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Oscillatory behavior of solutions of odd-order nonlinear delay differential equations

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Abstract

The objective of this study is to establish new sufficient criteria for oscillation of solutions of odd-order nonlinear delay differential equations. Based on creating comparison theorems that compare the odd-order equation with a couple of first-order equations, we improve and complement a number of related ones in the literature. To show the importance of our results, we provide an example.

Keywords: Odd-order; Delay differential equations; Oscillatory behavior

1 Introduction

In this study, we investigate the oscillatory behavior of solutions of the odd-order delay differential equation (DDE)

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}\left(\sigma(t)\right) = 0, \tag{1.1}$$

where $t \geq t_0$, $n \in \mathbb{Z}^+$ is odd, α is a ratio of odd positive integers, $r \in C^1([t_0,\infty),(0,\infty))$, $r'(t) \geq 0$, $\mu_{0,0}(t,t_0) := \int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s \to \infty$ as $t \to \infty$, $q \in C([t_0,\infty),[0,\infty))$, $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) < t$, and $\lim_{t\to\infty}\sigma(t) = \infty$.

Definition 1 Let $x \in C^{(n-1)}([t_x,\infty))$, $t_x \geq t_0$, and $r(x^{(n-1)})^{\alpha} \in C^1([t_x,\infty))$. The function x is called a *solution* of (1.1) on $[t_x,\infty)$ if x satisfies (1.1) for all t in $[t_x,\infty)$.

Definition 2 A nontrivial solution x of (1.1) is said to be *oscillatory* if there exists a sequence of zeros $\{t_n\}_{n=0}^{\infty}$ (i.e., $x(t_n) = 0$) of x such that $\lim_{n\to\infty} t_n = \infty$; otherwise, it is said to be *nonoscillatory*.

Although differential equations of even-order have been studied extensively, the study of qualitative behavior of odd-order differential equations has received considerably less attention in the literature, especially the third-order DDEs. However, certain results for third-order equations have been known for a long time and have some applications in

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mathematical modeling in biology and physics, see [17, 23, 25]. As a matter of fact, equation (1.1) under study is a so-called odd-order half-linear DDE, which has numerous applications in the research area of porous medium, see [13].

Different techniques have been used in studying the asymptotic behavior of DDEs. The articles [1, 3–9, 14–16, 27] were concerned with (in the canonical case and noncanonical case) the oscillation and asymptotic behavior of equation (1.1) and its special cases.

Based on creating comparison theorems that compare the odd-order DDEs with one or a couple of first-order DDEs, Agarwal et al. [1], Baculikova and Dzurina [3, 4] and Chatzarakis et al. [8] studied the oscillatory and asymptotic behavior of special cases of the third-order DDE

$$\left(a(t)\left(\left(b(t)x'(t)\right)'\right)^{\alpha}\right)' + q(t)f\left(x\left(\sigma(t)\right)\right) = 0,$$

where $a, b \in C^1([t_0,\infty),(0,\infty))$. By using the integral averaging technique, Bohner et al. [6] and Moaaz et al. [20] studied the asymptotic behavior of DDE with damping

$$\left(a(t)\left(b(t)\left(x'(t)\right)^{\alpha}\right)'\right)' + p(t)\left(x'(t)\right)^{\alpha} + q(t)f\left(x\left(\sigma(t)\right)\right) = 0,$$

where $\alpha \geq 1$ and $p \in C([t_0,\infty),[0,\infty))$. On the other hand, [5] used the Riccati transformation to study the asymptotic properties of the odd-order advanced equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}\left(g(t)\right) = 0,$$

where $g(t) > t$. The results concerned with the asymptotic properties and oscillation of the higher-order neutral DDEs were presented in [11, 18, 19, 21, 22, 26].

In this paper, by using an iterative method, we create sharper estimates for increasing and decreasing positive solutions of (1.1). Thus, we create sharper criteria for oscillation of (1.1). Moreover, iterative technique allows us to test the oscillation, even when the related results fail to apply. The results reported in this paper generalize, complement, and improve those in [7–9, 14–16, 27]. To show the importance of our results, we provide an example.

Remark 1.1 We restrict our discussion to those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \in [t_0,\infty)$.

Remark 1.2 All functional inequalities and properties, such as increasing, decreasing, positive, and so on, are assumed to hold eventually, that is, they are satisfied for all t large enough.

2 Main results

Lemma 2.1 ([2, Lemma 2.2.3]) *Let* $F \in C^n([t_0,\infty),(0,\infty))$, $F^{(n-1)}(t)F^{(n)}(t) \leq 0$ *for* $t \geq t_F$, *and* $\lim_{t\to\infty} F(t) \neq 0$. *Then, for every* $\delta \in (0,1)$, *there exists* $t_\delta \in [t_F,\infty)$ *such that*

$$F(t) \geq \frac{\delta}{(n-1)!}t^{n-1}\left|F^{(n-1)}(t)\right| \quad \text{for all } t \in [t_\delta,\infty).$$

Lemma 2.2 ([5, Lemma 2]) *If x is a positive solution of (1.1), then all derivatives $x^{(k)}(t)$, $1 \leq k \leq n-1$, are of constant signs, $r(t)(x^{(n-1)}(t))^{\alpha}$ is nonincreasing, and x satisfies either*

$$x'(t) > 0, \qquad x''(t) > 0, \qquad x^{(n-1)}(t) > 0, \qquad x^{(n)}(t) < 0 \tag{2.1}$$

or

$$(-1)^m x^{(m)} > 0, \quad m = 1, 2, \ldots, n. \tag{2.2}$$

Definition 3 The set of all positive solutions of (1.1) with property (2.1) or (2.2) is denoted by X_I^+ or X_D^+, respectively.

Lemma 2.3 *Assume that $x \in X_I^+$. Then*

$$x\big(\sigma(t)\big) \geq \eta_k\big(\sigma(t)\big)x^{(n-1)}\big(\sigma(t)\big), \tag{2.3}$$

where

$$\eta_0(t) := \frac{\delta_0}{(n-1)!}t^{n-1},$$

and

$$\eta_{k+1}(t) := \frac{\delta_k}{(n-2)!}r^{1/\alpha}(t)\int_{t_1}^{t} s^{n-2}\left(\frac{1}{r(s)}\exp\left(\int_s^t \frac{1}{r(u)}q(u)\eta_k^{\alpha}\big(\sigma(u)\big)\,\mathrm{d}u\right)\right)^{1/\alpha}\mathrm{d}s$$

for all $\delta_k \in (0,1)$ and $k = 0, 1, \ldots$.

Proof Let $x \in X_I^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Next, we will prove (2.3) using induction. For $k = 0$, using Lemma 2.1, we see that

$$x\big(\sigma(t)\big) \geq \frac{\delta_0}{(n-1)!}\sigma^{n-1}(t)x^{(n-1)}\big(\sigma(t)\big) \geq \eta_0\big(\sigma(t)\big)x^{(n-1)}\big(\sigma(t)\big).$$

Now, we assume that $x(\sigma(t)) \geq \eta_k(\sigma(t))x^{(n-1)}(\sigma(t))$ for $k > 0$. Since $x^{(n)} < 0$ and $\sigma(t) < t$, we have that

$$x\big(\sigma(t)\big) \geq \eta_k\big(\sigma(t)\big)x^{(n-1)}(t). \tag{2.4}$$

Then, from (1.1) and (2.4), we get

$$\Big(r(t)\big(x^{(n-1)}(t)\big)^{\alpha}\Big)' + q(t)\eta_k^{\alpha}\big(\sigma(t)\big)\big(x^{(n-1)}(t)\big)^{\alpha} \leq 0. \tag{2.5}$$

If we set $w := r(t)(x^{(n-1)}(t))^{\alpha}$, then (2.5) becomes

$$w'(t) \leq -\frac{1}{r(t)}q(t)\eta_k^{\alpha}\big(\sigma(t)\big)w(t).$$

Applying the Grönwall inequality, we find

$$w(s) \geq w(t) \exp\left(\int_s^t \frac{1}{r(u)} q(u) \eta_k^\alpha \big(\sigma(u)\big) \, \mathrm{d}u \right)$$

or

$$x^{(n-1)}(s) \geq r^{1/\alpha}(t) x^{(n-1)}(t) \left(\frac{1}{r(s)} \exp\left(\int_s^t \frac{1}{r(u)} q(u) \eta_k^\alpha \big(\sigma(u)\big) \, \mathrm{d}u \right) \right)^{1/\alpha}. \tag{2.6}$$

Using Lemma 2.1 with $F := x' > 0$, we see that

$$x'(t) \geq \frac{\delta_k t^{n-2}}{(n-2)!} x^{(n-1)}(t) \quad \text{for all } \delta_k \in (0,1).$$

By integrating this inequality from t_1 to t and taking into account (2.6), we see that

$$\begin{aligned} x(t) &\geq \frac{\delta_k}{(n-2)!} \int_{t_1}^t s^{n-2} x^{(n-1)}(s) \, \mathrm{d}s \\ &\geq x^{(n-1)}(t) \frac{\delta_k}{(n-2)!} r^{1/\alpha}(t) \int_{t_1}^t s^{n-2} \left(\frac{1}{r(s)} \exp\left(\int_s^t \frac{1}{r(u)} q(u) \eta_k^\alpha \big(\sigma(u)\big) \, \mathrm{d}u \right) \right)^{1/\alpha} \mathrm{d}s \\ &\geq \eta_{k+1}(t) x^{(n-1)}(t). \end{aligned}$$

Therefore, we have that

$$x\big(\sigma(t)\big) \geq \eta_{k+1}\big(\sigma(t)\big) x^{(n-1)}\big(\sigma(t)\big).$$

The proof is complete. □

Lemma 2.4 *Assume that $x \in X_D^+$. Then*

$$x(u) \geq r^{1/\alpha}(v) x^{(n-1)}(v) \mu_{l,n-2}(v,u), \tag{2.7}$$

where

$$\mu_{l,k+1}(v,u) := \int_u^v \mu_{l,k}(v,s) \, \mathrm{d}s$$

and

$$\mu_{l+1,0}(v,u) := \int_u^v \frac{1}{r^{1/\alpha}(s)} \exp\left(\int_s^v q(u) \mu_{l,n-2}^\alpha \big(u, \sigma(u)\big) \, \mathrm{d}u \right)^{1/\alpha} \mathrm{d}s$$

for $k = 0, 1, \ldots, n-3$, and $l = 0, 1, 2, \ldots$.

Proof Let $x \in X_D^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Next, we will prove (2.7) using induction. For $l = 0$, since $(r(z^{(n-1)}))' \leq 0$, we get that

$$-x^{(n-2)}(u) \geq x^{(n-2)}(v) - x^{(n-2)}(u) = \int_u^v \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s) x^{(n-1)}(s) \, \mathrm{d}s$$

$$\geq r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{0,0}(v,u). \tag{2.8}$$

Integrating (2.8) from u to v, we have

$$-x^{(n-3)}(u) \leq x^{(n-3)}(v) - x^{(n-3)}(u) = r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{0,1}(v,u). \tag{2.9}$$

Integrating (2.9) $n-3$ times from u to v, we get

$$x(u) \geq r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{0,n-2}(v,u).$$

Now, we assume that $x(u) \geq r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{l,n-2}(v,u)$ for $l>0$. Thus, we find

$$x\big(\sigma(t)\big) \geq r^{1/\alpha}(t)x^{(n-1)}(t)\mu_{l,n-2}\big(t,\sigma(t)\big),$$

which, with (1.1), gives

$$\big(r(t)\big(x^{(n-1)}(t)\big)^{\alpha}\big)' + q(t)r(t)\big(x^{(n-1)}(t)\big)^{\alpha}\mu^{\alpha}_{l,n-2}\big(t,\sigma(t)\big) \leq 0. \tag{2.10}$$

If we set $\psi := r(t)(x^{(n-1)}(t))^{\alpha}$, then (2.10) becomes

$$\psi'(t) \leq -q(t)\mu^{\alpha}_{l,n-2}\big(t,\sigma(t)\big)\psi(t).$$

Applying the Grönwall inequality, we find

$$\psi(s) \geq \psi(v)\exp\left(\int_s^v q(u)\mu^{\alpha}_{l,n-2}\big(u,\sigma(u)\big)\,\mathrm{d}u\right)$$

or

$$x^{(n-1)}(s) \geq r^{1/\alpha}(v)x^{(n-1)}(v)\left(\frac{1}{r(s)}\exp\left(\int_s^v q(u)\mu^{\alpha}_{l,n-2}\big(u,\sigma(u)\big)\,\mathrm{d}u\right)\right)^{1/\alpha}.$$

Thus, from (2.8), we see that

$$\begin{aligned}-x^{(n-2)}(u) &\geq r^{1/\alpha}(v)x^{(n-1)}(v)\int_u^v \frac{1}{r^{1/\alpha}(s)}\exp\left(\int_s^v q(u)\mu^{\alpha}_{l,n-2}\big(u,\sigma(u)\big)\,\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s\\ &\geq r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{l+1,0}(v,u).\end{aligned}$$

Integrating this inequality $n-2$ times from u to v, we get

$$x(u) \geq r^{1/\alpha}(v)x^{(n-1)}(v)\mu_{l+1,n-2}(v,u).$$

Thus, the proof is complete. □

Theorem 2.1 *Assume that x is a positive solution of (1.1) and η_k is defined as in Lemma 2.3. If the delay differential equation*

$$w'(t) + \frac{1}{r(\sigma(t))}q(t)\eta^{\alpha}_k\big(\sigma(t)\big)w\big(\sigma(t)\big) = 0 \tag{2.11}$$

is oscillatory for some $\delta_k \in (0,1)$ and some $k \in \mathbb{N}$, then X^+_I is empty.

Proof Assume to the contrary that $x \in X_I^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. From Lemma 2.3, we have that (2.3) holds. Combining (1.1) and (2.3), we obtain

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\eta_k^{\alpha}\left(\sigma(t)\right)\left(x^{(n-1)}\left(\sigma(t)\right)\right)^{\alpha} \leq 0. \tag{2.12}$$

If we set $w := r(x^{(n-1)})^{\alpha}$, then (2.12) becomes

$$w'(t) + \frac{1}{r(\sigma(t))}q(t)\eta_k^{\alpha}\left(\sigma(t)\right)w\left(\sigma(t)\right) \leq 0.$$

In view of [24, Theorem 1], we have that (2.11) also has a positive solution, a contradiction. Thus, the proof is complete. □

Corollary 2.1 *Assume that x is a positive solution of (1.1) and η_k is defined as in Lemma 2.3. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \frac{1}{r(\sigma(u))}q(u)\eta_k^{\alpha}\left(\sigma(u)\right)\mathrm{d}u > \frac{1}{\mathrm{e}} \tag{2.13}$$

for some $\delta_k \in (0,1)$ and some $k \in \mathbb{N}$, then X_I^+ is empty.

Proof In view of [12, Theorem 2], condition (2.13) guarantees that the delay equation (2.11) is oscillatory. □

Theorem 2.2 *Assume that x is a positive solution of (1.1), $\sigma'(t) > 0$, and $\mu_{l,k}$ is defined as in Lemma 2.4. If*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(u)\mu_{l,n-2}^{\alpha}\left(\sigma(t),\sigma(u)\right)\mathrm{d}u > 1 \tag{2.14}$$

for some $l \in \mathbb{N}$, then X_D^+ is empty.

Proof Assume to the contrary that $x \in X_D^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. From Lemma 2.4, we have that (2.7) holds. Integrating (1.1) from $\sigma(t)$ to t, we obtain

$$r\left(\sigma(t)\right)\left(x^{(n-1)}\left(\sigma(t)\right)\right)^{\alpha} - r(t)\left(x^{(n-1)}(t)\right)^{\alpha} = \int_{\sigma(t)}^{t} q(u)x^{\alpha}\left(\sigma(u)\right)\mathrm{d}u,$$

and so

$$r\left(\sigma(t)\right)\left(x^{(n-1)}\left(\sigma(t)\right)\right)^{\alpha} \geq \int_{\sigma(t)}^{t} q(u)x^{\alpha}\left(\sigma(u)\right)\mathrm{d}u. \tag{2.15}$$

Using (2.7) with $u = \sigma(u)$ and $v = \sigma(t)$, we get that

$$x\left(\sigma(u)\right) \geq r^{1/\alpha}\left(\sigma(t)\right)x^{(n-1)}\left(\sigma(t)\right)\mu_{l,n-2}\left(\sigma(t),\sigma(u)\right),$$

with (2.15), gives

$$\int_{\sigma(t)}^{t} q(u)\mu_{l,n-2}^{\alpha}\big(\sigma(t),\sigma(u)\big)\,\mathrm{d}u \leq 1,$$

which contradicts condition (2.14). This completes the proof. □

Theorem 2.3 *Assume that x is a positive solution of (1.1) and $\mu_{l,k}$ is defined as in Lemma 2.4. If there exists a function $\theta \in C([t_0,\infty),(0,\infty))$ satisfying $\theta(t) < t$ and $\sigma(t) < \theta(t)$ such that the delay differential equation*

$$\varphi'(t) + q(t)\mu_{l,n-2}^{\alpha}\big(\theta(t),\sigma(t)\big)\varphi\big(\theta(t)\big) = 0 \tag{2.16}$$

is oscillatory for some $l \in \mathbb{N}$, then X_D^+ is empty.

Proof Assume to the contrary that $x \in X_D^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. From Lemma 2.4, we have that (2.7) holds. Using (2.7) with $u = \sigma(t)$ and $v = \theta(t)$, we get that

$$x\big(\sigma(t)\big) \geq r^{1/\alpha}\big(\theta(t)\big)x^{(n-1)}\big(\theta(t)\big)\mu_{l,n-2}\big(\theta(t),\sigma(t)\big).$$

Thus, from (1.1), we obtain

$$\big(r(t)\big(x^{(n-1)}(t)\big)^{\alpha}\big)' + q(t)\mu_{l,n-2}^{\alpha}\big(\theta(t),\sigma(t)\big)r\big(\theta(t)\big)\big(x^{(n-1)}\big(\theta(t)\big)\big)^{\alpha} \leq 0. \tag{2.17}$$

If we set $\varphi := r(x^{(n-1)})^{\alpha}$, then (2.17) becomes

$$\varphi'(t) + q(t)\mu_{l,n-2}^{\alpha}\big(\theta(t),\sigma(t)\big)\varphi\big(\theta(t)\big) \leq 0.$$

In view of [24, Theorem 1], we have that (2.16) also has a positive solution, a contradiction. Thus, the proof is complete. □

Theorem 2.4 *Assume that x is a positive solution of (1.1), $(\sigma^{-1}(t))' > 0$ and $\mu_{l,k}$ is defined as in Lemma 2.4. If there exists a function $\vartheta \in C([t_0,\infty),(0,\infty))$ satisfying $\vartheta(t) > t$ and $\sigma(\vartheta(t)) < t$ such that the delay differential equation*

$$\varphi'(t) + \big(\sigma^{-1}(t)\big)'q\big(\sigma^{-1}(t)\big)\mu_{l,n-2}^{\alpha}\big(\vartheta(t),t\big)\varphi\big(\sigma\big(\vartheta(t)\big)\big) = 0 \tag{2.18}$$

is oscillatory for some $l \in \mathbb{N}$, then X_D^+ is empty.

Proof Assume to the contrary that $x \in X_D^+$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. From Lemma 2.4, we have that (2.7) holds. From (1.1), we get

$$\big(r\big(\sigma^{-1}(t)\big)\big(x^{(n-1)}\big(\sigma^{-1}(t)\big)\big)^{\alpha}\big)' + \big(\sigma^{-1}(t)\big)'q\big(\sigma^{-1}(t)\big)x^{\alpha}(t) = 0. \tag{2.19}$$

Using (2.7) with $u = t$ and $v = \vartheta(t)$, we have

$$x(t) \geq r^{1/\alpha}\big(\vartheta(t)\big)x^{(n-1)}\big(\vartheta(t)\big)\mu_{l,n-2}\big(\vartheta(t),t\big),$$

which with (2.19) gives

$$0 \geq \left(r\left(\sigma^{-1}(t)\right)\left(x^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^{\alpha}\right)' + \left(\sigma^{-1}(t)\right)' q\left(\sigma^{-1}(t)\right)\mu_{l,n-2}^{\alpha}\left(\vartheta(t),t\right) r\left(\vartheta(t)\right)\left(x^{(n-1)}\left(\vartheta(t)\right)\right)^{\alpha}. \tag{2.20}$$

If we set $\varphi(t) := r(x^{(n-1)})^{\alpha}(\sigma^{-1}(t))$, then (2.20) becomes

$$\varphi'(t) + \left(\sigma^{-1}(t)\right)' q\left(\sigma^{-1}(t)\right)\mu_{l,n-2}^{\alpha}\left(\vartheta(t),t\right)\varphi\left(\sigma\left(\vartheta(t)\right)\right) \leq 0.$$

In view of [24, Theorem 1], we have that (2.18) also has a positive solution, a contradiction. Thus, the proof is complete. □

Applying a well-known criterion [12, Theorem 2] for delay equations (2.16) and (2.18) to be oscillatory, we obtain the following two corollaries.

Corollary 2.2 *Assume that x is a positive solution of (1.1) and $\mu_{l,k}$ is defined as in Lemma 2.4. If there exists a function $\theta \in C([t_0,\infty),(0,\infty))$ satisfying $\theta(t) < t$ and $\sigma(t) < \theta(t)$ such that*

$$\liminf_{t\to\infty} \int_{\theta(t)}^{t} q(u)\mu_{l,n-2}^{\alpha}\left(\theta(u),\sigma(u)\right) \mathrm{d}u > \frac{1}{\mathrm{e}} \tag{2.21}$$

for some $l \in \mathbb{N}$, then X_D^+ is empty.

Corollary 2.3 *Assume that x is a positive solution of (1.1), $(\sigma^{-1}(t))' > 0$ and $\mu_{l,k}$ is defined as in Lemma 2.4. If there exists a function $\vartheta \in C([t_0,\infty),(0,\infty))$ satisfying $\vartheta(t) > t$ and $\sigma(\vartheta(t)) < t$ such that*

$$\liminf_{t\to\infty} \int_{\sigma(\vartheta(t))}^{t} \left(\sigma^{-1}(u)\right)' q\left(\sigma^{-1}(u)\right)\mu_{l,n-2}^{\alpha}\left(\vartheta(u),u\right) \mathrm{d}u > \frac{1}{\mathrm{e}} \tag{2.22}$$

for some $l \in \mathbb{N}$, then X_D^+ is empty.

Theorem 2.5 *Assume that η_k and $\mu_{l,k}$ are defined as in Lemmas 2.3 and 2.4, respectively. Then every solution of (1.1) is oscillatory if one of the following conditions is satisfied for some $\delta_k \in (0,1)$ and some $k,l \in \mathbb{N}$:*

(a) *There exists a function $\theta \in C([t_0,\infty),(0,\infty))$ satisfying $\theta(t) < t$ and $\sigma(t) < \theta(t)$ such that the delay differential equations (2.11) and (2.16) are oscillatory;*

(b) *There exists a function $\vartheta \in C([t_0,\infty),(0,\infty))$ satisfying $\vartheta(t) > t$, $(\sigma^{-1}(t))' > 0$ and $\sigma(\vartheta(t)) < t$ such that the delay differential equations (2.11) and (2.18) are oscillatory.*

Corollary 2.4 *Assume that η_k and $\mu_{l,k}$ are defined as in Lemmas 2.3 and 2.4, respectively. Then every solution of (1.1) is oscillatory if one of the following conditions is satisfied for some $\delta_k \in (0,1)$ and some $k,l \in \mathbb{N}$:*

(a) *Conditions (2.13) and (2.14) hold;*

(b) *Conditions (2.13) and (2.21) hold;*

(c) *Conditions (2.13) and (2.22) hold.*

Remark 2.1 The article [10] was concerned with the oscillation of equations (2.11), (2.16), and (2.18). Thus, one can obtain a number of oscillation criteria for (1.1) by using related results reported in [10].

Example 2.1 Consider the third-order differential equation

$$x''' + \frac{q_0}{t^3}x(\lambda t) = 0, \tag{2.23}$$

where $t \geq 1$, $q_0 > 0$, and $\lambda \in (0, 2/3)$. It is easy to verify that $\eta_0(t) := \frac{\delta_0}{2}\lambda^2 t^2$, $\mu_{0,0}(v,u) = v - u$, $\mu_{0,1}(v,u) = \frac{1}{2}(v-u)^2$,

$$\mu_{1,0}(v,u) = q_0\frac{(1-\lambda)^2}{2}v\ln\frac{v}{u}$$

and

$$\mu_{1,1}(v,u) = q_0\frac{(1-\lambda)^2}{2}v\left(v - u\left(1 + \ln\frac{v}{u}\right)\right).$$

Thus, by choosing $k = 0$, $l = 1$ and $\theta(t) := \frac{3}{2}\lambda t$, conditions (2.13) and (2.21) reduce to

$$q_0\lambda^2\ln\frac{1}{\lambda} > \frac{2}{\mathrm{e}} \tag{2.24}$$

and

$$q_0^2\frac{3}{4}\lambda^2(\lambda-1)^2\left(\frac{1}{2} - \ln\frac{3}{2}\right)\ln\frac{2}{3\lambda} > \frac{1}{\mathrm{e}}, \tag{2.25}$$

respectively. Using Corollary 2.4(b), we see that every solution of (2.23) is oscillatory if (2.24) and (2.25) hold.

Remark 2.2 Apparently, Corollary 2.4(a) and Theorem 2 in [8] are the same for $n = 3$. Consider a particular case of (2.23), namely $x''' + q_0 t^{-3}x(0.5t) = 0$. By using the results in Example 2.1, this equation is oscillatory if $q_0 > 16.988$.

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