Corcino and Corcino *Advances in Difference Equations* (2020) 2020:346
https://doi.org/10.1186/s13662-020-02802-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Fourier expansions for higher-order Apostol–Genocchi, Apostol–Bernoulli and Apostol–Euler polynomials

Cristina B. Corcino[1,2] and Roberto B. Corcino[1,2*]

*Correspondence:
rcorcino@yahoo.com
[1]Research Institute for Computational Mathematics and Physics (RICMP), Cebu Normal University, Cebu City, Philippines
[2]Mathematics Department, Cebu Normal University, Cebu City, Philippines

Abstract

Fourier expansions of higher-order Apostol–Genocchi and Apostol–Bernoulli polynomials are obtained using Laurent series and residues. The Fourier expansion of higher-order Apostol–Euler polynomials is obtained as a consequence.

Keywords: Fourier expansion; Bernoulli polynomials; Euler polynomials; Genocchi polynomials; Apostol–Genocchi polynomials; Apostol–Bernoulli polynomials; Apostol–Euler polynomials

1 Introduction

Higher-order Apostol–Genocchi, Apostol–Bernoulli, and Apostol–Euler polynomials are defined by the following relations, respectively (see [7]):

$$\sum_{n=0}^{\infty} G_n^m(z;\lambda)\frac{w^n}{n!} = \left(\frac{2w}{\lambda e^w + 1}\right)^m e^{wz}, \quad |w| < \pi \text{ when } \lambda = 1 \tag{1.1}$$

$$\text{and } |w| < \left|\log(-\lambda)\right| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C},$$

$$\sum_{n=0}^{\infty} B_n^m(z;\lambda)\frac{w^n}{n!} = \left(\frac{w}{\lambda e^w - 1}\right)^m e^{wz}, \quad |w| < \pi \text{ when } \lambda = 1 \tag{1.2}$$

$$\text{and } |w| < \left|\log(-\lambda)\right| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C},$$

$$\sum_{n=0}^{\infty} E_n^m(z;\lambda)\frac{w^n}{n!} = \left(\frac{2}{\lambda e^w + 1}\right)^m e^{wz}, \quad |w| < \pi \text{ when } \lambda = 1 \tag{1.3}$$

$$\text{and } |w| < \left|\log(-\lambda)\right| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C}.$$

When $m = 1$, the above equations give the generating functions for the Apostol–Genocchi, Apostol–Bernoulli, and Apostol–Euler polynomials, respectively (see [3]). When $m = 1$

and $\lambda = 1$, the equations give the generating functions for the classical Genocchi, Bernoulli, and Euler polynomials (see [4, 10]).

New formulas for the product of an arbitrary number of the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi polynomials were established in [6] where these polynomials were referred to as Apostol-type polynomials. Further, higher-order convolutions for these polynomials were established in [7]. New identities for the Apostol–Bernoulli polynomials and Apostol–Genocchi polynomials were also presented in [8].

Fourier expansion, being a sum of multiple of sines and cosines, is easily differentiated and integrated, which often simplifies analysis of functions such as saw waves which are common signals in experimentation [9]. Real world applications of Fourier series include the use for audio compression [5].

Fourier expansions of Genocchi polynomials and Apostol–Genocchi polynomials were obtained by Luo (see [11, 12]) using Lipschitz summation, while Bayad [3] obtained Fourier expansion for the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi polynomials using complex analysis theory of residues. Following Luo [12] and Bayad [3], the Fourier expansion of Apostol Frobenius–Euler polynomials was derived by Araci and Acikgoz [2]. Fourier series of periodic Genocchi functions and construction of good links between Genocchi functions and zeta function were also obtained in [1]. Fourier series of higher-order Bernoulli and Euler polynomials were used by López and Temme [10] to obtain asymptotic approximations of these polynomials. Using the method in [10], approximations for higher-order Genocchi polynomials were derived in [4].

In this paper, Fourier expansions for higher-order Apostol–Genocchi, Apostol–Bernoulli, and Apostol–Euler polynomials are derived as no Fourier expansions of these polynomials are available in the literature. The method of López and Temme [10] is used to derive the desired Fourier expansions. It is found out that the method using Lipschitz summation is not applicable to these higher-order polynomials. Moreover, it is shown that for $m = 1$ the Fourier series obtained reduce to those obtained in [3] and [12] . Exceptional values of the parameter λ are also considered.

2 Fourier expansions

In this section Fourier expansions for higher-order Apostol-type polynomials mentioned above are presented and proved.

Theorem 2.1 *For $\lambda \in \mathbb{C}$, $\lambda \neq 0, -1$, $0 < z < 1$, and $n \geq m$,*

$$G_n^m(z;\lambda) = \frac{2^m n e^{\pi i n}}{\lambda^z}\binom{n-1}{m-1} \times \sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(n-\nu-1)!B_\nu^m(z)\frac{e^{(2k+1)\pi i z}}{[\log\lambda-(2k+1)\pi i]^{n-\nu}}, \tag{2.1}$$

where $B_\nu^m(z) = B_\nu^m(z;1)$ denotes the Bernoulli polynomials of higher order defined in (1.2).

Proof Applying the Cauchy integral formula to (1.1),

$$\frac{G_n^m(z;\lambda)}{n!} = \frac{1}{2\pi i}\int_C\left(\frac{2w}{\lambda e^w+1}\right)^m e^{wz}\frac{dw}{w^{n+1}}, \tag{2.2}$$

where C is a circle about zero with radius $< |i\pi - \log\lambda|$. Let

$$f(w) = \left(\frac{2w}{\lambda e^w + 1}\right)^m \frac{e^{wz}}{w^{n+1}}. \tag{2.3}$$

Note that 0 is a pole of order $n - m + 1$, while the values w_k such that $\lambda e^{w_k} + 1 = 0$ are poles of order m. For $k \in \mathbb{Z}$,

$$w_k = -\log\lambda + (2k + 1)\pi i. \tag{2.4}$$

Let C_k be a circle about 0 with radius $< |w_k|$. Letting $k \to \infty$ and using the residue theorem,

$$\lim_{k\to\infty} \frac{1}{2\pi i} \int_{C_k} \left(\frac{2w}{\lambda e^w + 1}\right)^m \frac{e^{wz}}{w^{n+1}}\,dz = \operatorname{Res}\bigl(f(w), 0\bigr) + \sum_{k=-\infty}^{\infty} R_k, \tag{2.5}$$

where $R_k = \operatorname{Res}(f(w), w_k)$.

For $0 < z < 1$, the limit on the left-hand side of (2.5) is 0. For $k = 0$,

$$R_0 = \operatorname{Res}\bigl(f(w), 0\bigr) = \frac{1}{2\pi i} \int_C f(w)\,dw = \frac{G_n^m(z;\lambda)}{n!}.$$

Then (2.5) becomes

$$\begin{aligned} 0 &= \frac{G_n^m(z;\lambda)}{n!} + \sum_{k=-\infty}^{\infty} R_k \\ \Leftrightarrow \quad & G_n^m(z;\lambda) = -(n!) \sum_{k=-\infty}^{\infty} R_k. \end{aligned} \tag{2.6}$$

To compute the residues R_k, $k \geq 1$, the Laurent series of $f(w)$ about w_k will be used. Since w_k is a pole of order m, its Laurent series is

$$f(w) = \sum_{r=0}^{\infty} a_r (w - w_k)^r + \sum_{r=-1}^{-m} a_r (w - w_k)^r, \tag{2.7}$$

where $a_{-1} = \operatorname{Res}(f(w), w_k)$.

Multiplying both sides of (2.7) by $(w - w_k)^m$, we have

$$\begin{aligned} (w - w_k)^m f(w) = {} & \sum_{r=0}^{\infty} a_r (w - w_k)^{m+r} + a_{-1}(w - w_k)^{m-1} \\ & + a_{-2}(w - w_k)^{m-2} + \cdots + a_{-m}, \end{aligned}$$

where a_{-1} is now the coefficient of $(w - w_k)^{m-1}$. That is, $a_{-1} = a_{m-1}$ in the expansion

$$(w - w_k)^m f(w) = \sum_{r=0}^{\infty} a_r (w - w_k)^r. \tag{2.8}$$

Let

$$G_n^m(z;\lambda) = \frac{2(n!)}{\lambda^z} \sum_{k=-\infty}^{\infty} \beta_k^m(n,z) \frac{e^{(2k+1)\pi i z}}{[-\log\lambda + (2k+1)\pi i]^n}, \tag{2.9}$$

where $\beta_k^m(n,z)$ are to be determined. From [3] and [12],

$$G_n(z;\lambda) = \frac{2(n!)}{\lambda^z} \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)\pi i z}}{[-\log\lambda + (2k+1)\pi i]^n}, \tag{2.10}$$

it is seen that $\beta_k^1(n,z) = 1$, $\forall k$.

To find an explicit formula for $\beta_k^m(n,z)$, substitute $w_k = -\log\lambda + (2k+1)\pi i$ to (2.8) and use $f(z)$ in (2.3) to give

$$\begin{aligned} &\bigl(w - [-\log\lambda + (2k+1)\pi i]\bigr)^m \frac{2^m e^{wz}}{(\lambda e^w + 1)^m w^{n-m+1}} \\ &\quad = \sum_{r=0}^{\infty} a_r \bigl(w - [-\log\lambda + (2k+1)\pi i]\bigr)^r. \end{aligned} \tag{2.11}$$

Let $s = w - [-\log\lambda + (2k+1)\pi i]$. Then $w = s - \log\lambda + (2k+1)\pi i$ and (2.11) becomes

$$\frac{(-2)^m e^{(2k+1)\pi i z} e^{-z\log\lambda}}{[s - \log\lambda + (2k+1)\pi i]^{n-m+1}} \cdot \frac{s^m e^{zs}}{(e^s - 1)^m} = \sum_{r=0}^{\infty} a_r s^r. \tag{2.12}$$

Using (1.2) and writing

$$\bigl[s - \log\lambda + (2k+1)\pi i\bigr]^{m-n-1} = \sum_{\nu=0}^{\infty} \binom{m-n-1}{\nu} s^\nu \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{m-n-1-\nu},$$

the left-hand side of (2.12) becomes

$$\begin{aligned} &(-2)^m \lambda^{-z} e^{(2k+1)\pi i z} \left(\sum_{\nu=0}^{\infty} \binom{m-n-1}{\nu} s^\nu \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{m-n-1-\nu} \right) \\ &\quad \times \left(\sum_{\nu=0}^{\infty} \frac{B_\nu^m(z)}{\nu!} s^\nu \right). \end{aligned} \tag{2.13}$$

Applying Cauchy-product on (2.13) will yield

$$\begin{aligned} &\frac{(-2)^m \lambda^{-z} e^{(2k+1)\pi i z}}{[-\log\lambda + (2k+1)\pi i]^{n+1-m}} \sum_{r=0}^{\infty} \sum_{\nu=0}^{r} \binom{m-n-1}{r-\nu} \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{\nu-r} \frac{B_\nu^m(z)}{\nu!} s^r \\ &\quad = \sum_{r=0}^{\infty} a_r s^r. \end{aligned} \tag{2.14}$$

Thus,

$$a_r = \frac{(-2)^m e^{(2k+1)\pi iz}}{\lambda^z[-\log\lambda + (2k+1)\pi i]^{n+1-m}} \times \sum_{\nu=0}^{r} \binom{m-n-1}{r-\nu} \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{\nu-r} \frac{B_\nu^m(z)}{\nu!}. \tag{2.15}$$

In particular,

$$a_{m-1} = \frac{(-2)^m e^{(2k+1)\pi iz}}{\lambda^z[-\log\lambda + (2k+1)\pi i]^{n}} \sum_{\nu=0}^{m-1} \binom{m-n-1}{m-1-\nu} \frac{B_\nu^m(z)}{\nu!} \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{\nu}. \tag{2.16}$$

Comparing (2.6) and (2.9),

$$\beta_k^m(n,z) = \frac{\lambda^z[-\log\lambda + (2k+1)\pi i]^n}{-2e^{(2k+1)\pi iz}} a_{m-1}. \tag{2.17}$$

Substituting (2.16) to (2.17),

$$\beta_k^m(n,z) = (-2)^{m-1} \sum_{\nu=0}^{m-1} \binom{m-n-1}{m-1-\nu} \frac{B_\nu^m(z)}{\nu!} \bigl[-\log\lambda + (2k+1)\pi i\bigr]^{\nu}. \tag{2.18}$$

Using the identity

$$(-1)^{m-1+\nu} \binom{n-1}{m-1} \binom{m-1}{\nu} \frac{(n-\nu-1)!}{(n-1)!} = \frac{1}{\nu!} \binom{m-n-1}{m-1-\nu}, \tag{2.19}$$

(2.18) becomes

$$\beta_k^m(n,z) = 2^{m-1} \binom{n-1}{m-1} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \frac{(n-\nu-1)!}{(n-1)!} B_\nu^m(z) \bigl[\log\lambda - (2k+1)\pi i\bigr]^{\nu}. \tag{2.20}$$

Substituting to (2.9), the desired Fourier expansion for $G_n^m(z;\lambda)$ is obtained. □

Remark 2.2 When $m = 1$, (2.1) reduces to

$$G_n(z;\lambda) = \frac{2(n!)}{\lambda^z} \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)\pi it}}{[-\log\lambda + (2k+1)\pi i]^n},$$

which coincides with that of Luo [12] and Bayad [3].

Theorem 2.3 *For* $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1$, $0 < z < 1$, *and* $n \geq m$,

$$B_n^m(z;\lambda) = \frac{ne^{(n-m)\pi i}}{\lambda^z} \binom{n-1}{m-1} \times \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (n-\nu-1)! B_\nu^m(z) \frac{e^{2k\pi iz}}{[\log\lambda - 2k\pi i]^{n-\nu}}. \tag{2.21}$$

Proof The method used in proving Theorem 2.1 will be applied here. Applying the Cauchy integral formula to (1.2), we obtain

$$\frac{B_n^m(z;\lambda)}{n!} = \frac{1}{2\pi i}\int_C \left(\frac{w}{\lambda e^w - 1}\right)^m e^{wz}\frac{dw}{w^{n+1}}, \tag{2.22}$$

where C is a circle about zero with radius $< |\log\lambda|$.

Let

$$g(w) = \left(\frac{w}{\lambda e^w - 1}\right)^m \frac{e^{wz}}{w^{n+1}}. \tag{2.23}$$

Note that zero is a pole of order $n - m + 1$, while the values u_k such that $\lambda e^{u_k} - 1 = 0$ are poles of order m. For $k \in \mathbb{Z}$,

$$u_k = -\log\lambda + 2k\pi i. \tag{2.24}$$

Let C_k be a circle about 0 with radius $< |w_k|$. Letting $k \to \infty$ and using the residue theorem,

$$\lim_{k\to\infty}\frac{1}{2\pi i}\int_{C_k}\left(\frac{w}{\lambda e^w - 1}\right)^m \frac{e^{wz}}{w^{n+1}}\,dw = \operatorname{Res}\bigl(g(w), 0\bigr) + \sum_{k=-\infty}^{\infty} S_k, \tag{2.25}$$

where $S_k = \operatorname{Res}(g(w), u_k)$.

For $0 < z < 1$, the limit on the left-hand side of (2.25) is 0 and

$$Res\bigl(g(w), 0\bigr) = \frac{1}{2\pi i}\int_C g(w)\,dw = \frac{B_n^m(z;\lambda)}{n!}.$$

Then (2.25) becomes

$$0 = \frac{B_n^m(z;\lambda)}{n!} + \sum_{k=-\infty}^{\infty} S_k$$

$$\Leftrightarrow \quad B_n^m(z;\lambda) = -(n!)\sum_{k=-\infty}^{\infty} S_k. \tag{2.26}$$

To compute the residues S_k, use the Laurent series of $g(w)$ about u_k. Since u_k is a pole of order m, the Laurent series of $g(w)$ about u_k is

$$g(w) = \sum_{r=0}^{\infty} b_r(w - u_k)^r + \sum_{r=-1}^{-m} b_r(w - u_k)^r, \tag{2.27}$$

where $b_{-1} = \operatorname{Res}(g(w), u_k)$.

Multiplying both sides of (2.27) by $(w - u_k)^m$,

$$(w - u_k)^m g(w) = \sum_{r=0}^{\infty} b_r(w - u_k)^{m+r} + b_{-1}(w - u_k)^{m-1} + b_{-2}(w - u_k)^{m-2} + \cdots + b_{-m},$$

where b_{-1} is now the coefficient of $(w - u_k)^{m-1}$. That is, $b_{-1} = b_{m-1}$ in the expansion

$$(w - u_k)^m g(w) = \sum_{r=0}^{\infty} b_r(w - u_k)^r. \tag{2.28}$$

Let

$$B_n^m(z;\lambda) = \frac{n!}{\lambda^z}\sum_{k=-\infty}^{\infty}\gamma_k^m(n,z)\frac{e^{2k\pi iz}}{[2k\pi i - \log\lambda]^n}, \tag{2.29}$$

where $\gamma_k^m(n,z)$ are to be determined. From [3],

$$B_n(z;\lambda) = \frac{-(n!)}{\lambda^z}\sum_{k=-\infty}^{\infty}\frac{e^{2k\pi iz}}{[-\log\lambda + 2k\pi i]^n} \quad \text{for } \lambda \neq 1, \tag{2.30}$$

it is seen that $\gamma_k^1(n,z) = -1$, $\forall k$.

To find an explicit formula for $\gamma_k^m(n,z)$, substitute $u_k = -\log\lambda + 2k\pi i$ and the function $g(w)$ in (2.23) to (2.28) to obtain

$$\bigl(w - [-\log\lambda + 2k\pi i]\bigr)^m \frac{e^{wz}}{(\lambda e^w - 1)^m w^{n-m+1}} = \sum_{r=0}^{\infty} b_r\bigl(w - [-\log\lambda + 2k\pi i]\bigr)^r. \tag{2.31}$$

Let $t = w - [-\log\lambda + 2k\pi i]$. Then $w = t - \log\lambda + 2k\pi i$ and (2.31) becomes

$$\frac{\lambda^{-z}e^{2k\pi iz}}{[t - \log\lambda + 2k\pi i]^{n-m+1}}\left(\frac{t}{e^t - 1}\right)^m e^{tz} = \sum_{r=0}^{\infty} b_r t^r. \tag{2.32}$$

Using (1.2) and writing

$$[t - \log\lambda + 2k\pi i]^{m-n-1} = \sum_{\nu=0}^{\infty}\binom{m-n-1}{\nu} t^\nu(-\log\lambda + 2k\pi i)^{m-n-1-\nu},$$

the left-hand side of (2.32) becomes

$$\lambda^{-z}e^{2k\pi iz}\left(\sum_{\nu=0}^{\infty}\binom{m-n-1}{\nu}t^\nu(-\log\lambda + 2k\pi i)^{m-n-1-\nu}\right)\left(\sum_{\nu=0}^{\infty}\frac{B_\nu^m(z)}{\nu!}t^\nu\right). \tag{2.33}$$

Applying Cauchy-product on (2.33) will yield

$$\frac{\lambda^{-z}e^{2k\pi iz}}{[-\log\lambda + 2k\pi i]^{n-m+1}}\sum_{r=0}^{\infty}\left\{\sum_{\nu=0}^{r}\binom{m-n-1}{r-\nu}\frac{B_\nu^m(z)}{\nu!}(-\log\lambda + 2k\pi i)^{\nu-r}\right\}t^r$$
$$= \sum_{r=0}^{\infty} b_r t^r. \tag{2.34}$$

Thus,

$$b_r = \frac{e^{2k\pi iz}}{\lambda^z(-\log\lambda + 2k\pi i)^{n-m+1}}\sum_{\nu=0}^{r}\binom{m-n-1}{r-\nu}\frac{B_\nu^m(z)}{\nu!}(-\log\lambda + 2k\pi i)^{\nu-r}. \tag{2.35}$$

In particular,

$$b_{m-1} = \frac{e^{2k\pi iz}}{\lambda^z(-\log\lambda + 2k\pi i)^n}\sum_{\nu=0}^{m-1}\binom{m-n-1}{m-\nu-1}\frac{B_\nu^m(z)}{\nu!}(-\log\lambda + 2k\pi i)^\nu. \tag{2.36}$$

Comparing (2.26) and (2.29),

$$\gamma_k^m(n,z) = \frac{-\lambda^z(-\log\lambda + 2k\pi i)^n}{e^{2k\pi iz}} \cdot b_{m-1}. \tag{2.37}$$

Substituting (2.36) to (2.37),

$$\gamma_k^m(n,z) = -\sum_{\nu=0}^{m-1} \binom{m-n-1}{m-\nu-1} \frac{B_\nu^m(z)}{\nu!}(-\log\lambda + 2k\pi i)^\nu. \tag{2.38}$$

Using the identity in (2.19), we have

$$\gamma_k^m(n,z) = (-1)^m \binom{n-1}{m-1} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \frac{(n-\nu-1)!}{(n-1)!} B_\nu^m(z)(\log\lambda - 2k\pi i)^\nu. \tag{2.39}$$

Substituting (2.39) to (2.29), the desired Fourier expansion of $B_n^m(z;\lambda)$ is obtained. □

Remark 2.4 When $m = 1$, (2.21) reduces to

$$B_n(z;\lambda) = \frac{-(n)!}{\lambda^z} \sum_{k=-\infty}^{\infty} \frac{e^{2k\pi iz}}{[-\log\lambda + 2k\pi i]^n},$$

which coincides with that in [3].

Theorem 2.5 *For* $\lambda \in \mathbb{C}$, $\lambda \neq 0, -1$, $0 < z < 1$, *and* $n \geq m$,

$$\begin{aligned} E_n^m(z;\lambda) = {}& \frac{2^m e^{\pi i(n+m)}}{(m-1)!\lambda^z} \sum_{k=-\infty}^{\infty} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (n+m-\nu-1)! B_\nu^{n+m}(z) \\ & \times \frac{e^{(2k+1)\pi iz}}{[\log\lambda - (2k+1)\pi i]^{n+m-\nu}}. \end{aligned} \tag{2.40}$$

Proof Multiplying both sides of (1.3) by w^m yields

$$\left(\frac{2w}{\lambda e^w + 1}\right)^m e^{zw} = \sum_{n=0}^{\infty} E_n^m(z;\lambda) \frac{w^{n+m}}{n!}, \tag{2.41}$$

$$\sum_{n=0}^{\infty} G_n^m(z;\lambda) \frac{w^n}{n!} = \sum_{n=0}^{\infty} E_n^m(z;\lambda) \frac{w^{n+m}}{n!}. \tag{2.42}$$

The left hand-side of (2.42) can be written

$$\sum_{n=0}^{\infty} G_n^m(z;\lambda) \frac{w^n}{n!} = \sum_{n=-m}^{\infty} G_{n+m}^m(z;\lambda) \frac{w^{n+m}}{(n+m)!} \tag{2.43}$$

$$= \sum_{n=-m}^{\infty} G_{n+m}^m(z;\lambda) \frac{n!}{(n+m)!} \cdot \frac{w^{n+m}}{(n)!}. \tag{2.44}$$

Thus,

$$\sum_{n=0}^{\infty} E_n^m(z;\lambda)\frac{w^{n+m}}{n!} = \sum_{n=-m}^{\infty} G_{n+m}^m(z;\lambda)\frac{n!}{(n+m)!}\cdot\frac{w^{n+m}}{(n)!}. \tag{2.45}$$

Comparing coefficients in (2.45) gives

$$E_n^m(z;\lambda) = \frac{n!}{(n+m)!}G_{n+m}^m(z;\lambda). \tag{2.46}$$

Using (2.1),

$$\begin{aligned} E_n^m(z;\lambda) = \frac{n!}{(n+m)!}\Bigg\{ & \frac{2^m(n+m)e^{(n+m)\pi i}}{\lambda^z}\binom{n+m-1}{m-1}\sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(n+m-\nu-1)! \\ & \times B_\nu^{n+m}(z)\frac{e^{(2k+1)\pi iz}}{[\log\lambda-(2k+1)\pi i]^{n+m-\nu}}\Bigg\}. \end{aligned} \tag{2.47}$$

Simplifying

$$\frac{n!}{(n+m)!}(n+m)\binom{n+m-1}{m-1} = \frac{1}{(m-1)!},$$

and substituting to (2.47), the desired result is obtained. □

Remark 2.6 If $m = 1$, (2.40) reduces to

$$E_n(z;\lambda) = \frac{2(n!)}{\lambda^z}\sum_{k=-\infty}^{\infty}\frac{e^{(2k+1)\pi iz}}{[-\log\lambda+(2k+1)\pi i]^{n+1}},$$

which coincides with the corresponding result in [3].

3 The cases $\lambda = -1$ and $\lambda = 1$

Theorem 2.1 does not apply when $\lambda = -1$ because for $\lambda = -1$, $w_k = 0$, $\forall k$, while Theorem 2.3 does not apply for $\lambda = 1$ for similar reason. So these cases are considered here. Using (1.2),

$$\sum_{n=0}^{\infty} B_n^m(z;1)\frac{w^n}{n!} = \left(\frac{w}{e^w-1}\right)^m e^{wz}, \quad |w| < 2\pi. \tag{3.1}$$

On the other hand, using (1.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} G_n^m(z;-1)\frac{w^n}{n!} &= \left(\frac{2w}{-e^w+1}\right)^m e^{wz} \\ &= (-2)^m\left(\frac{w}{e^w-1}\right)^m e^{wz}, \quad |w| < 2\pi \\ &= (-2)^m\sum_{n=0}^{\infty} B_n^m(z;1)\frac{w^n}{n!}. \end{aligned}$$

Thus,

$$G_n^m(z;-1) = (-2)^m B_n^m(z;1). \tag{3.2}$$

Also, from (2.43),

$$\begin{aligned} E_n^m(z;-1) &= \frac{n!}{(n+m)!} G_{n+m}^m(z;-1) \\ &= \frac{n!}{(n+m)!}(-2)^m B_{n+m}^m(z;1). \end{aligned} \tag{3.3}$$

We proceed to finding the Fourier expansion for $B_n^m(z;1)$. The method in the previous section will be applied. First consider $m = 1$. The Fourier expansion for $B_n^1(z;1) = B_n(z;1)$ is given in the following lemma.

Lemma 3.1 *For* $0 < z < 1$ *and* $n \geq 1$,

$$B_n(z;1) = -(n!) \sum_{k=-\infty,k\neq 0}^{\infty} \frac{e^{2k\pi iz}}{(2k\pi i)^n}. \tag{3.4}$$

Proof By (1.2)

$$B_n(z;1) = B_n^1(z;1) = \frac{n!}{2\pi i} \int_C \frac{e^{wz}}{e^w - 1} \frac{dw}{w^n},$$

where C is a circle about the origin with radius $< 2\pi$. Let $f(w) = \frac{e^{wz}}{(e^w-1)w^n}$. Following the method in the previous section, we obtain

$$B_n(z;1) = -(n!) \sum_{k=-\infty,k\neq 0}^{\infty} R_k,$$

where $R_k = \operatorname{Res}(f(w), 2k\pi i)$, $k = \pm 1, \pm 2, \ldots$.

These residues can be computed to be

$$R_k = \frac{e^{2k\pi i(z-1)}}{(2k\pi i)^n}.$$

Thus,

$$B_n(z;1) = -(n!) \sum_{k=-\infty,k\neq 0}^{\infty} \frac{e^{2k\pi iz}}{(2k\pi i)^n}.$$

□

For $m > 1$, the Fourier series of B_n^m(z;1) is given in the following theorem.

Theorem 3.2 *For* $0 < z < 1$ *and* $n \geq m > 1$,

$$B_n^m(z;1) = (-1)^m n \binom{n-1}{m-1} \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (n-\nu-1)! B_\nu^m(z)(-1)^\nu \frac{e^{2k\pi iz}}{(2k\pi i)^{n-\nu}}. \tag{3.5}$$

Proof By the Cauchy integral formula,

$$\frac{B_n^m(z;1)}{n!} = \frac{1}{2\pi i}\int_C \frac{e^{wz}}{(e^w-1)^m w^{n-m+1}}\,dw, \quad |w| < 2\pi, \tag{3.6}$$

where C is a circle about the origin with radius $< 2\pi$.

The complex numbers $u_k = 2k\pi i$, $k = \pm1, \pm2, \dots$ are poles of order m of the function

$$h(w) = \frac{e^{wz}}{(e^w-1)^m w^{n-m+1}}. \tag{3.7}$$

Then

$$B_n^m(z;1) = -(n!)\sum_{k=-\infty,k\neq 0}^{\infty} R_k, \tag{3.8}$$

where $R_k = \operatorname{Res}(h(w), 2k\pi i)$, $k = \pm1, \pm2, \dots$.

Let

$$h(w) = \sum_{r=0}^{\infty} c_r(w-u_k)^r + \sum_{r=-1}^{-m} c_r(w-u_k)^r \tag{3.9}$$

be the Laurent series of $h(w)$, where

$$c_{-1} = \operatorname{Res}\bigl(h(w); u_k\bigr). \tag{3.10}$$

Multiplying both sides of (3.9) by $(w-u_k)^m$ gives

$$(w-u_k)^m h(w) = \sum_{r=0}^{\infty} c_r(w-u_k)^{m+r} + c_{-1}(w-u_k)^{m-1} + \cdots + c_{-m},$$

where c_{-1} is now the coefficient of $(w-u_k)^{m-1}$.

That is, $c_{-1} = c_{m-1}$ in the expansion

$$(w-u_k)^m h(w) = \sum_{r=0}^{\infty} c_r(w-u_k)^r. \tag{3.11}$$

Following (3.4), write

$$B_n^m(z;1) = -(n!)\sum_{k=-\infty,k\neq 0}^{\infty} \gamma_k^m(n,z;1)\frac{e^{2k\pi iz}}{(2k\pi i)^n}, \tag{3.12}$$

where $\gamma_k^m(n,z;1)$ are to be determined. Note that $\gamma_k^1(n,z;1) = 1$ (see (3.4)). From (3.11),

$$(w-2k\pi i)^m \frac{e^{wz}}{(e^w-1)^m e^{n-m+1}} = \sum_{r=0}^{\infty} c_r(w-2k\pi i)^r. \tag{3.13}$$

Let $t = w - 2k\pi i$. Then $w = t + 2k\pi i$ and (3.13) becomes

$$\frac{t^m}{(e^t-1)^m}e^t z \cdot \frac{e^{2k\pi iz}}{(t+2k\pi i)^{n-m+1}} = \sum_{r=0}^{\infty} c_r t^r. \tag{3.14}$$

Writing

$$(t+2k\pi i)^{m-n-1} = \sum_{\nu=0}^{\infty} \binom{m-n-1}{\nu} t^{\nu}(2k\pi i)^{m-n-1-\nu} \tag{3.15}$$

and using (3.1), (3.14) yields

$$\left(\sum_{n=0}^{\infty} B_n^m(z;1)\frac{t^n}{n!}\right)\left(\sum_{\nu=0}^{\infty}\binom{m-n-1}{\nu} t^{\nu}(2k\pi i)^{m-n-1-\nu}\right)e^{2k\pi iz} = \sum_{r=0}^{\infty} c_r t^r. \tag{3.16}$$

Applying Cauchy-product, (3.15) becomes

$$\frac{e^{2k\pi iz}}{(2k\pi i)^{n-m+1}}\sum_{r=0}^{\infty}\left\{\sum_{\nu=0}^{r}\binom{m-n-1}{r-\nu}\frac{B_{\nu}^m(z)}{\nu!}(2k\pi i)^{\nu-r}\right\}t^r = \sum_{r=0}^{\infty} c_r t^r. \tag{3.17}$$

Thus,

$$c_r = \frac{e^{2k\pi iz}}{(2k\pi i)^{n-m+1}}\sum_{\nu=0}^{r}\binom{m-n-1}{r-\nu}\frac{B_{\nu}^m(z)}{\nu!}(2k\pi i)^{\nu-r}. \tag{3.18}$$

In particular,

$$c_{m-1} = \frac{e^{2k\pi iz}}{(2k\pi i)^{n}}\sum_{\nu=0}^{m-1}\binom{m-n-1}{m-\nu-1}\frac{B_{\nu}^m(z)}{\nu!}(2k\pi i)^{\nu}. \tag{3.19}$$

Comparing (3.8) and (3.12),

$$\gamma_k^m(n,z;1) = \sum_{\nu=0}^{m-1}\binom{m-n-1}{m-\nu-1}\frac{B_{\nu}^m(z)}{\nu!}(2k\pi i)^{\nu}. \tag{3.20}$$

Applying (2.19),

$$\gamma_k^m(n,z;1) = (-1)^{m-1}\binom{n-1}{m-1}\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}\frac{(n-\nu-1)}{(n-1)!}B_{\nu}^m(z)(-2k\pi i)^{\nu}. \tag{3.21}$$

Substituting to (3.12), the theorem follows. □

Remark 3.3 When $m = 1$, the formula in Lemma 3.1 and Theorem 3.2 agrees with that obtained in [3].

Using (3.2) and (3.3) the following corollary is a direct consequence of Theorem 3.2.

Corollary 3.4 *For* $0 < z < 1$ *and* $n \geq m > 1$,

$$G_n^m(z;-1) = 2^m n \binom{n-1}{m-1} \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (n-\nu-1)! B_\nu^m(z)(-1)^\nu \frac{e^{2k\pi iz}}{(2k\pi i)^{n-\nu}},$$

$$E_n^m(z;-1) = \frac{2^m}{(m-1)!} \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (n+m-\nu-1)! B_\nu^n(z)(-1)^\nu \frac{e^{2k\pi iz}}{(2k\pi i)^{n+m-\nu}}.$$

4 Conclusion

It is seen that the Fourier expansions for higher-order Apostol–Genocchi, Apostol–Bernoulli, and Apostol–Euler polynomials are readily obtained using the method of Lopez and Temme [10]. Following [12] and [10] it will be interesting to consider the integral representations and asymptotic approximations of these polynomials for future study.

Acknowledgements
The authors would like to thank Cebu Normal University for the financial support to this research project.

Funding
This research project is partially funded by Cebu Normal University.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CC was the one who conceptualized the problem and the method to be used in solving the problem. She did the introduction and derived the Fourier expansion of higher-order Apostol–Genocchi and Apostol–Bernoulli polynomials. RC derived the Fourier expansion of higher-order Apostol–Euler polynomials, and he wrote Sect. 3. All authors read and approved the final manuscript.

Publisher's Note

Received: 12 March 2020 Accepted: 26 June 2020 Published online: 10 July 2020

References

1. Araci, S., Acikgoz, M.: Applications of Fourier series and zeta functions to Genocchi polynomials. Appl. Math. Inf. Sci. **12**(5), 951–955 (2018)
2. Araci, S., Acikgoz, M.: Construction of Fourier expansion of Apostol Frobenius–Euler polynomials and its application. Adv. Differ. Equ. **2018**, Article ID 67 (2018). https://doi.org/10.1186/s13662-018-1526-x
3. Bayad, A.: Fourier expansions for Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials. Math. Comput. **80**(276), 2219–2221 (2011). https://doi.org/10.1090/S0025-5718-2011-02476-2
4. Corcino, C., Corcino, R.: Asymptotics of Genocchi polynomials and higher order Genocchi polynomials using residues. Afr. Math. (2020). https://doi.org/10.1007/s13370-019-00759-z
5. Fixed Point (https://math.stackechange.com/users/30261/fixed-point): Real world application of Fourier series. Nov. 24, 2013. https://math.stackexchange.com/q/579695
6. He, Y., Araci, S., Srivastava, H.M.: Some new formulas for the products of the Apostol type polynomials. Adv. Differ. Equ. **2016**, Article ID 287 (2016). https://doi.org/10.1186/s13662-016-1014-0
7. He, Y., Araci, S., Srivastava, H.M., Abdel-Aty, M.: Higher-order convolutions for Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials. Mathematics **6**, Article ID 329 (2019). https://doi.org/10.3390/math6120329
8. He, Y., Araci, S., Srivastava, H.M., Acikgoz, M.: Some new identities for the Apostol–Bernoulli polynomials and the Apostol–Genocchi polynomials. Appl. Math. Comput. **262**, 31–41 (2015). https://doi.org/10.1016/j.amc.2015.03.132
9. Hollingsworth, M.: Applications of the Fourier series (2019)
10. López, J.L., Temme, N.M.: Large degree asymptotics of generalized Bernoulli and Euler polynomials. J. Math. Anal. Appl. **363**(1), 197–208 (2010). https://doi.org/10.1016/j.jmaa.2009.08.034
11. Luo, Q.-M.: Fourier expansions and integral representations for Genocchi polynomials. J. Integer Seq. **12**, Article ID 09.1.4 (2009)
12. Luo, Q.-M.: Extensions of the Genocchi polynomials and their Fourier expansions and integral representations. Osaka J. Math. **48**, 291–309 (2011)