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On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals

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Abstract

In this paper, we establish the generalized Riemann–Liouville (RL) fractional integrals in the sense of another increasing, positive, monotone, and measurable function Ψ. We determine certain new double-weighted type fractional integral inequalities by utilizing the said integrals. We also give some of the new particular inequalities of the main result. Note that we can form various types of new inequalities of fractional integrals by employing conditions on the function Ψ given in the paper. We present some corollaries as particular cases of the main results.

Keywords: Fractional integrals; The generalized fractional integrals; Fractional integral inequalities; The Chebyshev functional

1 Introduction

The Chebyshev functional is given by (see [7, 10])

$$\mathcal{T}(\mathcal{U},\mathcal{V},\mu)=\int_{x_1}^{x_2}\mu(\tau)\,d\tau\int_{x_1}^{x_2}\mu(\tau)\mathcal{U}(\tau)\mathcal{V}(\tau)\,d\tau-\int_{x_1}^{x_2}\mu(\tau)\mathcal{U}(\tau)\,d\tau\int_{x_1}^{x_2}\mu(\tau)\mathcal{V}(\tau)\,d\tau, \tag{1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are integrable functions on $[x_1,x_2]$, and μ is a positive integrable function on $[x_1,x_2]$. Applications of functional (1) are found in probability and statistical problems. Further applications can be found in [6, 16, 36]. In [9, 35] the authors defined the following extended Chebyshev functional:

$$\mathcal{T}(\mathcal{U},\mathcal{V},\mu,\nu)=\int_{x_1}^{x_2}\nu(\tau)\,d\tau\int_{x_1}^{x_2}\mu(\tau)\mathcal{U}(\tau)\mathcal{V}(\tau)\,d\tau+\int_{x_1}^{x_2}\mu(\tau)\,d\tau\int_{x_1}^{x_2}\nu(\tau)\mathcal{U}(\tau)\mathcal{V}(\tau)\,d\tau$$

$$
\begin{aligned}
&- \int_{x_1}^{x_2} \mu(\tau)\mathscr{U}(\tau)\,d\tau \int_{x_1}^{x_2} \nu(\tau)\mathscr{V}(\tau)\,d\tau \\
&- \int_{x_1}^{x_2} \nu(\tau)\mathscr{U}(\tau)\,d\tau \int_{x_1}^{x_2} \mu(\tau)\mathscr{V}(\tau)\,d\tau, \qquad (2)
\end{aligned}
$$

where $\mathscr{U}$ and $\mathscr{V}$ are integrable functions on $[x_1, x_2]$, and μ and ν are positive integrable functions on $[x_1, x_2]$. The functions $\mathscr{U}$ and $\mathscr{V}$ are said to be synchronous on $[x_1, x_2]$ if

$$
\big(\mathscr{U}(\rho) - \mathscr{U}(\zeta)\big)\big(\mathscr{V}(\rho) - \mathscr{V}(\zeta)\big) \geq 0, \quad \rho, \zeta \in [x_1, x_2].
$$

The functions $\mathscr{U}$ and $\mathscr{V}$ are said to be asynchronous on $[x_1, x_2]$ if the inequality reversed, that is,

$$
\big(\mathscr{U}(\rho) - \mathscr{U}(\zeta)\big)\big(\mathscr{V}(\rho) - \mathscr{V}(\zeta)\big) \leq 0, \quad \rho, \zeta \in [x_1, x_2].
$$

If the functions $\mathscr{U}$ and $\mathscr{V}$ are synchronous on $[r, s]$, then $\mathscr{T}(\mathscr{U}, \mathscr{V}, \mu) \geq 0$ and $\mathscr{T}(\mathscr{U}, \mathscr{V}, \mu, \nu) \geq 0$. For further details, the reader may consult Kuang [27] and Mitrinovic [35]. If we consider $\mu(\vartheta) = \nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, then $\mathscr{T}(\mathscr{U}, \mathscr{V}, \mu) = \frac{1}{2}\mathscr{T}(\mathscr{U}, \mathscr{V}, \mu, \nu)$. In [3, 12, 34, 44], various researchers gave valuable consideration to functionals (1) and (2). Recently, Rahman et al. [57] defined fractional conformable inequalities for the Chebyshev functionals (1) and (2).

Awan et al. [2] presented the following result: If Φ is an absolutely continuous on $[x_1, x_2]$ such that $(\Phi')^2 \in L_1[x_1, x_2]$ and μ is a positive integrable function on $[x_1, x_2]$, then the following inequality holds;

$$
\begin{aligned}
\mathscr{T}(\Phi, \Phi, \mu) \leq \frac{1}{Q^2(x_2)} \int_{x_1}^{x_2} &\left[\int_{x_1}^{\tau} \mu(\tau)\,d\tau \int_{x_1}^{x_2} \tau\mu(\tau)\,d\tau - \int_{x_1}^{x_2} \mu(\tau)\,d\tau \int_{x_1}^{\tau} \tau\mu(\tau)\,d\tau \right] \\
&\times \big[\Phi'(\theta)\big]^2\,d\theta,
\end{aligned}
$$

where $Q(x_2) = \int_{x_1}^{x_2} \mu(\tau)\,d\tau$.

Bezziou et al. [5] presented the following result.

Theorem 1.1 *Let $\Phi : [x_1, x_2] \to \mathbb{R}$ be an absolutely continuous function such that $(\Phi')^2 \in L_1[x_1, x_2]$, and let $\mu : [x_1, x_2] \to \mathbb{R}^+$ be an integrable function. Then we have the following inequality for $\kappa > 0$:*

$$
\mathscr{J}_{x_1}^{\tau}\mu(x_2)\,\mathscr{J}_{x_1}^{\tau}\mu\Phi^2(x_2) - \big(\mathscr{J}_{x_1}^{\tau}\mu\Phi(x_2)\big)^2 \leq \int_{x_1}^{x_2} \Lambda(\theta)\big[\Phi'(\theta)\big]^2\,d\theta
$$

with

$$
\Lambda(\theta) = \frac{1}{2}\left[\mathscr{J}_{x_1}^{\tau} x_2\mu(x_2) \int_{x_1}^{\tau} (x_2 - \tau)^{\tau-1}\mu(\tau)\,d\tau - \mathscr{J}_{x_1}^{\tau}\mu(x_2) \int_{x_1}^{\tau} \tau(x_2 - \tau)^{\tau-1}\mu(\tau)\,d\tau \right],
$$

where $\mathscr{J}_{x_1}^{\tau}$ is the classical RL-fractional integral.

Dahmani and Bounoua [13] established the following result.

Theorem 1.2 *Let $\Phi : [x_1, x_2] \to \mathbb{R}$ be an absolutely continuous such that $(\Phi')^2 \in L_1[x_1, x_2]$, and let $\mu : [x_1, x_2] \to \mathbb{R}^+$ be an integrable function. Then for all $\kappa > 0$ and $\theta \in [x_1, x_2]$, the following inequality holds;*

$$\frac{1}{\mathcal{J}_{x_1}^{\tau}\mu(\theta)}\mathcal{J}_{x_1}^{\tau}\big(\mu\Phi^2\big)(\theta) - \left[\frac{1}{\mathcal{J}_{x_1}^{\tau}\mu(\theta)}\mathcal{J}_{x_1}^{\tau}(\mu\Phi)(\theta)\right]^2 \leq \frac{1}{[\mathcal{J}_{x_1}^{\tau}\mu(\theta)]^2}\int_{x_1}^{\theta} Q_\theta(\tau)\big[\Phi'(\tau)\big]^2\,d\tau$$

with

$$Q_\theta(\tau) = \frac{1}{\Gamma(\tau)}\left[\mathcal{J}_{x_1}^{\tau}\big(\theta\mu(\theta)\big)\int_{x_1}^{\rho}\mu(\vartheta)(\theta-\vartheta)^{\tau-1}\,d\vartheta - \mathcal{J}_{x_1}^{\kappa}\mu(\theta)\int_{x_1}^{\rho}\vartheta\,\mu(\vartheta)(\theta-\vartheta)^{\tau-1}\,d\vartheta\right],$$

where $\mathcal{J}_{x_1}^{\tau}$ is the classical Riemann–Liouville fractional integral.

In the last few decades, the researchers investigated different kinds of integral inequalities by considering various integral approaches. In [14] the authors gave weighted Grüss-type inequalities by taking RL-fractional integrals into account. Dahmani [8] proposed some new inequalities in the sense of fractional integrals. Several inequalities for the extended gamma function and confluent hypergeometric k-function are found by Nisar et al. [38]. Nisar et al. [39] used Riemann–Liouville and Hadamard k-fractional derivatives and investigated Gronwall-type inequalities with applications. Rahman et al. [55] studied (k,ρ)-fractional integrals and investigated the corresponding inequalities. Sarikaya and Budak [59] proposed Ostrowski-type inequalities by considering local fractional integrals. Sarikaya et al. [60] proposed the idea of generalized (k,s)-fractional integrals with applications. Set et al. [61] investigated Grüss-type inequalities for the generalized k-fractional integrals. Recently, Jarad et al. [22, 23] proposed the idea of fractional conformable and proportional fractional integral operators. Huang et al. [20] recently presented generalized Hermite–Hadamard-type inequalities for k-fractional conformable integrals. Qi et al. [45] proposed Chebyshev-type inequalities by using generalized k-fractional conformable integrals. Rahman et al. [56] investigated Chebyshev-type inequalities by utilizing fractional conformable integrals. Chebyshev-type inequalities and Minkowski-type inequalities involving generalized conformable integrals can be found in the work of Nisar et al. [42, 43]. Recently, Tassaddiq et al. [63] proposed certain inequalities for the weighted and extended Chebyshev functionals by using fractional conformable integrals. Nisar et al. [40] presented some new classes of inequalities for an n ($n \in \mathbb{N}$) family of positive continuous and decreasing functions via generalized conformable fractional integrals. Nisar et al. [41] established generalized fractional integral inequalities via the Marichev–Saigo–Maeda (MSM) fractional integral operators. Rahman et al. [54] recently investigated Grüss-type inequalities for generalized k-fractional conformable integrals. Minkowski's inequalities, fractional Hadamard proportional integral inequalities, and fractional proportional inequalities for convex functions by employing fractional proportional integrals can be found in [46–53]. In addition, various applications of fractional calculus can be found in [1, 17, 18, 28–32, 62, 64].

The paper is organized as follows. Some auxiliary results are presented in Sect. 2. In Sect. 3, we present double-weighted fractional integral inequalities for the Chebyshev functionals. In Sect. 4, we retrieve several particular cases of the results. A concluding remark is given in Sect. 5.

2 Auxiliary results

In this section, we present some well-known definitions and mathematical preliminaries of fractional calculus.

Definition 2.1 ([26, 58]) Let $\mathscr{U} \in L[x_1, x_2]$. Then the classical left- and right-sided RL-fractional integrals of order $\tau > 0$ and $x_1 \geq 0$ are respectively defined by

$$\left({}_{x_1}\mathscr{J}^{\tau}\mathscr{U}\right)(\vartheta) = \frac{1}{\Gamma(\tau)}\int_a^{\vartheta}(\vartheta-\varrho)^{\tau-1}\mathscr{U}(\varrho)\,d\varrho, \quad x_1 < \vartheta, \tag{3}$$

and

$$\left(\mathscr{J}^{\tau}_{x_2}\mathscr{U}\right)(\vartheta) = \frac{1}{\Gamma(\tau)}\int_{\vartheta}^{x_2}(\varrho-\vartheta)^{\tau-1}\mathscr{U}(\varrho)\,d\varrho, \quad \vartheta < x_2, \tag{4}$$

where Γ is the standard gamma function.

Definition 2.2 ([37]) Let $\mathscr{U} \in L[x_1, x_2]$. Then the generalized left- and right-sided RL κ-fractional integrals of order $\tau > 0$ and $x_1 \geq 0$ are respectively defined by

$$\left({}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mathscr{U}\right)(\vartheta) = \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_a^{\vartheta}(\vartheta-\varrho)^{\frac{\tau}{\kappa}-1}\mathscr{U}(\varrho)\,d\varrho, \quad x_1 < \vartheta, \tag{5}$$

and

$$\left(\mathscr{J}^{\tau}_{x_2,\kappa}\mathscr{U}\right)(\vartheta) = \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{\vartheta}^{x_2}(\varrho-\vartheta)^{\frac{\tau}{\kappa}-1}\mathscr{U}(\varrho)\,d\varrho, \quad \vartheta < x_2, \tag{6}$$

where Γ_{κ} is the κ-gamma function defined in [15].

Remark 2.1 Applying Definition 2.2 for $\kappa = 1$, we get Definition 2.1.

Definition 2.3 ([26]) Let $\mathscr{U} : [x_1, x_2] \to \mathbb{R}$ be an integrable function, and let Ψ be an increasing positive function on $(x_1, x_2]$ with continuous derivative Ψ' on (x_1, x_2). Then the left- and right-sided generalized RL fractional integrals of a function $\mathscr{U}$ concerning another function Ψ are respectively defined by

$$\left({}^{\Psi}_{x_1}\mathscr{J}^{\tau}\mathscr{U}\right)(\rho) = \frac{1}{\Gamma(\tau)}\int_{x_1}^{\vartheta}\left(\Psi(\vartheta)-\Psi(\varrho)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\mathscr{U}(\varrho)\,d\varrho, \quad x_1 < \vartheta, \tag{7}$$

and

$$\left({}^{\Psi}\mathscr{J}^{\tau}_{x_2}\mathscr{U}\right)(\rho) = \frac{1}{\Gamma(\tau)}\int_{\vartheta}^{x_2}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\mathscr{U}(\varrho)\,d\varrho, \quad \vartheta < x_2, \tag{8}$$

where $\kappa > 0$ and $\tau \in \mathbb{C}$ with $\Re(\tau) > 0$.

Definition 2.4 ([33]) Let $\mathscr{U} : [x_1, x_2] \to \mathbb{R}$ be an integrable function, and let Ψ be an increasing positive function on $(x_1, x_2]$ with continuous derivative Ψ' on (x_1, x_2). Then the left- and right-sided generalized RL κ-fractional integrals of a function $\mathscr{U}$ concerning another function Ψ are respectively defined by

$$\left({}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mathscr{U}\right)(\rho) = \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{\vartheta}\left(\Psi(\vartheta)-\Psi(\varrho)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\mathscr{U}(\varrho)\,d\varrho, \quad x_1 < \vartheta, \tag{9}$$

and

$$\left({}^{\Psi}\mathscr{J}^{\tau}_{x_2,\kappa}\mathscr{U}\right)(\rho) = \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{\vartheta}^{x_2}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\mathscr{U}(\varrho)\,d\varrho, \quad \vartheta < x_2, \tag{10}$$

where $\kappa > 0$ and $\tau \in \mathbb{C}$ with $\Re(\tau) > 0$.

Remark 2.2 The following particular cases are easily derived:

i. Applying Definition 2.4 for $\Psi(\vartheta) = \vartheta$, we get Definition 2.2,
ii. Applying Definition 2.4 for $\kappa = 1$, we get Definition 2.3,
iii. Applying Definition 2.4 for $\Psi(\vartheta) = \ln\vartheta$, we get the generalized Hadamard κ-fractional integrals defined in [21],
iv. Applying Definition 2.4 for $\Psi(\vartheta) = \ln\vartheta$ and $\kappa = 1$ leads to the Hadamard fractional integrals defined in [26],
v. Applying Definition 2.4 for $\Psi(\vartheta) = \frac{\vartheta^{\tau}}{\tau}$, $\tau > 0$, and $\kappa = 1$ leads to the Katugampola fractional integrals [24],
vi. Applying Definition 2.4 for $\Psi(\vartheta) = \frac{\vartheta^{\alpha+s}}{\alpha+s}$ and $\kappa = 1$ (where $\alpha \in (0,1]$, $s \in \mathbb{R}$, and $\mu + s \neq 0$) leads to the generalized fractional conformable integrals defined by Khan and Khan [25],
vii. Applying Definition 2.4 for $\Psi(\vartheta) = \frac{(\vartheta-x_1)^{\alpha}}{\alpha}$ and $\Psi(\vartheta) = \frac{-(x_2-\vartheta)^{\alpha}}{\alpha}$, $\alpha > 0$, leads to the (k,α)-fractional conformable integrals defined by Habib et al. [19].
viii. Applying Definition 2.4 for $\Psi(\vartheta) = \frac{(\vartheta-x_1)^{\alpha}}{\alpha}$, $\Psi(\vartheta) = \frac{-(x_2-\vartheta)^{\alpha}}{\alpha}$, $\alpha > 0$, and $\kappa = 1$ leads to the conformable fractional integrals defined by Jarad et al. [23],
ix. Applying Definition 2.4 for $\Psi(\vartheta) = \vartheta$ and $\kappa = 1$, we get Definition 2.1.

3 Some double-weighted generalized fractional integral inequalities

In this section, we present some double-weighted generalized fractional integral inequalities. We start by proving the following lemma.

Lemma 3.1 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $\mathscr{V} : [x_1, x_2] \to \mathbb{R}$ be continuous on $[x_1, x_2]$, and let $\mu, \nu : [x_1, x_2] \to \mathbb{R}^+$ be positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mu(x_2)\right]\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}x_2(\nu\mathscr{V})(x_2)\right] + \left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mathscr{V}(x_2)\right]\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}x_2(\mu\mathscr{V})(x_2)\right] \\
&\quad - \left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mu(x_2)x_2\mu(x_2)\right]\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}(\nu\mathscr{V})(x_2)\right] - \left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}x_2\nu(x_2)\right]\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}(\mu\mathscr{V})(x_2)\right] \\
&\leq \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\left[{}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}x_2\mu(x_2)\int_{x_1}^{\vartheta}\left(\Psi(x_2)-\Psi(\varrho)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{J}^{\tau}_{\kappa}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\left(\Psi(x_2)-\Psi(\varrho)\right)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\left(\Psi'(\vartheta)\right)d\vartheta.
\end{aligned} \tag{11}$$

Proof Suppose that $\mathscr{U} : [x_1, x_2] \to \mathbb{R}$ is a continuous function on $[x_1, x_2]$. Then we get

$$\begin{aligned}
&\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U}\mathscr{V})(x_2)\big] + \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mathscr{V}(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U}\mathscr{V})(x_2)\big] \\
&\quad - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{V})(x_2)\big] - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{V})(x_2)\big] \\
&= \frac{1}{\kappa^2\Gamma^2_{\kappa}(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \\
&\quad \times \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\big[\big(\mathscr{U}(\xi)-\mathscr{U}(\varrho)\big)\big(\mathscr{V}(\xi)-\mathscr{V}(\varrho)\big)\big]\,d\xi\,d\varrho.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U}\mathscr{V})(x_2)\big] + \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mathscr{V}(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U}\mathscr{V})(x_2)\big] \\
&\quad - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{V})(x_2)\big] - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{V})(x_2)\big] \\
&= \frac{1}{\kappa^2\Gamma^2_{\kappa}(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \\
&\quad \times \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\big(\mathscr{U}(\xi)-\mathscr{U}(\varrho)\big)\bigg(\int_{\varrho}^{\xi}\mathscr{V}'(\vartheta)\,d\vartheta\bigg)\,d\xi\,d\varrho. \qquad (12)
\end{aligned}$$

Utilizing the condition $x_1 \le \varrho \le \vartheta \le \xi \le x_2$, we conclude that

$$\begin{aligned}
&\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U}\mathscr{V})(x_2)\big] + \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mathscr{V}(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U}\mathscr{V})(x_2)\big] \\
&\quad - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{V})(x_2)\big] - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{U})(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{V})(x_2)\big] \\
&= \frac{1}{\kappa^2\Gamma^2_{\kappa}(\tau)}\int_{x_1}^{x_2}\bigg[\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho) \\
&\quad \times \int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\mathscr{U}(\xi)-\mathscr{U}(\varrho)\big)\Psi'(\xi)\mu(\xi)\,d\xi\,d\varrho\bigg]\big(\mathscr{V}'(\vartheta)\big)\,d\vartheta. \qquad (13)
\end{aligned}$$

Applying (13) to the particular case $\mathscr{U}(x) = x$, we can write

$$\begin{aligned}
&\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2(\nu\mathscr{V})(x_2)\big] + \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mathscr{V}(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2(\mu\mathscr{V})(x_2)\big] \\
&\quad - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2(\mu)(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\mathscr{V})(x_2)\big] - \big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2(\nu)(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\mathscr{V})(x_2)\big] \\
&= \frac{1}{\kappa^2\Gamma^2_{\kappa}(\tau)}\int_{x_1}^{x_2}\bigg[\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho) \\
&\quad \times \int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}(\xi-\varrho)\Psi'(\xi)\mu(\xi)\,d\xi\,d\varrho\bigg]\big(\mathscr{V}'(\vartheta)\big)\,d\vartheta \\
&= \frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\bigg[\frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\xi\mu(\xi)\,d\xi \\
&\quad \times \int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho - \frac{1}{\kappa\Gamma_{\kappa}(\tau)} \\
&\quad \times \int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\,d\xi \times \int_{x_1}^{\vartheta} \\
&\quad \times \big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\varrho\nu(\varrho)\,d\varrho\bigg]\big(\mathscr{V}'(\vartheta)\big)\,d\vartheta.
\end{aligned}$$

The latter by (9) gives

$$\begin{aligned}
&\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2(\nu\mathscr{V})(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mathscr{V}(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2(\mu\mathscr{V})(x_2)\big]\\
&\quad-\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2(\mu)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu\mathscr{V})(x_2)\big]-\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2(\nu)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu\mathscr{V})(x_2)\big]\\
&=\frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\Bigg[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\\
&\quad-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\Bigg]\big(\mathscr{V}'(\vartheta)\big)\,d\vartheta,
\end{aligned}$$

which completes the proof. □

Based on Lemma 3.1, we prove the following theorem.

Theorem 3.1 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $\Phi:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(\Phi')^2\in L_1[x_1,x_2]$, and let $\mu,\nu:[x_1,x_2]\to\mathbb{R}^+$ be positive integrable functions. Then for all $\tau,\kappa>0$, we have*

$$\begin{aligned}
&\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\nu\Phi^2\big)(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\nu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\mu\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu\Phi)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu\Phi)(x_2)\big]\\
&\leq\frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\Bigg[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\\
&\quad-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\Bigg]\big(\Phi'(\vartheta)\big)^2\,d\vartheta.
\end{aligned}\tag{14}$$

Proof By employing definition (9) and Lemma 3.1 we obtain

$$\begin{aligned}
&\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\nu\Phi^2\big)(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\nu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\mu\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu\Phi)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu\Phi)(x_2)\big]\\
&=\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\big(\Phi(\xi)-\Phi(\varrho)\big)^2\,d\xi\,d\varrho\\
&=\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)^2\left(\frac{\Phi(\xi)-\Phi(\varrho)}{\xi-\varrho}\right)^2d\xi\,d\varrho.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\nu\Phi^2\big)(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\nu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\big(\mu\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu\Phi)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu\Phi)(x_2)\big]
\end{aligned}$$

$$\leq \frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}$$
$$\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)^2\left(\frac{\int_\varrho^\xi \Phi'(\vartheta)\,d\vartheta}{\xi-\varrho}\right)^2 d\xi\,d\varrho.$$

By the Cauchy–Schwarz inequality [11] we get

$$\begin{aligned}
&\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\mu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\big(\nu\Phi^2\big)(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\nu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\big(\mu\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\mu\Phi)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu\Phi)(x_2)\big]\\
&=\frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)^2\left(\frac{\int_\varrho^\xi \Phi'(\vartheta)\,d\vartheta}{\xi-\varrho}\right)^2 d\xi\,d\varrho\\
&\leq\frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)^2\left(\frac{(\int_\varrho^\xi d\vartheta)^{\frac12}(\int_\varrho^\xi(\Phi'(\vartheta))^2\,d\vartheta)^{\frac12}}{\xi-\varrho}\right)^2 d\xi\,d\varrho\\
&\leq\frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)^2\left(\frac{(\int_\varrho^\xi(\Phi'(\vartheta))^2\,d\vartheta)}{\xi-\varrho}\right)d\xi\,d\varrho\\
&\leq\frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\\
&\quad\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)\left(\int_\varrho^\xi\big(\Phi'(\vartheta)\big)^2\,d\vartheta\right)d\xi\,d\varrho. \quad (15)
\end{aligned}$$

Hence using (13) and (15), we conclude the proof. □

Corollary 3.1 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $\Phi:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(\Phi')^2\in L_1[x_1,x_2]$, and let $\nu:[x_1,x_2]\to\mathbb{R}^+$ be positive integrable. Then for all $\tau,\kappa>0$, we have*

$$\begin{aligned}
&\left[\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)}\right]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\big(\nu\Phi^2\big)(x_2)\big]+\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\nu(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\big(\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\Phi)(x_2)\big]\big[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu\Phi)(x_2)\big]\\
&\leq\frac{1}{\kappa\Gamma_\kappa(\tau)}\int_{x_1}^{x_2}\left[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau x_2\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right.\\
&\quad\left.-\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)}\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\big(\Phi'(\vartheta)\big)^2\,d\vartheta.
\end{aligned}$$

Proof By considering $\mu(\vartheta)=1$, $\vartheta\in[x_1,x_2]$, in Theorem 3.1 we obtain the desired result. □

Corollary 3.2 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $\Phi:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(\Phi')^2\in L_1[x_1,x_2]$, and let $\mu:[x_1,x_2]\to\mathbb{R}^+$ be positive integrable. Then for all $\tau,\kappa>0$, we have*

$$\begin{aligned}
&\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\big(\Phi^2\big)(x_2)\big]+\left[\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_{\kappa}(\tau+\kappa)}\right]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\big(\Phi^2\big)(x_2)\big]\\
&\quad-2\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\Phi)(x_2)\big]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\Phi)(x_2)\big]\\
&\le\frac{1}{\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right.\\
&\quad\left.-{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right]\big(\Phi'(\vartheta)\big)^2\,d\vartheta.
\end{aligned}$$

Proof By considering $\nu(\vartheta)=1$, $\vartheta\in[x_1,x_2]$, in Theorem 3.1 we get the desired result. □

Corollary 3.3 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $\Phi:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(\Phi')^2\in L_1[x_1,x_2]$. Then for all $\tau,\kappa>0$, we have*

$$\begin{aligned}
&\left[\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_{\kappa}(\tau+\kappa)}\right]\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\big(\Phi^2\big)(x_2)\big]-\big[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\Phi)(x_2)\big]^2\\
&\le\frac{1}{2\kappa\Gamma_{\kappa}(\tau)}\int_{x_1}^{x_2}\left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right.\\
&\quad\left.-\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_{\kappa}(\tau+\kappa)}\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right]\big(\Phi'(\vartheta)\big)^2\,d\vartheta.
\end{aligned}$$

Proof Taking $\mu(\vartheta)=\nu(\vartheta)=1$, $\vartheta\in[x_1,x_2]$, in Theorem 3.1, we obtain the desired result. □

Theorem 3.2 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $f_1,f_2:[x_1,x_2]\to\mathbb{R}$ be absolutely continuous functions with $(f_1')^2\in L_1[x_1,x_2]$ and $(f_2')^2\in L_1[x_1,x_2]$, and let $\mu,\nu:[x_1,x_2]\to\mathbb{R}^+$ be positive integrable. Then for all $\tau,\kappa>0$, we have*

$$\begin{aligned}
&\big|{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2){}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1f_2)(x_2)+{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2){}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1f_2)(x_2)\\
&\quad-{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2){}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2)-{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2){}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2)\big|\\
&\le\frac{1}{\kappa\Gamma_{\kappa}(\tau)}\left(\int_{x_1}^{x_2}\left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right.\right.\\
&\quad\left.\left.-{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\big(f_1'(\vartheta)\big)^2\,d\vartheta\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{x_1}^{x_2}\left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right.\right.\\
&\quad\left.\left.-{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\big(f_2'(\vartheta)\big)^2\,d\vartheta\right)^{\frac{1}{2}}.
\end{aligned}\tag{16}$$

Proof Considering the left-hand side of (16), we have

$$
\begin{aligned}
&\left|{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_1 f_2)(x_2)+{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\nu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_1 f_2)(x_2)\right.\\
&\quad\left.-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_2)(x_2)-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_2)(x_2)\right|\\
&\le\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\left(\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\right.\\
&\quad\left.\times\Psi'(\varrho)\nu(\varrho)\big(f_1(\xi)-f_1(\varrho)\big)^2\,d\xi\,d\varrho\right)^{\frac{1}{2}}\times\left(\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\right.\\
&\quad\left.\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\big(f_2(\xi)-f_2(\varrho)\big)^2\,d\xi\,d\varrho\right)^{\frac{1}{2}}\\
&\le\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\left(\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\right.\\
&\quad\left.\times\Psi'(\varrho)\nu(\varrho)\left(\int_{\varrho}^{\xi}f_1'(\vartheta)\,d\vartheta\right)^2d\xi\,d\varrho\right)^{\frac{1}{2}}\times\left(\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\right.\\
&\quad\left.\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\left(\int_{\varrho}^{\xi}f_2'(\vartheta)\,d\vartheta\right)^2d\xi\,d\varrho\right)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality [11] to this inequality, we get

$$
\begin{aligned}
&\left|{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\mu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_1 f_2)(x_2)+{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}\nu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_1 f_2)(x_2)\right.\\
&\quad\left.-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_2)(x_2)-{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\mu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_{\kappa}^{\tau}(\nu f_2)(x_2)\right|\\
&\le\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\left[\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\right.\\
&\quad\left.\times\Psi'(\varrho)\nu(\varrho)\left(\left(\int_{\varrho}^{\xi}d\vartheta\right)^{\frac{1}{2}}\left(\int_{\varrho}^{\xi}\big(f_1'(\vartheta)\big)^2d\vartheta\right)^{\frac{1}{2}}\right)^2d\xi\,d\varrho\right]^{\frac{1}{2}}\\
&\quad\times\left[\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\right.\\
&\quad\left.\times\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\left(\left(\int_{\varrho}^{\xi}d\vartheta\right)^{\frac{1}{2}}\left(\int_{\varrho}^{\xi}\big(f_2'(\vartheta)\big)^2d\vartheta\right)^{\frac{1}{2}}\right)^2d\xi\,d\varrho\right]^{\frac{1}{2}}\\
&\le\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\left[\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\right.\\
&\quad\left.\times\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)\left(\int_{\varrho}^{\xi}\big(f_1'(\vartheta)\big)^2d\vartheta\right)d\xi\,d\varrho\right]^{\frac{1}{2}}\times\left[\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\right.\\
&\quad\left.\times\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)(\xi-\varrho)\left(\int_{\varrho}^{\xi}\big(f_2'(\vartheta)\big)^2d\vartheta\right)d\xi\,d\varrho\right]^{\frac{1}{2}}\\
&\le\frac{1}{\kappa^2\Gamma_{\kappa}^2(\tau)}\left[\int_{x_1}^{x_2}\left(\int_{x_1}^{x_2}\xi\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\,d\xi\right.\right.\\
&\quad\left.\left.\times\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right)\big(f_1'(\vartheta)\big)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&- \left[\int_{x_1}^{x_2} \left(\int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi)\mu(\xi)\,d\xi \right.\right. \\
&\left.\left. \times \int_{x_1}^{\varrho} \varrho\big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\,d\varrho \right) \big(f_2'(\vartheta)\big)^2 \right]^{\frac{1}{2}}.
\end{aligned}
$$

In view of (9), we get the desired proof of (16). □

Corollary 3.4 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions with $(f_1')^2 \in L_1[x_1, x_2]$ and $(f_2')^2 \in L_1[x_1, x_2]$, and let $\nu : [x_1, x_2] \to \mathbb{R}^+$ be positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\le \frac{1}{\kappa\Gamma_\kappa(\tau)} \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\,d\varrho \right.\right. \\
&\quad \left.\left. - \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)} \int_{x_1}^{\vartheta} \varrho\big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\,d\varrho \right] \big(f_1'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\,d\varrho \right.\right. \\
&\quad \left.\left. - \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)} \int_{x_1}^{\vartheta} \varrho\big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\,d\varrho \right] \big(f_2'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}}.
\end{aligned}
$$

Proof Applying Theorem 3.2 with $\mu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, we obtain the desired result. □

Corollary 3.5 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions with $(f_1')^2 \in L_1[x_1, x_2]$ and $(f_2')^2 \in L_1[x_1, x_2]$, and let $\mu : [x_1, x_2] \to \mathbb{R}^+$ be positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) + \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) \right| \\
&\le \frac{1}{\kappa\Gamma_\kappa(\tau)} \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2\mu(x_2) \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\,d\varrho \right.\right. \\
&\quad \left.\left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho\big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\,d\varrho \right] \big(f_1'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2\mu(x_2) \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\,d\varrho \right.\right. \\
&\quad \left.\left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho\big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\,d\varrho \right] \big(f_2'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}}.
\end{aligned}
$$

Proof Applying Theorem 3.2 with $\nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, we obtain the desired result. □

Corollary 3.6 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions with $(f_1')^2 \in L_1[x_1, x_2]$ and $(f_2')^2 \in L_1[x_1, x_2]$, and let $\mu : [x_1, x_2] \to \mathbb{R}^+$ be positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) \right| \\
&\le \frac{1}{2\kappa\Gamma_\kappa(\tau)} \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right. \right. \\
&\quad \left. \left. - \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right] \big(f_1'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{x_1}^{x_2} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right. \right. \\
&\quad \left. \left. - \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right] \big(f_2'(\vartheta)\big)^2 d\vartheta \right)^{\frac{1}{2}}.
\end{aligned}
$$

Proof Applying Theorem 3.2 with $\mu(\vartheta) = \nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, we obtain the desired result. □

Theorem 3.3 *Let Ψ be measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1 : [x_1, x_2] \to \mathbb{R}$ be an absolutely continuous function with $(f_1') \in L^\infty[x_1, x_2]$, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. Moreover, let $\mu, \nu : [x_1, x_2] \to \mathbb{R}^+$ be positive integrable, Then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\le \frac{\|f_1'\|_\infty}{\kappa\Gamma_\kappa(\tau)} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \mu(x_2) \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right] \int_{x_1}^{x_2} f_2'(\vartheta)\, d\vartheta. \qquad (17)
\end{aligned}
$$

Proof Considering the left-hand side of (17), we have

$$
\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\le \frac{1}{\kappa^2\Gamma_\kappa^2(\tau)} \left| \int_{x_1}^{x_2} \int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho) \right. \\
&\quad \left. \times \big[\big(f_1(\xi) - f_1(\varrho)\big)\big(f_2(\xi) - f_2(\varrho)\big) \big]\, d\xi\, d\varrho \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\\
&\quad\times\left|\frac{(f_1(\xi)-f_1(\varrho))}{\xi-\varrho}\right|\big|(\xi-\varrho)\big(f_2(\xi)-f_2(\varrho)\big)\big|\,d\xi\,d\varrho\\
&\leq \frac{\|f_1'\|_\infty}{\kappa^2\Gamma_\kappa^2(\tau)}\int_{x_1}^{x_2}\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)\\
&\quad\times(\xi-\varrho)\left(\int_{x_1}^{x_2}f_2'(\vartheta)\,d\vartheta\right)d\xi\,d\varrho\\
&\leq \frac{\|f_1'\|_\infty}{\kappa^2\Gamma_\kappa^2(\tau)}\left[\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\xi\Psi'(\xi)\mu(\xi)\,d\xi\right.\\
&\quad\times\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho-\int_{x_1}^{x_2}\big(\Psi(x_2)-\Psi(\xi)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\xi)\mu(\xi)\,d\xi\\
&\quad\left.\times\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\int_{x_1}^{x_2}f_2'(\vartheta)\,d\vartheta.
\end{aligned}
$$

Hence taking (9) into account, we complete the proof of (17). □

Corollary 3.7 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $f_1:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(f_1')\in L^\infty[x_1,x_2]$, and let $f_2:[x_1,x_2]\to\mathbb{R}$ be a nondecreasing function. Suppose that $\nu:[x_1,x_2]\to\mathbb{R}^+$ is positive integrable. Then for all $\tau,\kappa>0$, we have*

$$
\begin{aligned}
&\left|\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)}{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu f_1f_2)(x_2)+{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\nu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(f_1f_2)(x_2)\right.\\
&\quad\left.-{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(f_2)(x_2)-{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu f_2)(x_2)\right|\\
&\leq\frac{\|f_1'\|_\infty}{\kappa\Gamma_\kappa(\tau)}\left[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau x_2\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right.\\
&\quad\left.-\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)}\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\nu(\varrho)\,d\varrho\right]\int_{x_1}^{x_2}f_2'(\vartheta)\,d\vartheta.
\end{aligned}
$$

Proof Applying Theorem 3.3 with $\mu(\vartheta)=1$, $\vartheta\in[x_1,x_2]$, we obtain the desired result. □

Corollary 3.8 *Let Ψ be a measurable increasing positive function on (x_1,x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1,x_2]$. Let $f_1:[x_1,x_2]\to\mathbb{R}$ be an absolutely continuous function with $(f_1')\in L^\infty[x_1,x_2]$, and let $f_2:[x_1,x_2]\to\mathbb{R}$ be nondecreasing. Suppose that $\mu:[x_1,x_2]\to\mathbb{R}^+$ is positive integrable. Then for all $\tau,\kappa>0$, we have*

$$
\begin{aligned}
&\left|{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\mu(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\nu f_1f_2)(x_2)+\frac{(\Psi(x_2)-\Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau+\kappa)}{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\mu f_1f_2)(x_2)\right.\\
&\quad\left.-{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\mu f_2)(x_2)-{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(\mu f_1)(x_2)\,{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau(f_2)(x_2)\right|\\
&\leq\frac{\|f_1'\|_\infty}{\kappa\Gamma_\kappa(\tau)}\left[{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau x_2\mu(x_2)\int_{x_1}^{\vartheta}\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right.\\
&\quad\left.-{}_{x_1}^{\Psi}\mathscr{I}_\kappa^\tau\mu(x_2)\int_{x_1}^{\vartheta}\varrho\big(\Psi(x_2)-\Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1}\Psi'(\varrho)\,d\varrho\right]\int_{x_1}^{x_2}f_2'(\vartheta)\,d\vartheta.
\end{aligned}
$$

Proof Applying Theorem 3.3 with $\nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, we obtain the desired result. □

Corollary 3.9 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1 : [x_1, x_2] \to \mathbb{R}$ be an absolutely continuous function with $(f_1') \in L^\infty[x_1, x_2]$, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. Then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\left| \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) \right| \\
&\quad \le \frac{\|f_1'\|_\infty}{2\kappa\Gamma_\kappa(\tau)} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right. \\
&\qquad \left. - \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right] \int_{x_1}^{x_2} f_2'(\vartheta)\, d\vartheta.
\end{aligned}$$

Proof Applying Theorem 3.3 with $\mu(\vartheta) = \nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, we obtain the desired result. □

Theorem 3.4 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. Suppose that $\mu, \nu : [x_1, x_2] \to \mathbb{R}^+$ are positive integrable. If $f_1', f_2' \in L^\infty[x_1, x_2]$, then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\qquad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\quad \le \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{\kappa\Gamma_\kappa(\tau)} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2^2 \mu(x_2) \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right. \\
&\qquad - 2\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \mu(x_2) \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \\
&\qquad \left. + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho^2 \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\nu(\varrho)\, d\varrho \right]. \qquad (18)
\end{aligned}$$

Proof Considering the left-hand side of (18), we have

$$\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\qquad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\quad \le \frac{1}{\kappa^2\Gamma_\kappa^2(\tau)} \left| \int_{x_1}^{x_2} \int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho) \right. \\
&\qquad \left. \times \big[\big(f_1(\xi) - f_1(\varrho)\big)\big(f_2(\xi) - f_2(\varrho)\big)\big]\, d\xi\, d\varrho \right| \\
&\quad \le \frac{1}{\kappa^2\Gamma_\kappa^2(\tau)} \int_{x_1}^{x_2} \int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho) \\
&\qquad \times \left| \frac{(f_1(\xi) - f_1(\varrho))}{\xi - \varrho} \right| \left| \frac{(f_2(\xi) - f_2(\varrho))}{\xi - \varrho} \right| (\xi - \varrho)^2\, d\xi\, d\varrho \\
&\quad \le \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{\kappa^2\Gamma_\kappa^2(\tau)} \int_{x_1}^{x_2} \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi)\mu(\xi)\Psi'(\varrho)\nu(\varrho)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\xi^2 - 2\xi\varrho + \varrho^2\right) d\xi\, d\varrho \\
&\leq \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{\kappa^2 \Gamma_\kappa^2(\tau)} \Bigg[\int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \xi^2 \Psi'(\xi) \mu(\xi)\, d\xi \\
&\quad \times \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \\
&\quad - 2 \int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \xi \Psi'(\xi) \mu(\xi)\, d\xi \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \\
&\quad + \int_{x_1}^{x_2} \big(\Psi(x_2) - \Psi(\xi)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\xi) \mu(\xi)\, d\xi \int_{x_1}^{\vartheta} \varrho^2 \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \Bigg].
\end{aligned}
$$

Hence by (9) we complete the proof. □

Corollary 3.10 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. Suppose that $\nu : [x_1, x_2] \to \mathbb{R}^+$ is positive integrable. If $f_1', f_2' \in L^\infty[x_1, x_2]$, then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2) \right| \\
&\leq \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{\kappa \Gamma_\kappa(\tau)} \Bigg[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2^2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \\
&\quad - 2 {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \\
&\quad + \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} \int_{x_1}^{\vartheta} \varrho^2 \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho) \nu(\varrho)\, d\varrho \Bigg].
\end{aligned}
$$

Proof Setting $\mu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, in Theorem 3.4, we obtain the desired result. □

Corollary 3.11 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. Suppose that $\mu : [x_1, x_2] \to \mathbb{R}^+$ is positive integrable. If $f_1', f_2' \in L^\infty[x_1, x_2]$, then for all $\tau, \kappa > 0$, we have*

$$
\begin{aligned}
&\left| {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) + \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \right. \\
&\quad \left. - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\, {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) \right| \\
&\leq \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{\kappa \Gamma_\kappa(\tau)} \Bigg[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2^2 \mu(x_2) \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \\
&\quad - 2 {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \mu(x_2) \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \\
&\quad + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho^2 \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \Bigg].
\end{aligned}
$$

Proof Setting $\nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, in Theorem 3.4, we obtain the desired result. □

Corollary 3.12 *Let Ψ be a measurable increasing positive function on (x_1, x_2) with continuous derivative $\Psi'(\varrho)$ on $[x_1, x_2]$. Let $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ be absolutely continuous functions, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be nondecreasing. If $f_1', f_2' \in L^\infty[x_1, x_2]$, then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\left| \frac{(\Psi(x_2) - \Psi(x_1))^{\frac{\tau}{\kappa}}}{\Gamma_\kappa(\tau + \kappa)} {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1 f_2)(x_2) - {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_1)(x_2)\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa}(f_2)(x_2) \right| \\
&\quad \le \frac{\|f_1'\|_\infty \|f_2'\|_\infty}{2\kappa\Gamma_\kappa(\tau)} \left[{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2^2 \int_{x_1}^{\vartheta} \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right. \\
&\qquad - 2\,{}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2 \int_{x_1}^{\vartheta} \varrho \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \\
&\qquad \left. + {}^{\Psi}_{x_1}\mathscr{I}^{\tau}_{\kappa} \mu(x_2) \int_{x_1}^{\vartheta} \varrho^2 \big(\Psi(x_2) - \Psi(\varrho)\big)^{\frac{\tau}{\kappa}-1} \Psi'(\varrho)\, d\varrho \right].
\end{aligned}$$

Proof Setting $\mu(\vartheta) = \nu(\vartheta) = 1$, $\vartheta \in [x_1, x_2]$, in Theorem 3.4, we obtain the desired result. □

4 Particular cases

Here we present some inequalities in terms of the Riemann–Liouville κ-fractional integrals, which are the particular cases of the main results.

Theorem 4.1 *Suppose that $\Phi : [x_1, x_2] \to \mathbb{R}$ is absolutely continuous on $[x_1, x_2]$ with $(\Phi')^2 \in L_1[x_1, x_2]$ and that $\mu, \nu : [x_1, x_2] \to \mathbb{R}^+$ are positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\big]\big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}\big(\nu\Phi^2\big)(x_2)\big] + \big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\big]\big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}\big(\mu\Phi^2\big)(x_2)\big] \\
&\qquad - 2\big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu\Phi)(x_2)\big]\big[{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu\Phi)(x_2)\big] \\
&\quad \le \frac{1}{\kappa\Gamma_\kappa(\tau)} \int_{x_1}^{x_2} \left[{}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2\mu(x_2) \int_{x_1}^{\vartheta} (x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right. \\
&\qquad \left. - {}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho(x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right] \big(\Phi'(\vartheta)\big)^2\, d\vartheta.
\end{aligned}$$

Proof Applying Theorem 3.1 with $\Psi(\vartheta) = \vartheta$ gives the proof of the theorem. □

Theorem 4.2 *Suppose that $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$ are absolutely continuous functions with $(f_1')^2 \in L_1[x_1, x_2]$ and $(f_2')^2 \in L_1[x_1, x_2]$ and that $\mu, \nu : [x_1, x_2] \to \mathbb{R}^+$ are positive integrable. Then for all $\tau, \kappa > 0$, we have*

$$\begin{aligned}
&\big|{}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2)\,{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1 f_2)(x_2) + {}_{x_1}\mathscr{I}^{\tau}_{\kappa}\nu(x_2)\,{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1 f_2)(x_2) \\
&\qquad - {}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_1)(x_2)\,{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_2)(x_2) - {}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\mu f_1)(x_2)\,{}_{x_1}\mathscr{I}^{\tau}_{\kappa}(\nu f_2)(x_2)\big| \\
&\quad \le \frac{1}{\kappa\Gamma_\kappa(\tau)} \left(\int_{x_1}^{x_2} \left[{}_{x_1}\mathscr{I}^{\tau}_{\kappa} x_2\mu(x_2) \int_{x_1}^{\vartheta} (x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right. \right. \\
&\qquad \left. \left. - {}_{x_1}\mathscr{I}^{\tau}_{\kappa}\mu(x_2) \int_{x_1}^{\vartheta} \varrho(x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right] \big(f_1'(\vartheta)\big)^2\, d\vartheta \right)^{\frac{1}{2}}
\end{aligned}$$

$$\times \left(\int_{x_1}^{x_2} \left[{}_{x_1}\mathscr{I}_{\kappa}^{\tau} x_2 \mu(x_2) \int_{x_1}^{\vartheta} (x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right.\right.$$

$$\left.\left. - {}_{x_1}\mathscr{I}_{\kappa}^{\tau} \mu(x_2) \int_{x_1}^{\vartheta} \varrho (x_2 - \varrho)^{\frac{\tau}{\kappa}-1} \nu(\varrho)\, d\varrho \right] \left(f_2'(\vartheta)\right)^2 d\vartheta \right)^{\frac{1}{2}}. \tag{19}$$

Proof Applying Theorem 3.2 with $\Psi(\vartheta) = \vartheta$ gives the proof of the theorem. □

Similarly, we can get several new inequalities in terms of the Riemann–Liouville κ-fractional integrals for $\Psi(\vartheta) = \vartheta$ in Theorems 3.3–3.4. Also, employing Corollaries 3.1–3.12 for $\Psi(\vartheta) = \vartheta$ results in various new inequalities.

Remark 4.1 We can also establish other types of new inequalities by taking the following assumptions:

i. Setting $\mu(\vartheta) = \nu(\vartheta)$ and $\Psi(\vartheta) = \vartheta$ throughout the paper.
ii. Setting $\mu(\vartheta) = \nu(\vartheta) = 1$ and $\Psi(\vartheta) = \vartheta$ throughout the paper.

Remark 4.2 If we take $\kappa = 1$, then all established results reduce to the work of Bezziou et al. [5].

Remark 4.3 Setting $\mu(\vartheta) = \nu(\vartheta)$, $\kappa = 1$, and $\Psi(\vartheta) = \vartheta$ in Theorems 3.1–3.4 restores the results of Bezziou et al. [4].

5 Concluding remarks

In this present paper, we derived some double-weighted generalized fractional integral inequalities by employing the generalized Riemann–Liouville κ-fractional integrals containing another function Ψ in the kernels, where Ψ is integrable, measurable, positive, and monotone. We can quickly form many new fractional integral inequalities for different fractional definitions by considering Remark 2.2.

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