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Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications

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Abstract

Fractal analysis is one of interesting research areas of computer science and engineering, which depicts a precise description of phenomena in modeling. Visual beauty and self-similarity has made it an attractive field of research. The fractal sets are the effective tools to describe the accuracy of the inequalities for convex functions. In this paper, we employ linear fractals $\mathbb{R}^{\alpha}$ to investigate the (s, m)-convexity and relate them to derive generalized Hermite–Hadamard (HH) type inequalities and several other associated variants depending on an auxiliary result. Under this novel approach, we aim at establishing an analog with the help of local fractional integration. Meanwhile, we establish generalized Simpson-type inequalities for (s, m)-convex functions. The results in the frame of local fractional showed that among all comparisons, we can only see the correlation between novel strategies and the earlier consequences in generalized s-convex, generalized m-convex, and generalized convex functions. We obtain application in probability density functions and generalized special means to confirm the relevance and computational effectiveness of the considered method. Similar results in this dynamic field can also be widely applied to other types of fractals and explored similarly to what has been done in this paper.

MSC: 26E60

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1 Introduction and prelimnaries

Fractional calculus based on differential and difference equations is of considerable importance due to their connection with real-world problems that depend not only on the instant time but also on the previous time, in particular, modeling the phenomena by means of fractals, random walk processes, control theory, signal processing, acoustics, and so on (see [1–12]). It has been shown that fractional-order models are much more adequate than integer-order models. A number of methods used to solve nonlinear partial differential equations have been successfully generalized to fractional differential equations, such as the Adomian decomposition method, homotopy analysis method, variational iteration

method, transform method, symmetry group method, and invariant subspace method. The concepts of fractional differentiation and fractional integration were examined by Riemann, Liouville, Abel, Laurent, Hardy, and Littlewood. Detailed discussions of fractional calculus and related work can be found in [13–16]. Fractal analysis is an entirely new field of research based on fractional calculus. It has introduced some fascinating complex graphs, picture compressions, and computer graphics. In 1982, Benoit Mandelbrot [17], the father of fractal geometry, in his book "The Fractal Geometry of Nature" predicted that "clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." Individuals accept that the items in nature can be made or can be depicted by images, for example, lines, circles, conic areas, polygons, circles, and quadratic surfaces. The utilization of new scientific tools and concepts in this field of research will have an inordinate impression on enlightening image compression, where fractals and fractal-concerned techniques have demonstrated applications [18–20]. It is interesting that the authors [21, 22] investigated the local fractional functions on fractal space deliberately, which comprises of local fractional calculus and the monotonicity of functions. Numerous analysts contemplated the characteristics of functions on fractal space and built numerous sorts of fractional calculus by utilizing various strategies [23–25].

The connection among fractal sets, integral inequalities, and convexity is very strong. Therefore it is essential to create mathematical inequalities that inspect the fractal sets and their significance in various areas of mathematics and engineering problems. Convexity is utilized to portray the functional values of a framework that we normally deal with inequalities. Convex functions are firmly identified with the most celebrated HH inequality [26, 27], which is the principal essential consequence for convex functions with natural geometrical interpretation and numerous applications. It has attained considerably much interest in elementary mathematics and is stated as follows:

$$(\lambda_2-\lambda_1)\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right)\leq\int_{\lambda_1}^{\lambda_2}\mathcal{H}(u)\,du\leq(\lambda_2-\lambda_1)\frac{\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)}{2}, \tag{1.1}$$

provided that $\mathcal{H}:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function on an interval $\mathcal{I}$ of reals with $\lambda_1,\lambda_2\in\mathcal{I}$ defined by

$$\mathcal{H}\big(\zeta\lambda_1+(1-\zeta)\lambda_2\big)\leq\zeta\mathcal{H}(\lambda_1)+(1-\zeta)\mathcal{H}(\lambda_2) \tag{1.2}$$

for $\lambda_1,\lambda_2\in\mathcal{I}$ and $\zeta\in[0,1]$. For a concave function $\mathcal{H}$, the inequalities in (1.1) hold in reverse direction. Over the last two decades, these types of generalizations have led to many novel testimonies, stimulating extensions, conspicuous generalizations, innovative HH-type inequalities, and a lot of applications of inequalities (1.1) in the literature of mathematical inequalities and in other branches of pure and applied mathematics; see [28–30] and the references therein.

Simpson's inequality is widely studied in the literature occupying a significant place in numerical analysis and inequality theory due to its systematic nature and is stated as follows:

$$\left|\frac{1}{3}\left[\frac{\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)}{2}+2\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\mathcal{H}(u)\,du\right|$$

$$\leq \frac{1}{2880}\left\|\mathcal{H}^4\right\|_\infty(\lambda_2-\lambda_1)^4, \tag{1.3}$$

where the mapping $\mathcal{H}:[\lambda_1,\lambda_2]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval, and $\mathcal{H}^4$ to be bounded on (λ_1,λ_2), that is, $\|\mathcal{H}^4\|_\infty = \sup_{\zeta\in(\lambda_1,\lambda_2)}|\mathcal{H}^4(\zeta)|<\infty$. For more generalizations of Simpson-type inequalities, see [31–36]. Adopting the aforementioned trend, we intend to establish some novel results on (s,m)-convex functions via local fractional integrals. To be more precise, Simpson-type inequalities in local fractional integrals are also presented.

Toader [37] gave the idea of m-convexity as follows.

Definition 1.1 ([37]) A mapping $\mathcal{H}:[0,\mathfrak{b}^*]\to\mathbb{R}$ is known to be m-convex if

$$\mathcal{H}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\leq\zeta\mathcal{H}(\lambda_1)+m(1-\zeta)\mathcal{H}(\lambda_2) \tag{1.4}$$

for $m\in[0,1]$, $\lambda_1,\lambda_2\in[0,\mathfrak{b}^*]$, and $\zeta\in[0,1]$.

Also, $\mathcal{K}_m(\mathfrak{b})$ denotes the set of m-convex functions on $[\lambda_1,\lambda_2]$ for which $\mathcal{H}(0)\leq 0$. For some modifications and generalizations related to m-convex functions, we refer to [28, 38].

Hudzik and Maligranda [39] proposed, among others, a class of functions, known as s-convex functions, defined as follows.

Definition 1.2 ([39]) A mapping $\mathcal{H}:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is said be s-convex if

$$\mathcal{H}\big(\zeta\lambda_1+(1-\zeta)\lambda_2\big)\leq\zeta^s\mathcal{H}(\lambda_1)+(1-\zeta)^s\mathcal{H}(\lambda_2) \tag{1.5}$$

for all $\lambda_1,\lambda_2\in\mathcal{I},\zeta\in[0,1]$ and some fixed $s\in(0,1]$.

Clearly, we see that for $s=1$, s-convexity becomes the classical convexity of functions on $[0,\infty)$. For generalizations and refinements on s-convex and m-convex functions, we refer the readers to [28, 38–42].

Now we mention some preliminaries from the theory of local fractional calculus. These ideas and important consequences associated with the local fractional derivative and local fractional integral are mainly due to Yang [22].

Let $\nu_1^{\alpha^*}$, $\nu_2^{\alpha^*}$, and $\nu_3^{\alpha^*}$ belong to $\mathbb{R}^{\alpha^*}(0<\alpha^*\leq 1)$. Then

(1) $\nu_1^{\alpha^*}+\nu_2^{\alpha^*}$ and $\nu_1^{\alpha^*}\nu_2^{\alpha^*}$ belong to $\mathbb{R}^{\alpha^*}$;
(2) $\nu_1^{\alpha^*}+\nu_2^{\alpha^*}=\nu_2^{\alpha^*}+\nu_1^{\alpha^*}=(\nu_1+\nu_2)^{\alpha^*}=(\nu_2+\nu_1)^{\alpha^*}$;
(3) $\nu_1^{\alpha^*}+(\nu_2^{\alpha^*}+\nu_3^{\alpha^*})=(\nu_1^{\alpha^*}+\nu_2^{\alpha^*})+\nu_3^{\alpha^*}$;
(4) $\nu_1^{\alpha^*}\nu_2^{\alpha^*}=\nu_2^{\alpha^*}\nu_1^{\alpha^*}=(\nu_1\nu_2)^{\alpha^*}=(\nu_2\nu_1)^{\alpha^*}$;
(5) $\nu_1^{\alpha^*}(\nu_2^{\alpha^*}\nu_3^{\alpha^*})=(\nu_1^{\alpha^*}\nu_2^{\alpha^*})\nu_3^{\alpha^*}$;
(6) $\nu_1^{\alpha^*}(\nu_2^{\alpha^*}+\nu_3^{\alpha^*})=\nu_1^{\alpha^*}\nu_2^{\alpha^*}+\nu_1^{\alpha^*}\nu_3^{\alpha^*}$;
(7) $\nu_1^{\alpha^*}+0^{\alpha^*}=0^{\alpha^*}+\nu_1^{\alpha^*}=\nu_1^{\alpha^*}$ and $\nu_1^{\alpha^*}1^{\alpha^*}=1^{\alpha^*}\nu_1^{\alpha^*}=\nu_1^{\alpha^*}$.

Definition 1.3 A nondifferentiable mapping $\mathcal{H}:\mathbb{R}\to\mathbb{R}^{\alpha^*},\theta\to\mathcal{H}(\epsilon)$, is said to be local fractional continuous at $\epsilon_\circ$ if for any $\epsilon>0$, there exists $l>0$ such that

$$\big|\mathcal{H}(\epsilon)-\mathcal{H}(\epsilon_\circ)\big|<\epsilon^{\alpha^*}$$

for $|\epsilon-\epsilon_\circ|<\kappa$. If $\mathcal{H}(\epsilon)$ is local continuous on $(\lambda1,\lambda2)$, then we write $\mathcal{H}(\epsilon)\in\mathbb{C}_{\alpha^*}(\lambda1,\lambda2)$.

Definition 1.4 The local fractional derivative of $\mathcal{H}(\epsilon)$ of order α^* at $\epsilon = \epsilon_\circ$ is defined by

$$\mathcal{H}^{(\alpha^*)}(\epsilon_\circ) = {}_{\epsilon_\circ}\mathcal{D}_{\epsilon}^{\alpha^*}\mathcal{H}(\epsilon) = \left.\frac{d^{\alpha^*}\mathcal{H}(\epsilon)}{d\epsilon^{\alpha^*}}\right|_{\epsilon=\epsilon_\circ} = \lim_{\epsilon\to\epsilon_\circ}\frac{\Delta^{\alpha^*}(\mathcal{H}(\epsilon)-\mathcal{H}(\epsilon_\circ))}{(\epsilon-\epsilon_\circ)^{\alpha^*}},$$

where $\Delta^{\alpha^*}(\mathcal{H}(\epsilon)-\mathcal{H}(\epsilon_\circ)) = \Gamma(\alpha^*+1)(\mathcal{H}(\epsilon)-\mathcal{H}(\epsilon_\circ))$. Let $\mathcal{H}^{(\alpha^*)}(\epsilon) = \mathcal{D}_{\epsilon}^{\alpha^*}\mathcal{H}(\epsilon)$. If there exists $\mathcal{H}^{(k+1)\alpha^*}(\epsilon) = \overbrace{\mathcal{D}_{\epsilon}^{\alpha^*}\ldots\mathcal{D}_{\epsilon}^{\alpha^*}}^{(k+1)times}\mathcal{H}(\epsilon)$ for any $\epsilon \in \Omega \subseteq \mathbb{R}$, then we write $\mathcal{H} \in \mathcal{D}_{(k+1)\alpha^*}(\mathcal{I})$, where $k = 0, 1, 2, \ldots$.

Definition 1.5 Let $\mathcal{H}(\epsilon) \in \mathbb{C}_{\alpha^*}[\lambda 1, \lambda 2]$, and let $\Delta = \{\eta_0, \eta_1, \ldots, \eta_N\}$ $(N \in \mathbb{N})$ be a partition of $[\lambda 1, \lambda 2]$ such that $\lambda 1 = \eta_0 < \eta_1 < \cdots < \eta_N = \lambda 2$. Then the local fractional integral of $\mathcal{H}$ on $[\lambda 1, \lambda 2]$ of order α^* is defined as follows:

$$\begin{aligned}{}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(\epsilon) &= \frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda 1}^{\lambda 2}\mathcal{H}(\eta)(d\eta)^{\alpha^*} \\ &:= \frac{1}{\Gamma(1+\alpha^*)}\lim_{\delta\eta\to 0}\sum_{j=0}^{N-1}\mathcal{H}(\eta_j)(\Delta\eta_j),\end{aligned}$$

where $\delta\eta := \max\{\Delta\eta_1, \Delta\eta_2, \ldots, \Delta\eta_{N-1}\}$ and $\Delta\eta_j := \eta_{j+1} - \eta_j, j = 0, \ldots, N-1$.

It follows that ${}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(\epsilon) = 0$ if $\lambda 1 = \lambda 2$ and ${}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(\epsilon) = -{}_{\lambda 2}\mathcal{I}_{\lambda 1}^{(\alpha^*)}\mathcal{H}(\epsilon)$ if $\lambda 1 < \lambda 2$. For any $\epsilon \in [\lambda 1, \lambda 2]$, if there exists ${}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(\epsilon)$, then we write $\mathcal{H}(\epsilon) \in \mathcal{I}_{\epsilon}^{\alpha^*}[\lambda 1, \lambda 2]$.

Lemma 1.6 ([22])

(1) *Suppose that* $\mathcal{H}(u) = \mathcal{G}^{(\alpha^*)}(u) \in \mathbb{C}_{\alpha^*}[\lambda 1, \lambda 2]$. *Then*

$${}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(u) = \mathcal{G}(\lambda 2) - \mathcal{G}(\lambda 1).$$

(2) *Suppose that* $\mathcal{H}(u), \mathcal{G}(u) \in \mathcal{D}_{\alpha^*}[\lambda 1, \lambda 2]$ *and* $\mathcal{H}^{(\alpha^*)}(u), \mathcal{G}^{(\alpha^*)}(u) \in \mathbb{C}_{\alpha^*}[\lambda 1, \lambda 2]$. *Then*

$${}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}(u)\mathcal{G}^{(\alpha^*)}(u) = \mathcal{H}(u)\mathcal{G}(u)|_{\lambda 1}^{\lambda 2} - {}_{\lambda 1}\mathcal{I}_{\lambda 2}^{(\alpha^*)}\mathcal{H}^{(\alpha^*)}(u)\mathcal{G}(u).$$

Lemma 1.7 ([22])

$$\frac{d^{\alpha^*}u^{k\alpha^*}}{du^{\alpha^*}} = \frac{\Gamma(1+k\alpha^*)}{\Gamma(1+(k-1)\alpha^*)}u^{(k-1)\alpha^*},$$

$$\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda 1}^{\lambda 2}u^{k\alpha^*}(du)^{\alpha^*} = \frac{\Gamma(1+k\alpha^*)}{\Gamma(1+(k+1)\alpha^*)}\left(\lambda_2^{(k+1)\alpha^*} - \lambda_1^{(k+1)\alpha^*}\right), \quad k > 0.$$

Lemma 1.8 ([43] Generalized Hölder's inequality) *Let* $s, q > 1$ *with* $s^{-1} + q^{-1} = 1$, *and let* $\mathcal{H}, \mathcal{G} \in \mathbb{C}_{\alpha^*}[\lambda 1, \lambda 2]$. *Then*

$$\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda 1}^{\lambda 2}\left|\mathcal{H}(u)\mathcal{G}(u)\right|(du)^{\alpha^*}$$

$$\leq \left(\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{\lambda_2}\left|\mathcal{H}(u)\right|^s(du)^{\alpha^*}\right)^{\frac{1}{s}}\left(\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{\lambda_2}\left|\mathcal{H}(u)\right|^q(du)^{\alpha^*}\right)^{\frac{1}{q}}.$$

Mo et al. [24] derived the following generalized HH inequality for generalized s-convex functions:

$$\frac{2^{(s-1)\alpha^*}}{\Gamma(1+\alpha^*)}\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right) \leq \frac{{}_{\lambda_1}\mathcal{I}_{\lambda_2}^{(\alpha^*)}\mathcal{H}(u)}{(\lambda_2-\lambda_1)^{\alpha^*}} \leq \frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left[\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)\right]. \tag{1.6}$$

In 1994, Hudzik and Maligranda [39] provided several generalizations via s-convexity and presented intriguing outcomes about the HH inequality for s-convex functions. In 1915, Bernstein and Doetsch [40] established a variant of the HH inequality for s-convex functions in the second sense. Moreover, some well-known integral inequalities via local fractional integral have been studied by several researchers; for instance, Kilicman and Saleh [41, 42] derived generalized HH inequalities for generalized s-convex functions. Du et al. [38] contemplated certain inequalities for generalized m-convex functions on fractal sets with utilities. Also, Vivas et al. [44] explored generalized Jensen and HH inequalities for h-convex functions. For results associated with local fractional inequalities, we refer the interested readers to [24, 45–47] and the references therein.

Owing to the phenomena mentioned, the principal purpose of this research is exploring a novel concept of (s,m)-convex functions, and we address important properties for such functions. Also, we establish some novel variants, which interact between (s,m)-convex functions and local fractional integrals. In fractal sets, we carry out two novel generalized identities to investigate the local differentiability of (s,m)-convex functions, s-convex functions, and generalized m-convex functions. Meanwhile, we present some new generalized Simpson-type inequalities for (s,m)-convexity. Generalized new special cases show the impressive performance of the local fractional integration. Some special cases correlate with existing results in classical convexity.

2 Generalized (s,m)-convex functions

We now present the concept of generalized (s,m)-convex functions on a fractal space.

Definition 2.1 Let $s\in(0,1]$. A function $\mathcal{H}:[0,\mathfrak{b}^*]\to\mathbb{R}^{\alpha^*}$ with $\mathfrak{b}^*>0$ is said to be generalized (s,m)-convex if

$$\mathcal{H}\left(\zeta u+m(1-\zeta)v\right)\leq \zeta^{s\alpha^*}\mathcal{H}(u)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(v) \tag{2.1}$$

for $u,v\in[0,\mathfrak{b}^*]$, $s\in(0,1]$, and some fixed $m\in[0,1]$.

Remark 2.2 Definition 2.1 leads to the conclusion that

1. If we take $s=1$, then we get Definition 2.1 in [38].
2. If we take $s=1$ and $\alpha^*=1$, then we get Definition in [39].
3. If we take $m=1$, then we get Definition in [23].
4. If we take $m=1$ and $\alpha^*=1$, then we get Definition in [37].
5. If we take $m=1$ and $s=1$, then we get Definition in [24].
6. If we take $m=s=1$ and $\alpha^*=1$, then we get the concept of classical convex functions.

Moreover, if we take $\zeta = \frac{1}{2}$ in (2.1), then the generalized (s,m)-convex functions become Jensen-type generalized (s,m)-convex functions as follows:

$$\mathcal{H}\left(\frac{u+mv}{2}\right) \leq \frac{1}{2^{s\alpha^*}}\left[\mathcal{H}(u) + m^{\alpha^*}\mathcal{H}(v)\right] \tag{2.2}$$

for $u, v \in [0, \mathfrak{b}^*]$, $s \in (0,1]$, and for some fixed $m \in [0,1]$.

It is worth mentioning that (s,m)-convex functions reduce to generalized convex, generalized m-convex functions, and generalized s-convex functions as particular cases. This shows that outcomes derived in the present paper continue to hold for these classes of convex functions and their variant forms.

Proposition 2.3 *For $m \in [0,1]$ and $s \in (0,1]$, if $\mathcal{H}, \mathcal{G} : \Omega \to \mathbb{R}^{\alpha^*}$ is a generalized (s,m)-convex functions, then*

(1) *$\mathcal{H} + \mathcal{G}$ is a generalized (s,m)-convex function;*

(2) *$\lambda^{\alpha^*}\mathcal{H}$ is a generalized (s,m)-convex function.*

Proof (1) Since $\mathcal{H}$ and $\mathcal{G}$ are generalized (s,m)-convex functions on Ω and $\zeta \in [0,1]$, we have

$$\begin{aligned}
&(\mathcal{H}+\mathcal{G})\big(\zeta u + m(1-\zeta)v\big) \\
&\quad = \mathcal{H}\big(\zeta u + m(1-\zeta)v\big) + \mathcal{G}\big(\zeta u + m(1-\zeta)v\big) \\
&\quad \leq \zeta^{s\alpha^*}\mathcal{H}(u) + m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(v) + \zeta^{s\alpha^*}\mathcal{G}(u) \\
&\qquad + m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{G}(v) \\
&\quad = \zeta^{s\alpha^*}(\mathcal{H}+\mathcal{G})(u) + m^{\alpha^*}(1-\zeta)^{s\alpha^*}(\mathcal{H}+\mathcal{G})(v).
\end{aligned}$$

So, $\mathcal{H} + \mathcal{G}$ is a generalized (s,m)-convex function on Ω.

(2) Since $\mathcal{H}$ and $\mathcal{G}$ are generalized (s,m)-convex functions on Ω, for $\zeta \in [0,1]$ and $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned}
\lambda^{\alpha^*}\mathcal{H}\big(\zeta u + m(1-\zeta)v\big) &= \lambda^{\alpha^*}\mathcal{H}\big(\zeta u + m(1-\zeta)v\big) \\
&\leq \lambda^{\alpha^*}\left[\zeta^{s\alpha^*}\mathcal{H}(u) + m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(v)\right] \\
&= \zeta^{\alpha^*}\big(\lambda^{\alpha^*}\mathcal{H}\big)(u) + m^{\alpha^*}(1-\zeta)^{s\alpha^*}\big(\lambda^{\alpha^*}\mathcal{H}\big)(v),
\end{aligned}$$

and hence $\lambda^{\alpha^*}\mathcal{H}$ is a generalized (s,m)-convex function on Ω. □

Proposition 2.4 *Let $\mathcal{H}_n : \Omega \to \mathbb{R}^{\alpha^*}$, $n \in \mathbb{N}$, be a sequence of generalized (s,m)-convex functions converging pointwise to a function $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha^*}$. Then $\mathcal{H}$ is a generalized (s,m)-convex function on Ω.*

Proof Let $u, v \in \Omega, \zeta \in [0,1]$, and let $\lim_{n\to\infty}\mathcal{H}_n(u) = \mathcal{H}(u)$. Then

$$\mathcal{H}\big(\zeta u + (1-\zeta)v\big) = \lim_{n\to\infty}\mathcal{H}_n\big(\zeta u + (1-\zeta)v\big)$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}\left[\zeta^{s\alpha^*}\mathcal{H}_n(u)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}_n(v)\right]\\
&=\zeta^{s\alpha^*}\lim_{n\to\infty}\mathcal{H}_n(u)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\lim_{n\to\infty}\mathcal{H}_n(v)\\
&=\zeta^{s\alpha^*}\mathcal{H}(u)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(v),
\end{aligned}
$$

that is, $\mathcal{H}$ is a generalized (s,m)-convex function on Ω. □

Proposition 2.5 *For $m\in[0,1]$ and $s\in(0,1]$, let $\mathcal{H}:[0,\infty)\to\mathbb{R}^{\alpha^*}$ be a generalized (s,m)-convex function such that $0\leq\lambda_1<m\lambda_2<\infty$. If $\mathcal{H}\in\mathbb{C}_{\alpha^*}[\lambda_1,m\lambda_2]$, then*

$$
\frac{{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(z)}{(m\lambda_2-\lambda_1)^{\alpha^*}}+\frac{{}_{m\lambda_1}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(z)}{(\lambda_2-m\lambda_1)^{\alpha^*}}\leq\left[\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)\right]\frac{(1+m^{\alpha^*})}{2^{\alpha^*}\Gamma(1+\alpha^*)}. \tag{2.3}
$$

Proof Utilizing the generalized (s,m)-convexity of $\mathcal{H}$, for all $\zeta\in[0,1]$ and $\lambda_1,\lambda_2\in\Omega$, we have

$$
\begin{aligned}
&\mathcal{H}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\leq\zeta^{s\alpha^*}\mathcal{H}(\lambda_1)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(\lambda_2),\\
&\mathcal{H}\big(\zeta\lambda_2+m(1-\zeta)\lambda_1\big)\leq\zeta^{s\alpha^*}\mathcal{H}(\lambda_2)+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\mathcal{H}(\lambda_1),\\
&\mathcal{H}\big((1-\zeta)\lambda_1+m\zeta\lambda_2\big)\leq(1-\zeta)^{s\alpha^*}\mathcal{H}(\lambda_1)+m^{\alpha^*}\zeta^{s\alpha^*}\mathcal{H}(\lambda_2)
\end{aligned}
$$

and

$$
\mathcal{H}\big((1-\zeta)\lambda_2+m(1-\zeta)\lambda_1\big)\leq(1-\zeta)^{s\alpha^*}\mathcal{H}(\lambda_2)+m^{\alpha^*}\zeta^{s\alpha^*}\mathcal{H}(\lambda_1).
$$

Adding these inequalities, we get

$$
\begin{aligned}
&\mathcal{H}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)+\mathcal{H}\big(\zeta\lambda_2+m(1-\zeta)\lambda_1\big)\\
&\quad+\mathcal{H}\big((1-\zeta)\lambda_1+m\zeta\lambda_2\big)+\mathcal{H}\big((1-\zeta)\lambda_2+m(1-\zeta)\lambda_1\big)\\
&\leq\left[\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)\right]\big(1+m^{\alpha^*}\big).
\end{aligned} \tag{2.4}
$$

Integrating inequality (2.4) with respect to ζ over $(0,1)$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*}\\
&\quad+\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\big(\zeta\lambda_2+m(1-\zeta)\lambda_1\big)(d\zeta)^{\alpha^*}\\
&\quad+\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\big((1-\zeta)\lambda_1+m\zeta\lambda_2\big)(d\zeta)^{\alpha^*}\\
&\quad+\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\big((1-\zeta)\lambda_2+m(1-\zeta)\lambda_1\big)(d\zeta)^{\alpha^*}\\
&\leq\left[\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)\right]\frac{(1+m^{\alpha^*})}{\Gamma(1+\alpha^*)}.
\end{aligned}
$$

Therefore we get

$$\frac{{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(z)}{(m\lambda_2-\lambda_1)^{\alpha^*}}+\frac{{}_{m\lambda_1}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(z)}{(\lambda_2-m\lambda_1)^{\alpha^*}}\leq\big[\mathcal{H}(\lambda_1)+\mathcal{H}(\lambda_2)\big]\frac{(1+m^{\alpha^*})}{2^{\alpha^*}\Gamma(1+\alpha^*)},$$

the desired result. □

3 Certain new results on generalized (s,m)-convexity

This section is devoted to the generalized HH inequality for generalized (s,m)-convex functions via local fractional integrals.

Theorem 3.1 *For $s,m\in(0,1]$, let $\mathcal{H}:\Omega\to\mathbb{R}^{\alpha^*}$ be a generalized (s,m)-convex function defined on a fractal space. If $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,m\lambda_2]$ for some $0\leq\lambda_1<\lambda_2$, then*

$$\begin{aligned}&\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\\&\quad\leq\frac{2^{\alpha^*(1-s)}\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\big[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}_{\frac{\lambda_1}{m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{H}(u)\big]\\&\quad\leq\frac{1}{2^{(s+1)\alpha^*}}\frac{\Gamma(1+\alpha^*)\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left[\mathcal{H}(\lambda_1)-m^{2\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right]\\&\qquad+\left(\frac{m}{2}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_2)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right].\end{aligned}\tag{3.1}$$

Proof Taking into account inequality (2.2), for all $u,v\in\Omega$, we have

$$\mathcal{H}\left(\frac{u+mv}{2}\right)\leq\frac{1}{(2)^{s\alpha^*}}\big[\mathcal{H}(u)+m^{\alpha^*}\mathcal{H}(v)\big].\tag{3.2}$$

Substituting $u=\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2$, $v=\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2$, for all $\zeta\in[0,1]$, we have

$$2^{s\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\leq\left[\mathcal{H}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)+m^{\alpha^*}\mathcal{H}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\right].\tag{3.3}$$

Integrating this inequality with respect to ζ over $(0,1)$, we have

$$\begin{aligned}&\frac{2^{s\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)(d\zeta)^{\alpha^*}\\&\quad\leq\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\left(\frac{\zeta}{2}\lambda_1+m^{\alpha^*}\frac{2-\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right.\\&\qquad\left.+\frac{m^{\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right]\\&\quad=\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{\lambda_1+m\lambda_2}{2}}^{m\lambda_2}\frac{2^{\alpha^*}\mathcal{H}(u)}{(m\lambda_2-\lambda_1)^{\alpha^*}}(du)^{\alpha^*}+\frac{m^{2\alpha^*}}{\Gamma(1+\alpha^*)}\int_{\frac{\lambda_1}{m}}^{\frac{\lambda_1+m\lambda_2}{2m}}\frac{2^{\alpha^*}\mathcal{H}(v)}{(m\lambda_2-\lambda_1)^{\alpha^*}}(du)^{\alpha^*}\\&\quad=\frac{2^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\big[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}_{\frac{\lambda_1}{m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{H}(u)\big].\end{aligned}$$

Also, using the fact that

$$\frac{2^{s\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1 \mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)(d\zeta)^{\alpha^*} = \frac{2^{s\alpha^*}}{\Gamma(1+\alpha^*)}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right),$$

we have

$$\begin{aligned}&\frac{2^{(s-1)\alpha^*}}{\Gamma(1+\alpha^*)}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\\&\quad\le \frac{1}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1}{m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{H}(u)\right].\end{aligned}$$

For the proof of the second inequality in (3.1), noting that $\mathcal{H}$ is a generalized (s,m)-convex function, for $\zeta\in[0,1]$, we have

$$\begin{aligned}&\mathcal{H}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)+m^{\alpha^*}\mathcal{H}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\\&\quad\le\left(\frac{\zeta}{2}\right)^{s\alpha^*}\left[\mathcal{H}(\lambda_1)-m^{2\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right]+m^{\alpha^*}\left[\mathcal{H}(\lambda_2)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right].\end{aligned}$$

Integrating this inequality with respect to ζ over $(0,1)$, we have

$$\begin{aligned}&\frac{2^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1}{m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{H}(u)\right]\\&\quad\le\frac{1}{2^{s\alpha^*}}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left[\mathcal{H}(\lambda_1)-m^{2\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right]\\&\qquad+\frac{m^{\alpha^*}}{\Gamma(1+\alpha^*)}\left[\mathcal{H}(\lambda_2)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right],\end{aligned}$$

where we have used Lemma 1.7 and the fact that

$$\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}=\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}.$$

This completes the proof. □

We present some remarkable cases of Theorem 3.1 as corollaries and remarks.

I. If we take $s=1$, then we have a new result for generalized m-convex functions.

Corollary 3.2 *For $m\in(0,1]$, let $\mathcal{H}:\Omega\to\mathbb{R}^{\alpha^*}$ be a generalized m-convex function defined on a fractal space. If $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,m\lambda_2]$ for some $0\le\lambda_1<\lambda_2$, then*

$$\begin{aligned}&\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\\&\quad\le\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1}{m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{H}(u)\right]\\&\quad\le\frac{1}{2^{2\alpha^*}}\frac{\Gamma^2(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\left[\mathcal{H}(\lambda_1)-m^{2\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right]+\left(\frac{m}{2}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_2)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1}{m^2}\right)\right].\end{aligned}$$

II. If we take $m = 1$, then we have a new result for s-convex functions.

Corollary 3.3 *For $s \in (0,1]$, let $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha^*}$ be a generalized s-convex function defined on a fractal space. If $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for some $0 \le \lambda_1 < \lambda_2$, then*

$$\begin{aligned}
&\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right)\\
&\quad\le \frac{2^{\alpha^*(1-s)}\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}_{\lambda_2}^{(\alpha^*)}\mathcal{H}(u) +_{\lambda_1} \mathcal{I}_{\frac{\lambda_1+\lambda_2}{2}}^{(\alpha^*)}\mathcal{H}(u)\right]\\
&\quad\le \left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_2)+\mathcal{H}(\lambda_1)\right]. \qquad (3.4)
\end{aligned}$$

III. If we take $m = 1$ and $s = 1$, then we have a new result for generalized convex functions.

Corollary 3.4 *For $s \in (0,1]$, let $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha^*}$ be a generalized convex function defined on a fractal space. If $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for some $0 \le \lambda_1 < \lambda_2$, then*

$$\begin{aligned}
\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right) &\le \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}_{\lambda_2}^{(\alpha^*)}\mathcal{H}(u) +_{\lambda_1} \mathcal{I}_{\frac{\lambda_1+\lambda_2}{2}}^{(\alpha^*)}\mathcal{H}(u)\right]\\
&\le \left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_2)+\mathcal{H}(\lambda_1)\right]. \qquad (3.5)
\end{aligned}$$

Remark 3.5 If we choose $\alpha^* = 1$ and $m = s = 1$, then Theorem 3.1 reduces to inequality (1.1).

Further, we obtain novel bounds that refine the generalized HH inequality for functions whose first derivative in absolute value raised to a certain power greater than one, respectively, at least one, is a generalized (s,m)-convex function. For our further results, we need the following lemma.

Lemma 3.6 *For $m \in (0,1]$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ (Ω° is the interior of Ω) be a function such that $\mathcal{H} \in \mathcal{D}_{\alpha^*}(\Omega^\circ)$ and $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. Then*

$$\begin{aligned}
&\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}_{m\lambda_2}^{(\alpha^*)}\mathcal{H}(u) + m^{2\alpha^*}{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}_{\frac{\lambda_1}{m}}^{(\alpha^*)}\mathcal{H}(u)\right]\\
&\quad-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\\
&= \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1 \zeta^{\alpha^*}\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right.\\
&\quad\left.-\frac{1}{\Gamma(1+\alpha^*)}\int_0^1 \zeta^{\alpha^*}\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right]. \qquad (3.6)
\end{aligned}$$

Proof Utilizing local fractional integration by parts, we get

$$I_1 = \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1 \zeta^{\alpha^*}\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right]$$

$$
\begin{aligned}
&= \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{-2^{\alpha^*}\zeta^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\mathcal{H}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)|_0^1\right.\\
&\quad\left.-\left(\frac{2}{m\lambda_2-\lambda_1}\right)^{\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+\alpha^*)}\int_0^1\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right]\\
&= \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{-2^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\right.\\
&\quad\left.+\left(\frac{2}{m\lambda_2-\lambda_1}\right)^{2\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+\alpha^*)}\int_{\frac{\lambda_1+m\lambda_2}{2}}^{m\lambda_2}\mathcal{H}(u)(du)^{\alpha^*}\right]\\
&= \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{-2^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)\right.\\
&\quad\left.+\left(\frac{2}{m\lambda_2-\lambda_1}\right)^{2\alpha^*}\Gamma\left(1+\alpha^*\right)\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2}m\lambda_2}\mathcal{H}(u)\right].
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
I_2 &= -\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)(d\zeta)^{\alpha^*}\right]\\
&= -\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{(2m)^{\alpha^*}}{(m\lambda_2-\lambda_1)^{\alpha^*}}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right.\\
&\quad\left.-\left(\frac{2m}{m\lambda_2-\lambda_1}\right)^{2\alpha^*}\Gamma\left(1+\alpha^*\right)\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}\frac{\lambda_1}{m}}\mathcal{H}(u)\right].
\end{aligned}
$$

Adding I_1 and I_2, we get the desired result. □

Theorem 3.7 *For $s,m\in(0,1]$ and $p,q>1$ with $p^{-1}+q^{-1}=1$, let $\mathcal{H}:\Omega^\circ\to\mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,m\lambda_2]$ for $\lambda_1,\lambda_2\in\Omega^\circ$ with $\lambda_2>\lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized (s,m)-convex on Ω for $q>1$, then*

$$
\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2}m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2m}\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\quad\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\quad\left.+m^{\alpha^*}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right]^{\frac{1}{q}}\\
&\quad+\left[m^{\alpha^*}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}
\tag{3.7}
$$

Proof Using Lemma 3.6, the generalized power mean inequality, and the generalized (s,m)-convexity of $|\mathcal{H}^{(\alpha^*)}|^q$, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\quad\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)\right|(d\zeta)^{\alpha^*}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\right|(d\zeta)^{\alpha^*}\right]\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}(d\zeta)^{\alpha^*}\right]^{\frac{1}{p}}\\
&\quad\times\left[\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)\right|^q(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\right|^q(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\right]\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}}\\
&\quad\times\left[\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left\{\left(\frac{\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\right.\\
&\quad\left.\left.\left.+m^{\alpha^*}\left(\frac{2-\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right\}(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left\{m^{\alpha^*}\left(\frac{2-\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\right.\right.\\
&\quad\left.\left.\left.+\left(\frac{\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right\}(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\right].
\end{aligned}
\tag{3.8}
$$

Taking into consideration Lemma 1.7, we easily see that

$$
\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left(\frac{\zeta}{2}\right)^{s\alpha^*}(d\zeta)^{\alpha^*}:=\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)},
\tag{3.9}
$$

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left(\frac{2-\zeta}{2}\right)^{s\alpha^*}(d\zeta)^{\alpha^*}\\
&:=\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}.
\end{aligned}
\tag{3.10}
$$

Combining (3.8)–(3.10), we get the desired inequality (3.7). This completes the proof. □

Some particular cases of Theorem 3.7 are presented as follows.

I. If we choose $s=1$, then we get a new result for generalized m-convex functions.

Corollary 3.8 *For $m \in (0,1]$ and $p,q>1$ with $p^{-1}+q^{-1}=1$, let $\mathcal{H}:\Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1,\lambda_2 \in \Omega^\circ$ with $\lambda_2>\lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized m-convex on Ω for $q>1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}\,{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\quad\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\quad+m^{\alpha^*}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\Bigg]^{\frac{1}{q}}\\
&\quad+\left[m^{\alpha^*}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

II. If we choose $m=1$, then we get a new result for generalized s-convex functions.

Corollary 3.9 *For $s\in(0,1]$ and $p,q>1$ with $p^{-1}+q^{-1}=1$, let $\mathcal{H}:\Omega^\circ\to\mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,\lambda_2]$ for $\lambda_1,\lambda_2\in\Omega^\circ$ with $\lambda_2>\lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized s-convex on Ω for $q>1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(u)+{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_1}\mathcal{H}(u)\right]-\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|\\
&\le\frac{(\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\quad+\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\Bigg]^{\frac{1}{q}}\\
&\quad+\left[\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

III. If we choose $m=1$ and $s=1$, then we get a new result for generalized convex functions.

Corollary 3.10 *For $p,q>1$ with $p^{-1}+q^{-1}=1$, let $\mathcal{H}:\Omega^\circ\to\mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,\lambda_2]$ for $\lambda_1,\lambda_2\in\Omega^\circ$ with $\lambda_2>\lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized convex on Ω for $q>1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(u)+{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_1}\mathcal{H}(u)\right]-\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|\\
&\le\frac{(\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.
\end{aligned}$$

$$+\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\Bigg]^{\frac{1}{q}}$$
$$+\left[\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.$$
$$\left.+\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\Bigg].$$

Remark 3.11 If we choose $\alpha^* = 1$ and $s = m = 1$, then Theorem 3.7 reduces to the result in [48].

Theorem 3.12 *For $s, m \in (0,1]$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized (s,m)-convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\quad\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\quad+m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\Bigg](d\zeta)^{\alpha^*}\Bigg]^{\frac{1}{q}}\\
&\quad+\left[m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}\tag{3.11}$$

Proof Using Lemma 3.6, the generalized Hölder inequality, and the generalized (s,m)-convexity of $|\mathcal{H}^{(\alpha^*)}|^q$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\quad\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)\right|(d\zeta)^{\alpha^*}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\right|(d\zeta)^{\alpha^*}\right]\\
&\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\zeta^{p\alpha^*}(d\zeta)^{\alpha^*}\right]^{\frac{1}{p}}\\
&\quad\times\left[\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\zeta}{2}\lambda_1+m\frac{2-\zeta}{2}\lambda_2\right)\right|^q(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\left|\mathcal{H}^{(\alpha^*)}\left(\frac{2-\zeta}{2m}\lambda_1+\frac{\zeta}{2}\lambda_2\right)\right|^q(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\Bigg]\\
&\leq\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)}\right]^{\frac{1}{p}}\\
&\times\left[\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\left\{\left(\frac{\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\right.\\
&\left.+m^{\alpha^*}\left(\frac{2-\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right\}(d\zeta)^{\alpha^*}\Bigg]^{\frac{1}{q}}\\
&+\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^1\left\{m^{\alpha^*}\left(\frac{2-\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\left.\left.+\left(\frac{\zeta}{2}\right)^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right\}(d\zeta)^{\alpha^*}\right]^{\frac{1}{q}}\Bigg]\\
&=\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\left.+m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right](d\zeta)^{\alpha^*}\Bigg]^{\frac{1}{q}}\\
&+\left[m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\left.\left.+\left(\frac{1}{2}\right)^{s\alpha^*}\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right],
\end{aligned}
\tag{3.12}
$$

the required result. This completes the proof. □

We present some particular cases of Theorem 3.12.

I. If we take $s=1$, then we get a new result for generalized m-convex functions.

Corollary 3.13 *For $m\in(0,1]$ and $p,q>1$ with $p^{-1}+q^{-1}=1$, let $\mathcal{H}:\Omega^\circ\to\mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)}\in\mathbb{C}_{\alpha^*}[\lambda_1,m\lambda_2]$ for $\lambda_1,\lambda_2\in\Omega^\circ$ with $\lambda_2>\lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized m-convex on Ω for $q>1$, then*

$$
\begin{aligned}
&\left|\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\left[{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u)+m^{2\alpha^*}{}_{\frac{\lambda_1+m\lambda_2}{2m}}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1}{m}}\mathcal{H}(u)\right]\right.\\
&\left.-\left(\frac{1}{2}\right)^{\alpha^*}\left[\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+m^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2m}\right)\right]\right|\\
&\leq\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}}\left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)}\right]^{\frac{1}{p}}\left[\left[\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q\right.\right.\\
&\left.+m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\\
&+\left[m^{\alpha^*}\left[1-\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]\left|\mathcal{H}^{(\alpha^*)}\left(\frac{\lambda_1}{m^2}\right)\right|^q\right.\\
&\left.\left.+\left(\frac{1}{2}\right)^{\alpha^*}\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}
$$

II. If we take $m = 1$, then we get a new result for generalized s-convex functions.

Corollary 3.14 *For $s \in (0,1]$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized s-convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left| \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}} \left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(u) + {}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_1}\mathcal{H}(u) \right] - \mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2} \right) \right| \\
&\quad \le \frac{(\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}} \left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)} \right]^{\frac{1}{p}} \left[\left[\left(\frac{1}{2} \right)^{s\alpha^*} \frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)} \left| \mathcal{H}^{(\alpha^*)}(\lambda_1) \right|^q \right. \right. \\
&\qquad + \left. \left[1 - \left(\frac{1}{2} \right)^{s\alpha^*} \frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)} \right] \left| \mathcal{H}^{(\alpha^*)}(\lambda_2) \right|^q \right]^{\frac{1}{q}} \\
&\qquad + \left[\left[1 - \left(\frac{1}{2} \right)^{s\alpha^*} \frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)} \right] \left| \mathcal{H}^{(\alpha^*)}(\lambda_1) \right|^q \right. \\
&\qquad \left. \left. + \left(\frac{1}{2} \right)^{s\alpha^*} \frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)} \left| \mathcal{H}^{(\alpha^*)}(\lambda_2) \right|^q \right]^{\frac{1}{q}} \right].
\end{aligned}$$

III. If we take $m = 1 = s$, then we get a new result for generalized convex functions.

Corollary 3.15 *For $p, q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left| \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}} \left[{}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\mathcal{H}(u) + {}_{\frac{\lambda_1+\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{\lambda_1}\mathcal{H}(u) \right] - \mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2} \right) \right| \\
&\quad \le \frac{(\lambda_2-\lambda_1)^{\alpha^*}}{4^{\alpha^*}} \left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)} \right]^{\frac{1}{p}} \left[\left[\left(\frac{1}{2} \right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \left| \mathcal{H}^{(\alpha^*)}(\lambda_1) \right|^q \right. \right. \\
&\qquad + \left. \left[1 - \left(\frac{1}{2} \right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \right] \left| \mathcal{H}^{(\alpha^*)}(\lambda_2) \right|^q \right]^{\frac{1}{q}} \\
&\qquad + \left. \left[\left[1 - \left(\frac{1}{2} \right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \right] \left| \mathcal{H}^{(\alpha^*)}(\lambda_1) \right|^q + \left(\frac{1}{2} \right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \left| \mathcal{H}^{(\alpha^*)}(\lambda_2) \right|^q \right]^{\frac{1}{q}} \right].
\end{aligned}$$

Remark 3.16 If we choose $\alpha^* = 1$ and $s = m = 1$, then Theorem 3.12 reduces to the result in [48].

4 New estimates for generalized Simpson's type via (s, m)-convex functions

Before continuing toward our main results in regards to generalized Simpson's inequality utilizing m-convex functions, we start with the accompanying lemma.

Lemma 4.1 *For $m \in (0,1]$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ (Ω° is the interior of Ω) be such that $\mathcal{H} \in \mathcal{D}_{\alpha^*}(\Omega^\circ)$ and $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. Then*

$$\begin{aligned}
&\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) + \mathcal{H}(m\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(m\lambda_2 - \lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2} \\
&\quad = \frac{(m\lambda_2 - \lambda_1)^{\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1 \mu(\zeta)\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*}, \qquad (4.1)
\end{aligned}$$

$$\mu(\zeta) = \begin{cases} (\zeta - \frac{1}{6})^{\alpha^*}, & \zeta \in [0, \frac{1}{2}), \\ (\zeta - \frac{5}{6})^{\alpha^*}, & \zeta \in [\frac{1}{2}, 1]. \end{cases}$$

Proof Consider

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^1 \mu(\zeta)\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*} \\
&\quad = \frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left(\zeta - \frac{1}{6}\right)^{\alpha^*}\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*} \\
&\qquad + \frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left(\zeta - \frac{5}{6}\right)^{\alpha^*}\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*}. \qquad (4.2)
\end{aligned}$$

Using the local fractional integration by parts, we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left(\zeta - \frac{1}{6}\right)^{\alpha^*}\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*} \\
&\quad = \left(\frac{1}{m\lambda_2 - \lambda_1}\right)^{\alpha^*}\left[-\left(\frac{1}{3}\right)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) - \left(\frac{1}{6}\right)^{\alpha^*}\mathcal{H}(m\lambda_2)\right] \\
&\qquad + \frac{\Gamma(1+\alpha^*)}{(m\lambda_2 - \lambda_1)^{\alpha^*}}\,{}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\mathcal{H}(u) \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left(\zeta - \frac{5}{6}\right)^{\alpha^*}\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1 + m(1-\zeta)\lambda_2\big)(d\zeta)^{\alpha^*} \\
&\quad = \left(\frac{1}{m\lambda_2 - \lambda_1}\right)^{\alpha^*}\left[-\left(\frac{1}{3}\right)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) - \left(\frac{1}{6}\right)^{\alpha^*}\mathcal{H}(\lambda_1)\right] \\
&\qquad + \frac{\Gamma(1+\alpha^*)}{(m\lambda_2 - \lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{\frac{\lambda_1+m\lambda_2}{2}}\mathcal{H}(u). \qquad (4.4)
\end{aligned}$$

Combining (2.1), (4.3), and Definition 2.1, suitable rearrangements give the desired identity (4.1). □

Theorem 4.2 *For $m \in (0,1]$ and $s \in (0,1]$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|$ is generalized (s,m)-convex on Ω, then*

$$\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) + \mathcal{H}(m\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(m\lambda_2 - \lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right|$$

$$\leq (m\lambda_2-\lambda_1)^{\alpha^*}\left\{\frac{2^{\alpha^*}(5^{\alpha^*(s+2)}-3^{\alpha^*(s+1)})-5^{\alpha^*}(3^{\alpha^*(s+1)}+6^{\alpha^*(s+1)})}{6^{\alpha^*(s+2)}}\right\}$$
$$\times\left[\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}+\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left[\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|+m^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|\right]. \tag{4.5}$$

Proof Utilizing Lemma 4.1, the modulus property, and generalized (s,m)-convexity of $|\mathcal{H}^{\alpha^*}|$, we obtain

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1)+(4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+\mathcal{H}(m\lambda_2)\right]-\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right| \\
&\leq \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1\left|\mu(\zeta)\right|\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*} \\
&=(m\lambda_2-\lambda_1)^{\alpha^*}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{1}{6}\right|^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*}\right. \\
&\quad\left.+\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{5}{6}\right|^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*}\right] \qquad (4.6)\\
&\leq(m\lambda_2-\lambda_1)^{\alpha^*}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{1}{6}\right|^{\alpha^*}\left\{\zeta^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|\right.\right. \\
&\quad\left.+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\left|\mathcal{H}^{\alpha^*}(\lambda_2)\right|\right\}(d\zeta)^{\alpha^*} \\
&\quad+\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{5}{6}\right|^{\alpha^*}\left.\left\{\zeta^{s\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|+m^{\alpha^*}(1-\zeta)^{s\alpha^*}\left|\mathcal{H}^{\alpha^*}(\lambda_2)\right|\right\}(d\zeta)^{\alpha^*}\right] \\
&=(m\lambda_2-\lambda_1)^{\alpha^*}\left[\frac{|\mathcal{H}^{(\alpha^*)}(\lambda_1)|}{\Gamma(1+\alpha^*)}\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-\zeta\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\zeta-\frac{1}{6}\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}\right.\right. \\
&\quad+\left.\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-\zeta\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}+\int_{\frac{5}{6}}^{1}\left(\zeta-\frac{5}{6}\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}\right\} \\
&\quad+\frac{m^{\alpha^*}|\mathcal{H}^{(\alpha^*)}(\lambda_2)|}{\Gamma(1+\alpha^*)}\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-\zeta\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*}\right. \\
&\quad+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\zeta-\frac{1}{6}\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*} \\
&\quad\left.\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-\zeta\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*}+\int_{\frac{5}{6}}^{1}\left(\zeta-\frac{5}{6}\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*}\right\}\right]. \qquad (4.7)
\end{aligned}$$

Utilizing Lemma 1.7, we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{6}}\left(\frac{1}{6}-\zeta\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*} \\
&=\left(\frac{1}{6}\right)^{\alpha^*(s+2)}\left[\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}-\frac{\Gamma(1+\alpha^*(s+1))}{\Gamma(1+(s+2)\alpha^*)}\right], \qquad (4.8)\\
&\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\zeta-\frac{1}{6}\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*} \\
&=\left(\frac{3^{\alpha^*(s+2)}-1}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}-\left(\frac{3^{\alpha^*(s+1)}-1}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}, \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-\zeta\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}\\
&\quad=\left(\frac{5^{\alpha^*(s+2)}-5^{\alpha^*}(3)^{\alpha^*(s+1)}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}\\
&\qquad-\frac{\Gamma(1+\alpha^*(s+1))}{\Gamma(1+(s+2)\alpha^*)}\left(\frac{5^{\alpha^*(s+2)}-(3)^{\alpha^*(s+1)}}{6^{\alpha^*(s+2)}}\right)
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{5}{6}}^{1}\left(\zeta-\frac{5}{6}\right)^{\alpha^*}\zeta^{s\alpha^*}(d\zeta)^{\alpha^*}\\
&\quad=\left(\frac{6^{\alpha^*(s+2)}-5^{(s+2)\alpha^*}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\\
&\qquad-\left(\frac{5^{\alpha^*}6^{\alpha^*(s+1)}-5^{\alpha^*(s+2)}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}.
\end{aligned}\tag{4.11}$$

Again, using Lemma 1.7 and change of variable $u=1-\zeta$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-\zeta\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*}\\
&\quad=\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{5}{6}}^{\frac{1}{2}}\left(u-\frac{5}{6}\right)^{\alpha^*}u^{s\alpha^*}(du)^{\alpha^*}\\
&\quad=\left(\frac{6^{\alpha^*(s+2)}-5^{(s+2)\alpha^*}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\\
&\qquad-\left(\frac{5^{\alpha^*}6^{\alpha^*(s+1)}-5^{\alpha^*(s+2)}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}
\end{aligned}\tag{4.12}$$

and, similarly,

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\zeta-\frac{1}{6}\right)^{\alpha^*}(1-\zeta)^{\alpha^*s}(d\zeta)^{\alpha^*}\\
&\quad=\left(\frac{5^{\alpha^*(s+2)}-5^{\alpha^*}(3)^{\alpha^*(s+1)}}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)}\\
&\qquad-\frac{\Gamma(1+\alpha^*(s+1))}{\Gamma(1+(s+2)\alpha^*)}\left(\frac{5^{\alpha^*(s+2)}-(3)^{\alpha^*(s+1)}}{6^{\alpha^*(s+2)}}\right),
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-\zeta\right)^{\alpha^*}(1-\zeta)^{\alpha^*s}(d\zeta)^{\alpha^*}\\
&\quad=\left(\frac{3^{\alpha^*(s+2)}-1}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}-\left(\frac{3^{\alpha^*(s+1)}-1}{6^{\alpha^*(s+2)}}\right)\frac{\Gamma(1+\alpha^*s)}{\Gamma(1+(s+1)\alpha^*)},
\end{aligned}\tag{4.14}$$

and

$$\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{5}{6}}^{1}\left(\zeta-\frac{5}{6}\right)^{\alpha^*}(1-\zeta)^{s\alpha^*}(d\zeta)^{\alpha^*}\tag{4.15}$$

$$= \left(\frac{1}{6}\right)^{\alpha^*(s+2)} \left[\frac{\Gamma(1+\alpha^* s)}{\Gamma(1+(s+1)\alpha^*)} - \frac{\Gamma(1+\alpha^*(s+1))}{\Gamma(1+(s+2)\alpha^*)}\right].$$

Combining (4.6) and (4.8)–(4.15), we get the desired inequality (4.5). This completes the proof. □

We present aome particular cases of Theorem 4.2.

I. If we choose $s = 1$, then we get a new result for generalized m-convex functions.

Corollary 4.3 *For $m \in (0,1]$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|$ is generalized m-convex on Ω, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) + \mathcal{H}(m\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right| \\
&\quad\le \frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{12^{\alpha^*}}\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} + \frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left[\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right| + m^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|\right].
\end{aligned}$$

II. If we choose $m = 1$, then we get a new result for generalized s-convex functions.

Corollary 4.4 *For $s \in (0,1]$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|$ is generalized s-convex on Ω, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \mathcal{H}(\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\right| \\
&\quad\le (\lambda_2-\lambda_1)^{\alpha^*}\left\{\frac{2^{\alpha^*}(5^{\alpha^*(s+2)} - 3^{\alpha^*(s+1)}) - 5^{\alpha^*}(3^{\alpha^*(s+1)} + 6^{\alpha^*(s+1)})}{6^{\alpha^*(s+2)}}\right\} \\
&\qquad\times\left[\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)} + \frac{\Gamma(1+(s+1)\alpha^*)}{\Gamma(1+(s+2)\alpha^*)}\right]\left[\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right| + \left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|\right].
\end{aligned}$$

Remark 4.5 If we choose $s = 1 = m$, then Theorem 4.2 reduces to Theorem 7 in [36].

Theorem 4.6 *For $m, s \in (0,1]$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized (s,m)-convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1 + m\lambda_2}{2}\right) + \mathcal{H}(m\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right| \\
&\quad\le (m\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}} \\
&\qquad\times\left[\left(\left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + m^{\alpha^*}\left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + m^{\alpha^*}\left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{4.16}$$

Proof Utilizing Lemma 4.1, the modulus property, and generalized (s,m)-convexity of $|\mathcal{H}^{\alpha^*}|$, we obtain

$$
\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1)+(4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right)+\mathcal{H}(m\lambda_2)\right]-\frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right| \\
&\quad\le\frac{(m\lambda_2-\lambda_1)^{\alpha^*}}{\Gamma(1+\alpha^*)}\int_0^1\left|\mu(\zeta)\right|\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*} \\
&\quad=(m\lambda_2-\lambda_1)^{\alpha^*}\left[\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{1}{6}\right|^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*}\right. \\
&\qquad\left.+\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{5}{6}\right|^{\alpha^*}\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|(d\zeta)^{\alpha^*}\right] \\
&\quad\le(m\lambda_2-\lambda_1)^{\alpha^*}\left[\left(\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{1}{6}\right|^{\alpha^*p}(d\zeta)^{\alpha^*}\right)^{\frac{1}{p}}\right. \\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|^q\right)^{\frac{1}{q}} \\
&\qquad+\left(\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left|\zeta-\frac{5}{6}\right|^{\alpha^*p}(d\zeta)^{\alpha^*}\right)^{\frac{1}{p}} \\
&\qquad\left.\times\left(\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left|\mathcal{H}^{(\alpha^*)}\big(\zeta\lambda_1+m(1-\zeta)\lambda_2\big)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad\le(m\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}} \\
&\qquad\times\left[\left(\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left\{\zeta^{\alpha^*s}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+m^{\alpha^*}(1-\zeta)^{\alpha^*s}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right\}(d\zeta)^{\alpha^*}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left\{\zeta^{\alpha^*s}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+m^{\alpha^*}(1-\zeta)^{\alpha^*s}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right\}(d\zeta)^{\alpha^*}\right)^{\frac{1}{q}}\right] \\
&\quad=(m\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}} \\
&\qquad\times\left[\left(\left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+m^{\alpha^*}\left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+m^{\alpha^*}\left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right]. \qquad (4.17)
\end{aligned}
$$

Using Lemma 1.7, and change of variable technique, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\zeta-\frac{1}{6}\right|^{\alpha^*p}(d\zeta)^{\alpha^*} \\
&\quad=\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left|\zeta-\frac{5}{6}\right|^{\alpha^*p}(d\zeta)^{\alpha^*} \\
&\quad=\frac{1}{\Gamma(1+\alpha^*)}\int_0^{\frac{1}{2}}\left|\frac{1}{6}-\zeta\right|^{\alpha^*p}(d\zeta)^{\alpha^*}+\frac{1}{\Gamma(1+\alpha^*)}\int_{\frac{1}{2}}^1\left|\zeta-\frac{1}{6}\right|^{\alpha^*p}(d\zeta)^{\alpha^*}
\end{aligned}
$$

$$= \frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\alpha^*}. \tag{4.18}$$

□

I. If we choose $s = 1$, then we get a new result for generalized m-convex functions.

Corollary 4.7 *For $m \in (0,1]$ and $p,q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized m-convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+m\lambda_2}{2}\right) + \mathcal{H}(m\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(m\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{m\lambda_2}\right| \\
&\quad\le (m\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}} \\
&\qquad\times\left[\left(\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + m^{\alpha^*}\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + m^{\alpha^*}\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{4.19}$$

II. If we choose $m = 1$, then we get a new result for generalized s-convex functions.

Corollary 4.8 *For $s \in (0,1]$ and $p,q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, m\lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized s-convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right) + \mathcal{H}(\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\right| \\
&\quad\le (\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+s\alpha^*)}{\Gamma(1+(s+1)\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}} \\
&\qquad\times\left[\left(\left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + \left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left(\frac{2^{\alpha^*(s+1)}-1}{2^{\alpha^*(s+1)}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q + \left(\frac{1}{2}\right)^{\alpha^*(s+1)}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

III. If we choose $m = 1 = s$, then we get a new result for generalized convex functions.

Corollary 4.9 *For $p,q > 1$ with $p^{-1} + q^{-1} = 1$, let $\mathcal{H} : \Omega^\circ \to \mathbb{R}^{\alpha^*}$ be a differentiable function on Ω° such that $\mathcal{H}^{(\alpha^*)} \in \mathbb{C}_{\alpha^*}[\lambda_1, \lambda_2]$ for $\lambda_1, \lambda_2 \in \Omega^\circ$ with $\lambda_2 > \lambda_1$. If $|\mathcal{H}^{(\alpha^*)}|^q$ is generalized convex on Ω for $q > 1$, then*

$$\begin{aligned}
&\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[\mathcal{H}(\lambda_1) + (4)^{\alpha^*}\mathcal{H}\left(\frac{\lambda_1+\lambda_2}{2}\right) + \mathcal{H}(\lambda_2)\right] - \frac{\Gamma(1+\alpha^*)}{(\lambda_2-\lambda_1)^{\alpha^*}}\,{}_{\lambda_1}\mathcal{I}^{(\alpha^*)}_{\lambda_2}\right| \\
&\quad\le (\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}}
\end{aligned}$$

$$\times\left[\left(\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathcal{H}^{(\alpha^*)}(\lambda_1)\right|^q+\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathcal{H}^{(\alpha^*)}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right].$$

5 Applications

5.1 Probability density functions

Consider a random variable χ with generalized probability density function $\mathfrak{p}:[\lambda_1,\lambda_2]\to[0^{\alpha^*},1^{\alpha^*}]$, which is generalized convex and has the cumulative distribution function

$$P_{\alpha^*}(\chi\le x)=\mathcal{F}_{\alpha^*}(u):=\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{u}\mathfrak{p}(\zeta)(d\zeta)^{\alpha^*}.$$

Moreover, the generalized expectation can be expressed as

$$E_{\alpha^*}=\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{\lambda_2}\zeta^{\alpha^*}\mathfrak{p}(\zeta)(d\zeta)^{\alpha^*}.$$

For more information related to probability density functions, see [49].

Clearly, we see that

$$E_{\alpha^*}(u)=\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{x}\zeta^{\alpha^*}\,d\mathcal{F}_{\alpha^*}\zeta=\lambda_2^{\alpha^*}-\frac{1}{\Gamma(1+\alpha^*)}\int_{\lambda_1}^{\lambda_2}\mathcal{F}_{\alpha^*}(\zeta)(d\zeta)^{\alpha^*}.$$

The following results are associated with Sect. 4.

Proposition 5.1 *In Theorem 4.2, choosing $m=1=s$, we have*

$$\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[P_{\alpha^*}(\chi\le\lambda_1)+4^{\alpha^*}P_{\alpha^*}\left(\chi\le\frac{\lambda_1+\lambda_2}{2}\right)+P_{\alpha^*}(\chi\le\lambda_2)\right]-\frac{\lambda_2^{\alpha^*}-E_{\alpha^*}(\chi)}{(\lambda_2-\lambda_1)^{\alpha^*}}\right|$$
$$\le(\lambda_2-\lambda_1)^{\alpha^*}\left\{\frac{2^{\alpha^*}(5^{3\alpha^*}-3^{2\alpha^*})-5^{\alpha^*}(3^{2\alpha^*}+6^{2\alpha^*})}{6^{3\alpha^*}}\right\}$$
$$\times\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}+\frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right]\left[\left|\mathfrak{p}(\lambda_1)\right|+\left|\mathfrak{p}(\lambda_2)\right|\right].$$

Proposition 5.2 *In Theorem 4.6, choosing $m=1=s$, we have*

$$\left|\left(\frac{1}{6}\right)^{\alpha^*}\left[P_{\alpha^*}(\chi\le\lambda_1)+4^{\alpha^*}P_{\alpha^*}\left(\chi\le\frac{\lambda_1+\lambda_2}{2}\right)+P_{\alpha^*}(\chi\le\lambda_2)\right]-\frac{\lambda_2^{\alpha^*}-E_{\alpha^*}(\chi)}{(\lambda_2-\lambda_1)^{\alpha^*}}\right|$$
$$\le(\lambda_2-\lambda_1)^{\alpha^*}\left(\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(p+1)\alpha^*)}\right)^{\frac{1}{p}}\left(\frac{1+2^{p+1}}{6^{p+1}}\right)^{\frac{\alpha^*}{p}}$$
$$\times\left[\left(\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathfrak{p}(\lambda_1)\right|^q+\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathfrak{p}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2^{2\alpha^*}-1}{2^{2\alpha^*}}\right)\left|\mathfrak{p}(\lambda_1)\right|^q+\left(\frac{1}{2}\right)^{2\alpha^*}\left|\mathfrak{p}(\lambda_2)\right|^q\right)^{\frac{1}{q}}\right].$$

5.2 Generalized special means

Considering the following α^*-type special means [50]. For $\lambda_1 < \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^{\alpha^*}$, we have:

I. The generalized arithmetic mean

$$\mathcal{A}_{\alpha^*}(\lambda_1, \lambda_2) := \left(\frac{\lambda_1 + \lambda_2}{2}\right)^{\alpha^*} = \frac{\lambda_1^{\alpha^*} + \lambda_2^{\alpha^*}}{2^{\alpha^*}}.$$

II. The generalized logarithmic mean

$$\mathcal{L}_{\alpha^*}(\lambda_1, \lambda_2) := \left[\frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n+1)\alpha^*)} \frac{\lambda_2^{\alpha^*+1} - \lambda_1^{\alpha^*+1}}{(\lambda_2 - \lambda_1)^{\alpha^*}}\right],$$

$n \in \mathbb{Z} \setminus \{-1, 0\}; \lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$.

Considering $\mathcal{H}(u) = u^{n\alpha^*} (u \in \mathbb{R} : n \in \mathbb{Z}, |n| \geq 2)$ in Corollaries 3.10 and 3.15, we obtain the following inequalities stated by Propositions 5.3 and 5.4, respectively.

Proposition 5.3 *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2, 0 \notin [\lambda_1, \lambda_2]$, and $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned}
&\left|\Gamma\left(1+\alpha^*\right)\left[\mathcal{L}^n_{n\alpha^*}\left(\lambda_2, \mathcal{A}(\lambda_1, \lambda_2)\right) + \mathcal{L}^n_{n\alpha^*}\left(\mathcal{A}(\lambda_1, \lambda_2), \lambda_1\right)\right] - \mathcal{A}^n_{\alpha^*}(\lambda_1, \lambda_2)\right| \\
&\leq \frac{(\lambda_2 - \lambda_1)^{\alpha^*}}{4^{\alpha^*}} \left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right]^{\frac{1}{p}} \left[\left[\left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)} \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_1^{(n-1)\alpha^*}\right|^q\right.\right. \\
&\quad + \left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} - \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right] \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_2^{(n-1)\alpha^*}\right|^q\Big]^{\frac{1}{q}} \\
&\quad + \left[\left[\frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} - \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)}\right] \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_1^{(n-1)\alpha^*}\right|^q\right. \\
&\quad \left.\left. + \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+2\alpha^*)}{\Gamma(1+3\alpha^*)} \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_2^{(n-1)\alpha^*}\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Proposition 5.4 *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2, 0 \notin [\lambda_1, \lambda_2]$, and $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned}
&\left|\Gamma\left(1+\alpha^*\right)\left[\mathcal{L}^n_{n\alpha^*}\left(\lambda_2, \mathcal{A}(\lambda_1, \lambda_2)\right) + \mathcal{L}^n_{n\alpha^*}\left(\mathcal{A}(\lambda_1, \lambda_2), \lambda_1\right)\right] - \mathcal{A}^n_{\alpha^*}(\lambda_1, \lambda_2)\right| \\
&\leq \frac{(\lambda_2 - \lambda_1)^{\alpha^*}}{4^{\alpha^*}} \left[\frac{\Gamma(1+p\alpha^*)}{\Gamma(1+(1+p)\alpha^*)}\right]^{\frac{1}{p}} \left[\left[\left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_1^{(n-1)\alpha^*}\right|^q\right.\right. \\
&\quad + \left[1 - \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right] \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_2^{(n-1)\alpha^*}\right|^q\Big]^{\frac{1}{q}} \\
&\quad + \left[\left[1 - \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)}\right] \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_1^{(n-1)\alpha^*}\right|^q\right. \\
&\quad \left.\left. + \left(\frac{1}{2}\right)^{\alpha^*} \frac{\Gamma(1+\alpha^*)}{\Gamma(1+2\alpha^*)} \frac{\Gamma(1+n\alpha^*)}{\Gamma(1+(n-1)\alpha^*)} \left|\lambda_2^{(n-1)\alpha^*}\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

6 Conclusions

In this paper, we addressed a novel concept of (s, m)-convex functions on a fractal domain. Moreover, we have discussed some algebraic properties of the proposed technique.

Also, we established some appropriate results about generalized HH type inequalities and local fractional Simpson's-like type inequalities by using tools of fractal analysis and (s,m)-convexity. Several novel results have been captured for generalized s-convex, generalized m-convex, and generalized convex functions. The obtained results have been testified by two intriguing applications to show the effectiveness of the derived results. To the best of our knowledge, the said results are new for convexity theory involving fractal sets. In the future the above theory and analysis can be extended to more complicated and applicable problems of convexity involving fractal domains. Finally, our consequences have a potential connection in fractal theory and machine learning [19, 20]. This new concept will be opening new doors of investigation toward fractal differentiations and integrations in convexity, preinvexity, fractal image processing, and camouflage in the garments industry. We hope that the main results of this paper will inspire the interested readers and will stimulate further research in this field.

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