation

$$\omega_{n+1} = (1 - \delta)\omega_n + \delta\omega_{n-k}\left(1 + \eta\left(1 - \left(\frac{\omega_{n-k}}{M}\right)^\gamma\right)\right)_+, \tag{1.4}$$

where $\delta \in (0,1)$ and $M, \eta, \gamma \in (0,\infty)$, has been proposed. For the global stability of Eq. (1.4), see [19].

For, general equations, Stevo [32] investigated the periodic character of solutions of the equation

$$\omega_{n+1} = \frac{g(\omega_n, \omega_{n-1})}{A + \omega_n},$$

where A, ω_{-1} and ω_0 are positive real numbers and $g : (0,\infty)^2 \to (0,\infty)$ is continuous and satisfies

$$g(u,v) - g(v,w) = (u-w)h(u,v,w) - A(u-v),$$

for some continuous function $h : (0,\infty)^3 \to (0,\infty)$ such that

$$\frac{1}{u}h(u,v,w) \to 0 \quad \text{as } u,v,w \to \infty \quad \text{and} \quad \sup \frac{1}{A+u}h(u,v,w) < \infty.$$

The boundedness, global attractivity, oscillatory and asymptotic periodicity of the nonnegative solutions of the equation

$$\omega_{n+1} = \alpha + \frac{\omega_{n-k}}{f(\omega_n, \omega_{n-1}, \ldots, \omega_{n-k+1})}$$

is investigated in [17], where α is a nonnegative real number and f is a continuous function, nondecreasing in each variable and increasing in at least one. In [38], Sun and Xi studied the global behavior of the nonlinear equation

$$\omega_{n+} = f(\omega_{n-s}, \omega_{n-t}),$$

where s and t nonnegative integers, $s < t$ and f is decreasing in u and increasing in v.

Abdelrahman et al. [2] studied the asymptotic behavior of the solutions of a general class of difference equations,

$$\omega_{n+1} = a\omega_{n-l} + b\omega_{n-k} + f(\omega_{n-l}, \omega_{n-k}),$$

where a and b are nonnegative real numbers and $f : (0,\infty)^2 \to (0,\infty)$ is continuous real function and homogeneous with degree *zero*.

Moaaz et al. [27–29] investigated the qualitative behavior of solutions of the equations

$$\omega_{n+1} = f(\omega_{n-l}, \omega_{n-k}) \tag{1.5}$$

and

$$\omega_{n+1} = a\omega_{n-1}\mathbf{e}^{-f(\omega_n, \omega_{n-1})}, \tag{1.6}$$

where l and k are positive integers, a is a positive real number, f is a continuous real function and homogeneous with degree *zero*.

For many results, applications and open problems on higher-order equations and difference systems, see [1–38].

This paper is concerned with the investigation of the asymptotic behavior of the solutions of the general difference model

$$\omega_{n+1} = a\omega_n + b\omega_{n-k}\mathbf{e}^{-f(\omega_n,\omega_{n-k})}, \tag{E}$$

where k is a nonnegative integer, a and b are nonnegative real numbers, the function $f(u,v) : [0,\infty)^2 \to [0,\infty)$ is a continuous real function and homogeneous with degree κ and the initial conditions $\omega_{-k}, \omega_{-k+1}, \dots, \omega_0$ are positive real numbers.

Our aim in this paper is to give a complete picture regarding the stability of the equilibrium point of Eq. (E). Furthermore, we get sufficient conditions which ensure that the solutions of the studied equation are bounded, also sufficient conditions for global stability of equilibrium point. Moreover, we study the existence of periodic solutions of a prime period two. Finally, we apply our general results to the population model with two age classes and the flour beetle model.

In addition to the theoretical importance of studying the qualitative behavior of solutions to a general model of difference equations, this work is characterized by:

1. Study of some qualitative properties of biological models which have been previously partially verified or not verified.
2. The studied equation includes many special cases that were studied previously. It's easy to note that Eqs. (1.3), (1.5) and (1.6) are special cases of Eq. (E). So, our results extend and complement the results in [20, 27–29].
3. This work can be extended. Some other properties of the general model, such as bifurcations, can also be studied.

2 Dynamics of Eq. (E)
2.1 Stability of Eq. (E)

In the following, we state a necessary and sufficient condition for locally asymptotically stable of the equilibrium point of Eq. (E). For our next considerations, we define the function $\Phi : [0,\infty)^2 \to [0,\infty)$ by

$$\Phi(u,v) := au + bv\mathbf{e}^{-f(u,v)}. \tag{2.1}$$

An equilibrium point of Eq. (2.1) is a point ω^* such that $\omega^* = \Phi(\omega^*,\omega^*)$ (that is, ω^* is a fixed point of the function $\Phi(u,v)$). Then, equilibrium point of Eq. (E) is given by $\omega^* = a\omega^* + b\omega^*\mathbf{e}^{-f(\omega^*,\omega^*)}$, and hence

$$\omega^*\left(1 - a - b\mathbf{e}^{-(\omega^*)^\kappa f(1,1)}\right) = 0.$$

Thus, we have

$$\omega^* = 0,$$

and the positive equilibrium point

$$\omega^* = \left[\frac{1}{f(1,1)}\ln\left(\frac{b}{1-a}\right)\right]^{1/\kappa}, \quad \kappa \neq 0, b > 1 - a > 0. \tag{2.2}$$

The linearized equation of (E) of ω^* is

$$z_{n+1} - \mu_u z_n - \mu_v z_{n-1} = 0, \tag{2.3}$$

where $\mu_s = \Phi_s(\omega^*, \omega^*)$, $s = u, v$. A linear equation will be called stable, asymptotically stable, or unstable provided that the zero equilibrium has that property. From (2.1), we get

$$\Phi_u(u, v) = a - bv e^{-f(u,v)} f_u(u, v) \tag{2.4}$$

and

$$\Phi_v(u, v) = \left(1 - v f_v(u, v)\right) b e^{-f(u,v)}. \tag{2.5}$$

In the next theorems, we study the asymptotic stability for (E).

Theorem 2.1 *For local stability of the equilibrium point $\omega^* = 0$ of Eq. (E), we have the following cases*:

  (1) *If $a + b < 1$, then ω^* is locally asymptotically stable and sink.*
  (2) *If $b - 1 > a$, then ω^* is unstable and repeller.*
  (3) *If $|b - 1| < a$, then ω^* is an unstable saddle point.*
  (4) *If $|b - 1| = a$, then ω^* is a nonhyperbolic point.*

Proof If we put $\omega^* = 0$ in (2.4) and (2.5), then we have $\mu_u = a$ and $\mu_v = b$. The rest of the ς is immediate (from [21, Theorem 1.1.1]) and hence is omitted.   □

Theorem 2.2 *Assume that $a \neq 0$, $\kappa \neq 0$, and $b > 1 - a > 0$. For local stability of the equilibrium point (2.2) of Eq. (E), we have the following cases*:

  (1) *Equilibrium point ω^* is locally asymptotically stable and sink if and only if $\kappa > 0$ and*

$$\alpha - 2\sigma\gamma < \beta < (2/a - 1)\sigma\gamma. \tag{2.6}$$

  (2) *Equilibrium point ω^* is unstable and repeller if and only if*

$$\beta < \min\{-\sigma\gamma, \alpha - 2\sigma\gamma\} \quad \text{for } \kappa < 0, \tag{2.7}$$

$$\beta > \max\left\{\alpha - 2\sigma\gamma, (2/a - 1)\sigma\gamma\right\} \quad \text{for } \kappa > 0. \tag{2.8}$$

  (3) *Equilibrium point ω^* is an unstable saddle point if and only if*

$$\left((2/a - 1)\sigma\gamma + \alpha\right)^2 > \frac{4}{a}\sigma\kappa\gamma^2, \tag{2.9}$$

  and

$$\begin{aligned}
-\sigma\gamma < \beta < (\alpha - 2\sigma\gamma) \quad \boldsymbol{or} \quad \beta < -\sigma\gamma \quad \text{for } \kappa > 0, \\
-\sigma\gamma > \beta > (\alpha - 2\sigma\gamma) \quad \boldsymbol{or} \quad \beta > -\sigma\gamma \quad \text{for } \kappa < 0.
\end{aligned} \tag{2.10}$$

(4) *Equilibrium point ω^* is a nonhyperbolic point if and only if one of the following condition hold:*

$$\beta = \alpha - 2\sigma\gamma, \quad \textbf{\textit{or}} \tag{2.11}$$

$$\beta = (2/a - 1)\sigma\gamma \quad and \quad -(2/a - 1)\sigma\gamma \le \alpha \le (2/a + 1)\sigma\gamma, \tag{2.12}$$

where $\alpha = f_u(1,1)$, $\beta = f_v(1,1)$, $\gamma = f(1,1)$ and $\sigma = \frac{a}{(1-a)\ln(b/(1-a))}$.

Proof First, since f homogeneous with degree κ, we have from [7] that f_u and f_v homogeneous with degree $\kappa - 1$ and hence

$$
\begin{aligned}
\mu_u &= \Phi_u\big(\omega^*, \omega^*\big) \\
&= a - b\big(\omega^*\big)^\kappa e^{-(\omega^*)^\kappa f(1,1)} f_u(1,1) \\
&= a - (1-a)\frac{f_u(1,1)}{f(1,1)}\ln\left(\frac{b}{1-a}\right) \\
&= a - (1-a)\frac{\alpha}{\gamma}A,
\end{aligned}
\tag{2.13}
$$

where $A = \ln(\frac{b}{1-a})$, and

$$
\begin{aligned}
\mu_v &= \Phi_v\big(\omega^*, \omega^*\big) \\
&= \big(1 - \big(\omega^*\big)^\kappa f_v(1,1)\big)(1-a) \\
&= (1-a)\left(1 - \frac{f_v(1,1)}{f(1,1)}\ln\left(\frac{b}{1-a}\right)\right) \\
&= (1-a)\left(1 - \frac{\beta}{\gamma}A\right).
\end{aligned}
\tag{2.14}
$$

Thus, the characteristic equation of (2.3) is

$$\lambda^2 - \mu_u\lambda - \mu_v = 0. \tag{2.15}$$

For Case (1). From Euler's homogeneous function theorem, we have $uf_u + vf_v = \kappa f$, and hence $\alpha + \beta = \kappa\gamma$ (at $(u,v) = (1,1)$). Thus and from (2.6), we get

$$\alpha - \frac{2a\gamma}{(1-a)A} < \beta < \beta + \kappa\gamma,$$

where $\kappa\gamma > 0$. Then, we obtain (by adding $(-\alpha - \beta)$)

$$-\frac{2a\gamma}{(1-a)A} - \beta < -\alpha < \beta.$$

Next, we get (by multiplying $\times(\frac{1-a}{\gamma}A)$)

$$-2a - \frac{1-a}{\gamma}\beta A < -\frac{1-a}{\gamma}\alpha A < \frac{1-a}{\gamma}\beta A.$$

By adding a to the last inequality, we find

$$-a - \frac{1-a}{\gamma}\beta A < a - \frac{1-a}{\gamma}\alpha A < a + \frac{1-a}{\gamma}\beta A,$$

and hence

$$\left| a - (1-a)\frac{\alpha}{\gamma}A \right| < 1 - (1-a)\left(1 - \frac{\beta}{\gamma}A\right). \tag{2.16}$$

Also, from (2.6), we have

$$(1-a)\frac{\beta}{\gamma}A < 2 - a,$$

and hence

$$(1-a) - (1-a)\frac{\beta}{\gamma}A > -1,$$

so

$$1 - (1-a)\left(1 - \frac{\beta}{\gamma}A\right) < 2. \tag{2.17}$$

From (2.16) and (2.17), we obtain

$$|\mu_u| < 1 - \mu_v < 2.$$

Hence, and from [21, Theorem 1.1.1-(c)], we see that ω^* is a locally asymptotically stable and sink.

For Case (2). First, we let (2.7) hold. Thus,

$$\beta < -\frac{a\gamma}{(1-a)A},$$

and so

$$-(1-a)\frac{\beta}{\gamma}A > a.$$

Then we find

$$\mu_v = (1-a)\left(1 - \frac{\beta}{\gamma}A\right) > 1. \tag{2.18}$$

Also, from (2.7), we have

$$\alpha + 2\beta = \beta + \kappa\gamma < \beta < \alpha - \frac{2a\gamma}{(1-a)A},$$

where $\kappa < 0$. As in Case (1), we can prove that

$$|\mu_u| < \mu_v - 1. \tag{2.19}$$

Similarly, if we consider the condition (2.8), we can prove that

$$\mu_v < -1 \quad \text{and} \quad |\mu_u| < 1 - \mu_v.$$

Therefore, and from [21, Theorem 1.1.1-(d)], we see that ω^* is an unstable and repeller.

For Case (3). If we have (2.9) hold, then we get

$$\left(\left(\frac{2}{a} - 1\right)\sigma\gamma + \alpha\right)^2 > \frac{4}{a}\sigma\gamma(\alpha + \beta),$$

and so,

$$\left(\frac{2}{a} - 1\right)^2 \gamma^2\sigma^2 + \alpha^2 > \frac{4}{a}\sigma\gamma(\alpha + \beta) - 2\left(\frac{2}{a} - 1\right)\alpha\gamma\sigma$$

$$= \frac{2}{a}\sigma\gamma(2\beta + a\alpha).$$

Since $\sigma = a/(1 - a)A$, we have

$$\gamma^2\left(\frac{2 - a}{1 - a}\right)^2 + A^2\alpha^2 > 2\gamma\frac{A}{(1 - a)}(a\alpha + 2\beta),$$

so

$$\gamma^2(2 - a)^2 + A^2\alpha^2(1 - a)^2 > 2\gamma A(1 - a)(a\alpha + 2\beta).$$

This is equivalent (after performing some simple algebraic operations) to

$$\left(a - (1 - a)\frac{\alpha}{\gamma}A\right)^2 + 4(1 - a)\left(1 - \frac{\beta}{\gamma}A\right) > 0.$$

This implies

$$\mu_u^2 + 4\mu_v > 0. \tag{2.20}$$

Next, let (2.10) hold. Proceeding as in the proof of Case (1), we can prove that

$$|\mu_u| > |1 - \mu_v|. \tag{2.21}$$

Therefore, and from [21, Theorem 1.1.1-(e)], we see that ω^* is an unstable saddle point.

For Case (4), if $\beta = \alpha - 2\sigma\gamma$, then we find $\mu_u = \mu_v - 1$. Finally, let (2.12) hold. Then, we have $\beta = (2/a - 1)\sigma\gamma$, and hence $\mu_v = -1$. Also, from (2.12), we have

$$-\frac{2 - a}{(1 - a)A}\gamma \le \alpha \le \frac{2 + a}{(1 - a)A}\gamma,$$

which is equivalent to

$$-2 - a \le -(1 - a)\frac{\alpha}{\gamma}A \le 2 - a,$$

so,

$$\left| a - (1-a)\frac{\alpha}{\gamma}A \right| \le 2.$$

Then, from [21, Theorem 1.1.1-(f)], we see that ω^* is a nonhyperbolic point. The proof is complete. □

Theorem 2.3 *Assume that $a = 0$, $\kappa \ne 0$ and $b > 1$. For local stability of the equilibrium point (2.2) of Eq. (E), we have the following cases:*

(1) *Equilibrium point ω^* is locally asymptotically stable and sink if and only if*

$$|\alpha| < \beta < \frac{2}{\ln b}\gamma. \tag{2.22}$$

(2) *Equilibrium point ω^* is unstable and repeller if and only if*

$$\beta < -|\alpha|, \quad \textit{or} \quad \beta > \max\{|\alpha|, 2\gamma/\ln b\}. \tag{2.23}$$

(3) *Equilibrium point ω^* is an unstable saddle point if and only if*

$$\left(\frac{\alpha}{\gamma}\ln b + 2 \right)^2 > 4\kappa \ln b, \tag{2.24}$$

and

$$|\alpha| > -\beta > 0, \quad \textit{or} \quad 2\gamma/\ln b < \beta < |\alpha|. \tag{2.25}$$

(4) *Equilibrium point ω^* is a nonhyperbolic point if and only if*

$$|\alpha| = |\beta| \quad \textit{or} \quad |\alpha| \le \beta = 2\gamma/\ln b, \tag{2.26}$$

where α, β and γ are defined as in Theorem 2.2.

Proof The proof is similar to the proof of Theorem 2.2 and hence is omitted. □

2.2 Boundedness of Eq. (E)
In the following theorems, we study the boundedness of the solutions of Eq. (E).

Theorem 2.4 *Assume that $a = 0$ and $b \in (0, 1]$. Then every solution of Eq. (E) is bounded and*

$$0 < \omega_n \le \max\{\omega_{-k}, \omega_{-k+1}, \ldots, \omega_0\}, \tag{2.27}$$

for all $n > 0$.

Proof Assume that $\{\omega_n\}_{n=-k}^{\infty}$ is a solution of Eq. (E). From (E) and $f(u, v) \ge 0$, we note that

$$\omega_{n+1} = b\omega_{n-k}\mathrm{e}^{-f(\omega_n, \omega_{n-k})}$$

$$\le b\omega_{n-k}.$$

Since $b \leq 1$, we get $\omega_{n+1} \leq \omega_{n-k}$. Thus, we can divide the sequence $\{\omega_n\}_{n=-k}^{\infty}$ into $k+1$ subsequences bounded above by the initial conditions

$$\omega_{-k} \geq \omega_1 \geq \omega_{k+2} \geq \omega_{2k+3} \geq \cdots,$$

$$\omega_{-k+1} \geq \omega_2 \geq \omega_{k+3} \geq \omega_{2k+4} \geq \cdots,$$

$$\vdots$$

$$\omega_0 \geq \omega_{k+1} \geq \omega_{2k+2} \geq \omega_{3k+3} \geq \cdots.$$

Hence, we see that $\omega_n \leq \max\{\omega_{-k}, \omega_{-k+1}, \ldots, \omega_0\}$ for all $n > 0$. The proof is complete. $\qquad\square$

Theorem 2.5 *Assume that $a \in [0, 1)$ and there exists a constant λ such that $f(u, v) \geq \lambda v$. Then every solution of Eq. (E) is bounded and*

$$\limsup_{n \to \infty} \omega_n \leq \frac{b}{\lambda \mathbf{e}(1-a)}, \tag{2.28}$$

Proof Assume that $\{\omega_n\}_{n=-k}^{\infty}$ is a solution of Eq. (E). Since $f(u, v) \geq \lambda v$, we obtain

$$\omega_{n+1} = a\omega_n + b\omega_{n-k}\mathbf{e}^{-f(\omega_n, \omega_{n-k})}$$

$$\leq a\omega_n + b\omega_{n-k}\mathbf{e}^{-\lambda\omega_{n-k}}. \tag{2.29}$$

Now, we define the function $Q(u) = \lambda u\mathbf{e}^{1-\lambda u}$. Then, we have $Q'(u) = \lambda\mathbf{e}^{1-\lambda u}(1-\lambda u)$. Hence, the critical point u^* of the function $Q(u)$ is $u^* = 1/\lambda$, and $Q(1/\lambda) = 1$ is the maximum value of $Q(u)$. Thus, we obtain $\lambda u\mathbf{e}^{1-\lambda u} < 1$ and so

$$u\mathbf{e}^{-\lambda u} < 1/\lambda\mathbf{e}. \tag{2.30}$$

Consequently, and from (2.29), we see that

$$\omega_{n+1} \leq a\omega_n + \frac{b}{\lambda\mathbf{e}}.$$

Next, we let

$$y_{n+1} = ay_n + \frac{b}{\lambda\mathbf{e}}. \tag{2.31}$$

Thus, the solution of (2.31) is

$$y_n = a^n y_0 + \frac{b}{\lambda\mathbf{e}(1-a)}(1-a^n).$$

Since $a \in (0, 1)$, we get

$$\limsup_{n \to \infty} \omega_n \leq \limsup_{n \to \infty} y_n = \frac{b}{\lambda\mathbf{e}(1-a)}.$$

Then every solution of Eq. (E) is bounded and the proof is complete. $\qquad\square$

2.3 Global stability of equilibrium point

In the next theorem, we study the globally asymptotically stable of zero equilibrium point of Eq. (E) when $k = 1$.

Theorem 2.6 *If $a \in (0, 1)$, $b \in [0, 1)$, $a + b < 1$ and f has positive partial derivatives, then the zero equilibrium of (E) is globally asymptotically stable.*

Proof From Eq. (E), we have

$$\omega_{n+1} = a\omega_n + \left(b\mathbf{e}^{-f(\omega_n, \omega_{n-1})}\right)\omega_{n-1}.$$

Now, we let $f_0(u, v) = a$ and $f_1(u, v) = b\mathbf{e}^{-f(u,v)}$. Firstly, we have

$$\frac{\partial}{\partial s}f_1(u, v) = -bf_s\mathbf{e}^{-f(u,v)} \leq 0 \quad \text{for } s = u, v.$$

Then we have

$$f_0 \text{ and } f_1 \text{ are non-increasing in } u, v \text{ for all } u, v \in [0, \infty). \tag{2.32}$$

Secondly, we note that

$$f_0(u, u) = a > 0 \quad \text{for all } u \in [0, \infty). \tag{2.33}$$

Finally, since $f(u, v) \geq 0$, we obtain

$$\begin{aligned}
f_0(u, v) + f_1(u, v) &= a + b\mathbf{e}^{-f(u,v)} \\
&\leq a + b \\
&< 1,
\end{aligned} \tag{2.34}$$

for all $u, v \in (0, \infty)$. Therefore, from [21, Theorem 1.3.1] and (2.32)–(2.34), the zero equilibrium of (E) is globally asymptotically stable. $\qquad\square$

Theorem 2.7 *Assume that $\kappa > 0$, $a \in (0, 1)$, $b/(1 - a) \in (1, \mathbf{e})$, there exists a constant λ such that $f(u, v) \geq \lambda v$, $0 < f_u \leq \lambda a\mathbf{e}/b$ and $0 < f_v \leq \lambda$. Then every positive solution of Eq. (E) converges to ω^*.*

Proof Consider the function $\Phi : (0, \infty)^2 \to (0, \infty)$ defined as (2.1). Since $f(u, v) \geq \lambda v$ and $f_u \leq \lambda a\mathbf{e}/b$, we find

$$\begin{aligned}
\Phi_u(u, v) &= a - bv\mathbf{e}^{-f(u,v)}f_u(u, v) \\
&\geq a - bv\mathbf{e}^{-\lambda v}f_u(u, v) \\
&> a - \frac{b}{\lambda\mathbf{e}}f_u(u, v) \quad \left[\text{from } (2.30)\right] \\
&> a - \frac{b}{\lambda\mathbf{e}}\left(\frac{\lambda a\mathbf{e}}{b}\right) \\
&> 0.
\end{aligned} \tag{2.35}$$

Next, since $a \in (0, 1)$ and $f(u, v) \geq \lambda v$, we see that the equation $\omega_{n+1} = \Phi(\omega_n, \omega_{n-k})$ is bounded [from Theorem 2.5] and

$$\limsup_{n \to \infty} \omega_n \leq \frac{b}{\lambda \mathbf{e}(1-a)}.$$

Thus, for some integer $N > k$, we see that

$$\omega_{n-k} \leq \frac{b}{\lambda \mathbf{e}(1-a)} < \frac{1}{\lambda}, \quad \text{for } n > N,$$

where $b(1-a) < \mathbf{e}$. Hence, we obtain

$$
\begin{aligned}
\Phi_v(u, v) &= b\big(1 - v f_v(u, v)\big) \mathbf{e}^{-f(u,v)} \\
&> b\left(1 - \frac{1}{\lambda} f_v(u, v)\right) \mathbf{e}^{-f(u,v)} \\
&> b\left(1 - \frac{1}{\lambda} f_v(u, v)\right) \mathbf{e}^{-f(u,v)} \\
&> 0.
\end{aligned}
\tag{2.36}
$$

Now, we define the function

$$
\begin{aligned}
\Psi(\omega) &= \Phi(\omega, \omega) - \omega \\
&= \omega\big(b \mathbf{e}^{-f(\omega,\omega)} - (1-a)\big).
\end{aligned}
$$

We note that $\Psi(\omega) = 0$ if and only if $\omega = \omega^*$. Next, let $\omega < \omega^*$, then

$$\omega^\kappa < \frac{1}{f(1,1)} \ln\left(\frac{b}{1-a}\right),$$

and so

$$b \mathbf{e}^{-\omega^\kappa f(1,1)} > 1 - a.$$

Thus, $\Psi(\omega) < 0$. Similarly, if $\omega > \omega^*$, then we have $\Psi(\omega) > 0$. Consequently, we see that the function Φ satisfies the negative feedback condition

$$\big(\omega - \omega^*\big)\big(\Phi(\omega, \omega) - \omega\big) < 0 \quad \text{for all } \omega \in (0, \infty) \backslash \{\omega^*\}. \tag{2.37}$$

Hence, from [21, Theorem 1.4.1] and (2.35)–(2.37), every positive solution of Eq. (E) converges to ω^*. □

Theorem 2.8 *Assume that* $\kappa > 0$, $a = 0$, *there exists a constant* λ *such that* $f(u, v) \geq \lambda v$, $f_u > 0$, $0 < f_v \leq \lambda \mathbf{e}/b$ *and*

$$f(u, v) = f(v, u) \quad \Rightarrow \quad u = v. \tag{2.38}$$

Then every positive solution of Eq. (E) *converges to* ω^*.

Proof Consider the function $\Phi : (0, \infty)^2 \to (0, \infty)$ defined as (2.1). Since $f_u > 0$, we find

$$
\begin{aligned}
\Phi_u(u, v) &= -bv\mathbf{e}^{-f(u,v)}f_u(u, v) \\
&\le 0.
\end{aligned}
$$

From Theorem 2.5, we obtain, for some integer $N > k$,

$$
\omega_{n-k} \le \frac{b}{\lambda \mathbf{e}}, \quad \text{for } n > N.
$$

Hence, we see that

$$
\begin{aligned}
\Phi_v(u, v) &= b\big(1 - vf_v(u, v)\big)\mathbf{e}^{-f(u,v)} \\
&> b\left(1 - \frac{b}{\lambda \mathbf{e}}f_v(u, v)\right)\mathbf{e}^{-f(u,v)}.
\end{aligned}
$$

Since $f_v \le \lambda \mathbf{e}/b$, we have $\Phi_v(u, v) > 0$. Now, we will prove that Eq. (E) has no solutions of prime period two. Suppose otherwise, we assume that Eq. (E) has prime period two $\ldots, \rho, \sigma, \rho, \sigma, \ldots$. Thus, we get

$$
\rho = b\rho\mathbf{e}^{-f(\sigma,\rho)},
$$

$$
\sigma = b\sigma\mathbf{e}^{-f(\rho,\sigma)}.
$$

Thus, we have $f(\rho, \sigma) = f(\sigma, \rho) = \ln b$ and hence $\rho = \sigma$ [from (2.38)], and this is a contradiction. Therefore, from [21, Theorem 1.4.6], every positive solution of Eq. (E) converges to ω^*. □

Theorem 2.9 *Assume that $k = 1$, $\kappa > 0$, $a \in (0, 1)$, $b \in [0, 1)$, $a + b < 1$ and f has positive partial derivatives. If the solution $\{\omega_n\}_{n=-1}^{\infty}$ of (E) is a nonzero eventually, then*

$$
\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = \rho_+ \quad or \quad \lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = \rho_-, \tag{2.39}
$$

where

$$
\rho_\pm = \frac{1}{2}\big(a \pm \sqrt{a^2 + 4b}\big).
$$

Proof The linearized equation of (E) at $\omega^* = 0$ is

$$
z_{n+1} - az_n - bz_{n-1} = 0. \tag{2.40}
$$

The solutions of characteristic equation of (2.40) are $\rho_\pm$. Next, we can formulate (E) as

$$
\omega_{n+1} = A_n\omega_n + B_n\omega_{n-1}, \tag{2.41}
$$

where

$$
A_n = a \quad \text{and} \quad B_n = b\mathbf{e}^{-f(\omega_n,\omega_{n-k})}.
$$

From Theorem 2.6, we have $\lim_{n\to\infty} \omega_n = 0$, and so $\lim_{n\to\infty} f(\omega_n, \omega_{n-k}) = \lim_{n\to\infty} \omega_n^\kappa \times F(\frac{\omega_{n-k}}{\omega_n}) = 0$. Hence,

$$\lim_{n\to\infty} A_n = a \quad \text{and} \quad \lim_{n\to\infty} B_n = b.$$

Therefore, the limiting equation of (2.41) is exactly (2.40). From Poincaré's theorem, we see that (2.39) holds, and this completes the proof of the theorem. □

2.4 The existence of periodic solutions

Here, we give the periodicity character of the solution for Eq. (E).

Theorem 2.10 *Suppose that k is even and $\kappa \neq 0$. Then Eq. (E) has a periodic solution of prime period two $\{\ldots, \tau\sigma, \sigma, \tau\sigma, \sigma, \ldots\}$, $\tau \neq 1$, if and only if*

$$b^{(\tau^\kappa - 1)} = \frac{\tau(\tau - a)^{\tau^\kappa}}{1 - a\tau}. \tag{2.42}$$

Proof Assume that Eq. (E) has a periodic solution of prime period two $\ldots, \rho, \sigma, \rho, \sigma, \ldots$. Thus, and from (E), we get

$$\rho = a\sigma + b\sigma e^{-f(\sigma,\sigma)}$$
$$= \sigma\left(a + be^{-\sigma^\kappa f(1,1)}\right),$$

this implies that

$$\sigma^\kappa = \frac{1}{f(1,1)} \ln\left(\frac{b}{\rho/\sigma - a}\right).$$

Let $\tau = \rho/\sigma$ and $\tau \neq 1$, we find

$$\sigma^\kappa = \frac{1}{f(1,1)} \ln\left(\frac{b}{\tau - a}\right). \tag{2.43}$$

Also, from Eq. (E), we obtain

$$\sigma = a\rho + b\rho e^{-f(\rho,\rho)},$$

and hence

$$\rho^\kappa = \frac{1}{f(1,1)} \ln\left(\frac{b\tau}{1 - a\tau}\right). \tag{2.44}$$

Since $\rho^\kappa = \tau^\kappa \sigma^\kappa$, we have

$$\ln\left(\frac{b\tau}{1 - a\tau}\right) = \tau^\kappa \ln\left(\frac{b}{\tau - a}\right), \tag{2.45}$$

it follows that

$$b^{(\tau^\kappa - 1)} = \frac{\tau(\tau - a)^{\tau^\kappa}}{1 - a\tau}.$$

On the other hand, if (2.42) holds, then we choose

$$\omega_{-k} = \omega_{-k+2r} = \left(\frac{1}{f(1,1)} \ln\left(\frac{b}{\tau - a} \right) \right)^{1/\kappa}$$

and

$$\omega_{-k+2r-1} = \left(\frac{1}{f(1,1)} \ln\left(\frac{b\tau}{1 - a\tau} \right) \right)^{1/\kappa},$$

for $r = 1, 2, \ldots, k/2$, $t \in (0, \infty)$ and $t \neq 1$. Thus, we see that

$$
\begin{aligned}
\omega_1 &= a\omega_0 + b\omega_{-k} \exp\left(-f(\omega_0, \omega_{-k})\right) \\
&= \omega_0 \left(a + b \exp\left(-\omega_0^\kappa f(1,1)\right) \right) \\
&= \left(\frac{1}{f(1,1)} \ln\left(\frac{b}{\tau - a} \right) \right)^{1/\kappa} \left(a + b \exp\left(-\ln\left(\frac{b}{\tau - a} \right) \right) \right) \\
&= \left(\frac{1}{f(1,1)} \ln\left(\frac{b}{\tau - a} \right) \right)^{1/\kappa} \tau.
\end{aligned}
\tag{2.46}
$$

From (2.42), we see that (2.45) holds. By using (2.45), we obtain

$$
\begin{aligned}
\omega_1 &= \tau \left(\frac{1}{f(1,1)} \frac{1}{\tau^\kappa} \ln\left(\frac{b\tau}{1 - a\tau} \right) \right)^{1/\kappa} \\
&= \left(\frac{1}{f(1,1)} \ln\left(\frac{b\tau}{1 - a\tau} \right) \right)^{1/\kappa} = \omega_{-1}.
\end{aligned}
$$

Similarly, we can prove that $\omega_2 = \omega_0$. Therefore, it follows by induction that

$$\omega_{2r} = \omega_0 \quad \text{and} \quad \omega_{2r+1} = \omega_{-1} \quad \text{for all } r = 1, 2, \ldots.$$

Hence, Eq. (E) has a prime period two solution and the proof is complete. □

Theorem 2.11 *Suppose that k is odd $a \neq 0$ and $\kappa \neq 0$. Then, Eq. (E) has a periodic solution of prime period two $\{\ldots, \tau\sigma, \sigma, \tau\sigma, \sigma, \ldots\}$, $\tau \neq 1$, if and only if*

$$b^{(\eta_2 - \eta_1)} = \frac{\tau^{\eta_1}(1 - \tau a)^{\eta_2}}{(\tau - a)^{\eta_1}},
\tag{2.47}$$

where $\eta_1 = f(\tau, 1)$ and $\eta_2 = f(1, \tau)$.

Proof We proceed as in proof of Theorem 2.10. Thus, we get

$$\sigma^\kappa = \frac{1}{f(1, \tau)} \ln\left(\frac{b\tau}{\tau - a} \right)$$

and

$$\rho^\kappa = \frac{1}{f(1, 1/\tau)} \ln\left(\frac{b}{1 - a\tau} \right).$$

Since $\rho^\kappa = \tau^\kappa \sigma^\kappa$ and $\tau^\kappa f(1, 1/\tau) = f(\tau, 1)$, we have

$$f(\tau, 1) \ln\left(\frac{b\tau}{\tau - a} \right) = f(1, \tau) \ln\left(\frac{b}{1 - a\tau} \right).$$

Thus, we see that (2.47) holds. The rest of the proof proceeds as the proof of Theorem 2.10, and hence the proof is complete. □

Theorem 2.12 *Suppose that k is odd a = 0 and $\kappa \neq 0$. Then, Eq. (E) has a periodic solution of prime period two $\{\ldots, \tau\sigma, \sigma, \tau\sigma, \sigma, \ldots\}$, $\tau \neq 1$, if and only if*

$$f(\tau, 1) = f(1, \tau) = \frac{1}{\sigma^\kappa} \ln b. \tag{2.48}$$

Proof The proof is similar to the proof of Theorem 2.11, so it is omitted. □

3 Applications on biological models

The great importance of difference equations comes from their ability to describe natural phenomena, in particular its ability to describe and study biological models. In this section, by using our general results in the previous section, we study the qualitative behavior of two biological models, and we answer some of the problems that have been raised previously.

3.1 Population model with two age classes

The discrete model with two age classes, adults and juveniles

$$\begin{cases} \omega_{n+1} = \omega_{n-1} \mathbf{e}^{r - (\delta\omega_n + y_n)}; \\ y_{n+1} = \omega_n, \end{cases} \tag{3.1}$$

where $r, \delta \in (0, \infty)$. The term $\exp(r - (\delta\omega_n + y_n))$ represents the reproduction rate and is a decreasing exponential which captures the over crowding phenomenon as the population grows. To apply our results, we set system (3.1) as follows:
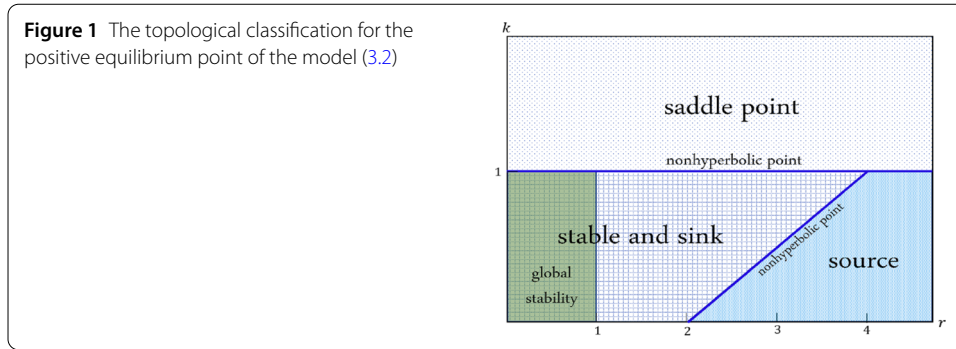
$$\omega_{n+1} = \omega_{n-1} \mathbf{e}^{r - (\delta\omega_n + \omega_{n-1})}. \tag{3.2}$$

Note that $k = 1$, $a = 0$, $b = \mathbf{e}^r > 1$ and $f(u, v) = \delta u + v$. Equilibrium points of Eq. (3.2) are $\omega^* = 0$ and positive equilibrium point $r/(\delta + 1)$ [for system (3.1) is $(r/(\delta + 1), r/(\delta + 1))$]. For stability of the equilibrium points, we introduce the following corollaries.

Corollary 3.1 *Zero equilibrium point of* (3.2) *is unstable and repeller.*

Corollary 3.2 *The positive equilibrium point $\omega^* = r/(\delta + 1)$ of* (3.2) *has one of the following cases; see Fig.* 1:
   (1) *ω^* is a locally asymptotically stable and sink if and only if $\delta < 1 < \frac{2}{r}(\delta + 1)$;*
   (2) *ω^* is an unstable and repeller if and only if $\max\{\delta, \frac{2}{r}(\delta + 1)\} < 1$;*
   (3) *ω^* is an unstable saddle point if and only if $\delta > 1$;*
   (4) *ω^* is a nonhyperbolic point if and only if $\delta = 1$ or $\delta = (\frac{1}{2}r - 1) \leq 1$.*

Figure 1 The topological classification for the positive equilibrium point of the model (3.2)

Corollary 3.3 *Every solution of* (3.2) *is bounded and* $\limsup_{n\to\infty} \omega_n \leq \mathbf{e}^{r-1}$.

Corollary 3.4 *Assume that* $0 < r \leq 1$ *and* $0 < \delta < 1$. *Then every positive solution of* (3.2) *converges to* ω^*.

Corollary 3.5 *Let* $\delta = 1$. *Then the model* (3.2) *has a periodic solution of prime period two* $\{\ldots, r - \sigma, \sigma, r - \sigma, \sigma, \ldots\}$.

3.2 The flour beetle model

Flour beetles are members of the darkling beetle genera Tribolium or Tenebrio. They are pests of cereal silos and are widely used as laboratory animals, as they are easy to keep. The flour beetles consume wheat and other grains, are adapted to survive in very dry environments, and can withstand even higher amounts of radiation than cockroaches [39]. They are a major pest in the agricultural industry and are highly resistant to insecticides. The flour beetle model obeys

$$\omega_{n+1} = a\omega_n + b\omega_{n-2}\mathbf{e}^{-(\eta_1\omega_n + \eta_2\omega_{n-2})}, \tag{3.3}$$

where $a, b \in (0, \infty)$, $\eta_1, \eta_2 \in [0, \infty)$ and $\eta_1 + \eta_2 > 0$. The equilibrium points of (3.3) are $\omega^* = 0$ and the positive point

$$\omega^* = \frac{1}{\eta_1 + \eta_2} \ln\left(\frac{b}{1-a}\right), \quad b > 1 - a > 0. \tag{3.4}$$
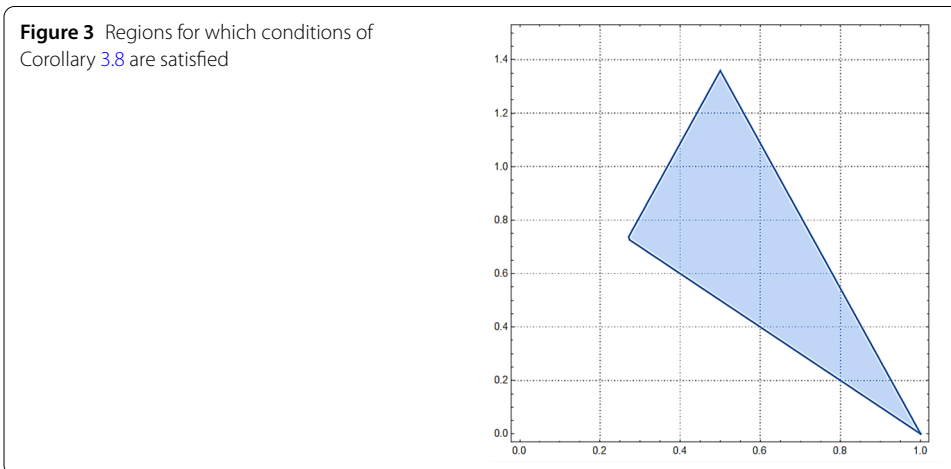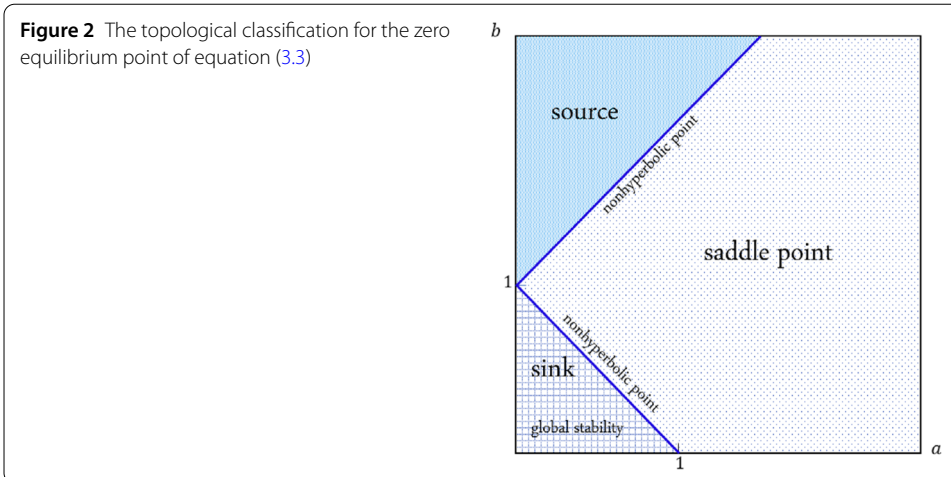
By Theorem 2.1, we see the local stability behavior of the zero equilibrium point of (3.3); see Fig. 2.

By Theorem 2.2, if we put $\kappa = 1$, $\alpha = \eta_1$, $\beta = \eta_2$ and $\gamma = \eta_1 + \eta_2$, then we see the local stability behavior of the positive equilibrium point of (3.3).

Corollary 3.6 *Assume that* $a \in (0, 1)$. *Then every solution of Eq.* (3.3) *is bounded and*

$$\limsup_{n\to\infty} \omega_n \leq \frac{b}{\eta_2\mathbf{e}(1-a)}.$$

Corollary 3.7 *Let* $a \in (0, 1)$ *and* $a + b < 1$. *Then the zero equilibrium of* (3.3) *is globally asymptotically stable.*

Figure 2 The topological classification for the zero equilibrium point of equation (3.3)

Figure 3 Regions for which conditions of Corollary 3.8 are satisfied

Corollary 3.8 *Assume that $a \in (0, 1)$, $b/(1-a) \in (1, \mathbf{e})$ and $b\eta_1 \leq \mathbf{e}a\eta_2$. Then every positive solution of* (3.3) *converges to ω^*.*

Remark 3.1 Let $\eta_1 = \eta_2 = 1$. Figure 3 shows the regions for which conditions of Corollary 3.8 are satisfied, that is, the regions for which the positive equilibrium point of the model (3.3) is globally asymptotically stable.

Corollary 3.9 *The model* (3.3) *has a periodic solution of prime period two* $\{\ldots, \tau\sigma, \sigma, \tau\sigma, \sigma, \ldots\}$, *if and only if*

$$b^{(\tau-1)} = \frac{\tau(\tau - a)^\tau}{1 - a\tau}.$$ (3.5)

Example 3.1 Consider the model (3.3) with $a = 1/4$, $b = 49/4$, $\omega_{-2} = \omega_{-1} = 1$ and $\omega_0 = 2$. By Corollary 3.9, we see that (3.3) has a periodic solution of prime period two $\{\ldots, \ln 7, \ln \sqrt{7}, \ln 7, \ln \sqrt{7}, \ldots\}$; see Fig. 4.

4 Conclusion

Difference equations are widely being used as mathematical models for describing real life situations in biology.
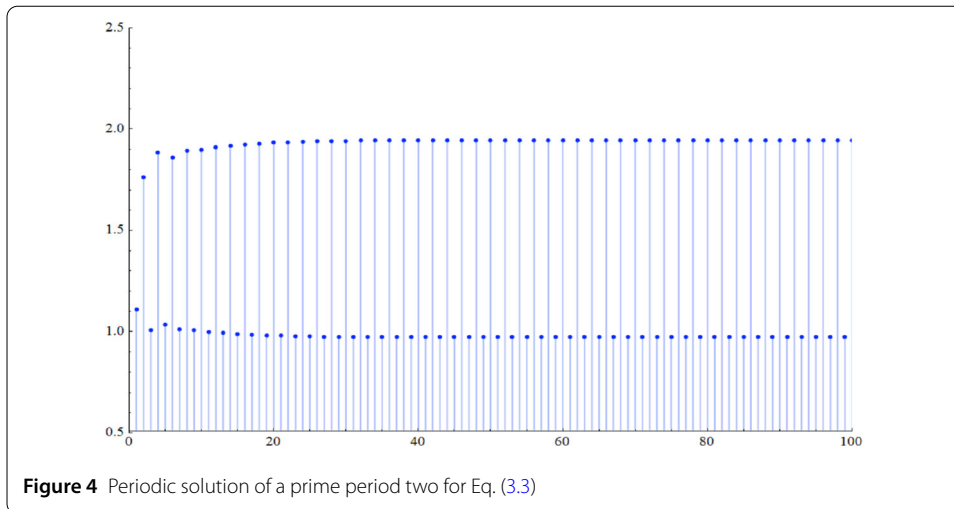
Figure 4 Periodic solution of a prime period two for Eq. (3.3)

In this work, we studied the global behavior and the periodic character of the solution of a general class of the nonlinear difference equations. In detail, we established criteria for stability (local and global), boundedness and periodicity character of the solution of (E). Moreover, by applying our general results on biological models (as special cases), we examined several qualitative behaviors of the solutions of these models.

For the discrete model with two age classes, Corollaries 3.1–3.4 set the criteria for local and global stability, and Corollary 3.5 studied the existence of periodic solutions for this model. On the other hand, Corollaries 3.6–3.8 gave the global behavior of equilibrium points of the flour beetle model (3.3). Furthermore, Corollary 3.9 gave the necessary and sufficient condition for the existence of periodic solutions.

We can use our results to study many special cases of (E). For example, if $f(u, v) = \ln(1/h(u, v))$ and h is homogeneous with degree zero, then Eq. (E) becomes

$$\omega_{n+1} = a\omega_n + b\omega_{n-k}h(\omega_n, \omega_{n-k}).$$

In particular, we can use our results to study the equation

$$\omega_{n+1} = a\omega_n + \frac{b\omega_n\omega_{n-k}}{c\omega_n + d\omega_{n-k}}.$$

Further, in future work, we can try to get some qualitative behavior of the more general equation

$$\omega_{n+1} = a\omega_n + b\omega_{n-k}\Phi(\omega_n, \omega_{n-k}),$$

where $\Phi(u, v)$ is a homothetic function, that is, there exist a strictly increasing function $G : \mathbb{R} \to \mathbb{R}$ and a homogeneous function $H : \mathbb{R}^2 \to \mathbb{R}$ with degree β, such that $\Phi = G \circ H$.

Author details
[1]Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt. [2]Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland. [3]Department of Mathematics, Faculty of Science, University of Benghazi, Benghazi, Libya.

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