Harisa et al. *Advances in Difference Equations* (2020) 2020:345
https://doi.org/10.1186/s13662-020-02800-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Shrinking Cesáro means method for the split equilibrium and fixed point problems in Hilbert spaces

Samy A. Harisa[1,3], M.A.A. Khan[2], F. Mumtaz[2], Nashat Faried[3], Ahmed Morsy[1], Kottakkaran Sooppy Nisar[1*] and Abdul Ghaffar[4,5]

*Correspondence:
n.sooppy@psau.edu.sa
[1]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, 11991, Wadi Aldawasir, Kingdom of Saudi Arabia
Full list of author information is available at the end of the article

Abstract

We propose a modified version of the classical Cesáro means method endowed with the hybrid shrinking projection method to solve the split equilibrium and fixed point problems (SEFPP) in Hilbert spaces. One of the main reasons to equip the classical Cesáro means method with the shrinking projection method is to establish strong convergence results which are often required in infinite-dimensional functional spaces. As a consequence, the convergence analysis is carried out under mild conditions on the underlying shrinking Cesáro means method. We emphasize that the results accounted in this manuscript can be considered as an improvement and generalization of various existing exciting results in this field of study.

MSC: Primary 47H05; 47H09; 47J05; secondary 49H05

Keywords: Cesáro means method; Shrinking projection method; Split equilibrium fixed point problem; Total asymptotically nonexpansive mapping

1 Introduction

Throughout the introduction, we first fix some necessary notions and concepts which will be required in the sequel. The inner product and the induced norm associated with a real Hilbert space H are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$, respectively. For a sequence $\{x_n\}_{n=1}^{\infty}$ in H, the strong convergence characteristics (resp. weak convergence characteristics) of $\{x_n\}_{n=1}^{\infty}$ is denoted as $x_n \to x$ (resp. $x_n \rightharpoonup x$). For a self-mapping T over a nonempty subset C of H, the set of all fixed points of the mapping T is denoted by $F(T)$. Recall that the self-mapping T is said to be total asymptotically nonexpansive mapping [1] if, for all $x, y \in C$, we have

$$\left\|T^n x - T^n y\right\| \leq \|x-y\| + \lambda_n \xi\left(\|x-y\|\right) + \mu_n \quad \text{for all } n \geq 1, \tag{1}$$

where $\{u_n\}$ and $\{v_n\}$ are nonnegative sequences satisfying $\lambda_n, \mu_n \overset{n\to\infty}{\longrightarrow} 0$ and $\xi : [0,\infty) \to [0,\infty)$ satisfying $\xi(0) = 0$ and $\xi(x) < \xi(y)$ for $x < y$.

Springer

It is remarked that the fixed points of certain nonlinear mappings can be constructed with the effective iterative algorithms by employing a suitable set of control conditions. On the other hand, computational investigation associated with the fixed points of generalized nonexpansive mappings is the only main tool to establish the consequent fixed point property in various framework of spaces. We remark that the convergence analysis of effective iterative algorithms associated with the class of total asymptotically nonexpansive mappings, as defined in (1), contributes significantly in metric fixed point theory. The introduction of the aforementioned class of mappings includes, inter alia, the generalization of nonlinear mappings as well as unification of different notions associated with the class of asymptotically nonexpansive mappings. Some useful results concerning the iterative construction of fixed points can be found in [18, 19, 22] and the references cited therein.

In 1975, Baillon [2] established the nonlinear version of a classical ergodic theorem involving a nonexpansive self-mapping T defined over a closed bounded convex subset C of a Hilbert space H. In fact, he proved that, for every $x \in C$, the Cesáro (-arithmetical) means method

$$\frac{x + Tx + \cdots + T^n x}{n+1}$$

exhibits weak convergence towards a fixed point of the mapping provided that $F(T)$ is nonempty. This nonlinear ergodic theorem was then generalized to Banach spaces in [4, 15, 26]. Moreover, Hirano and Takahashi [16] extended the Baillon's result to asymptotically nonexpansive mappings. Since then, the Cesáro means method has been employed for the construction of fixed points of (asymptotically) nonexpansive mappings; see, for instance, [27–29] and the references cited therein. We remark that the so-called ergodic average $\frac{1}{n+1}\sum_{i=0}^{n} T^i x$ converges weakly, whereas a strongly convergent iterative algorithm involving a nonlinear mapping is much more desirable than the weakly convergent iterative algorithm in infinite dimensional functional spaces. In 1967, Halpern [13] suggested the following strongly convergent iterative algorithm:

$$P_0^{\alpha} x = x, \qquad P_{n+1}^{\alpha} x = \alpha_{n+1} x + (1 - \alpha_{n+1}) T P_n^{\alpha} x,$$

where $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. Note that for a particularly important choice for T being positively homogeneous (i.e., $T(tx) = tTx$ for any $t \geq 0$ and $x \in C$) and $\{\alpha_n\}_{n=1}^{\infty}$ to be $\{\frac{1}{n+1}\}_{n=1}^{\infty}$, the so-called Halpern iterative algorithm is a nonlinear generalization of the Cesáro means method such that $P_n^{\alpha} x = \frac{1}{n+1} S_n x$, where $S_0 x := x$ and $S_{n+1} x := x + T(S_n x)$. In 1968, Haugazeau [14] proposed and analyzed a hybrid projection algorithm or outer-approximation methods in Hilbert spaces. This method has been modified in different ways to ensure the strong convergence characteristics of an iterative algorithm. In 2008, Takahashi et al. [32] firstly proposed a strongly convergent hybrid algorithm, based on the shrinking effect of the half-space, for nonexpansive mappings in Hilbert spaces. We are aiming to employ a modified version of the classical Cesáro means method endowed the hybrid shrinking projection method for the construction of common fixed points of a finite family of total asymptotically nonexpansive mappings.

In 2012, Censor et al. [8] coined the concept of split variational inequality problem (SVIP) in Hilbert spaces. The concept of SVIP was then generalized to split monotone variational inclusions (SMVI) in [25]. The concept of split equilibrium problem (SEP) is considered as a special case of SMVI which aims to solve a pair of equilibrium problems in such a way that the equilibrium points of an equilibrium problem solve another equilibrium problem under the action of a given bounded linear operator. The above stated theoretical problems have been successfully implemented to real world applications, for example, medical image reconstruction, see [6, 7]. Moreover, SEP can also be employed for problems in phase retrieval, data compression, sensor networks, inverse problems, and computerized tomography; see, for example, [5, 10]. We now introduce the concept of SEP:

Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions, where $\emptyset \neq C \subseteq H_1$ and $\emptyset \neq Q \subseteq H_2$, respectively. An SEP is as follows:

$$\text{find a point } x^* \in C \quad \text{which solves } F\bigl(x^*, x\bigr) \geq 0 \text{ for all } x \in C \tag{2}$$

and

$$\text{find the image } y^* = Ax^* \in Q \quad \text{which solves } G\bigl(y^*, y\bigr) \geq 0 \text{ for all } y \in Q, \tag{3}$$

where A is a bounded linear operator from H_1 onto H_2.

The set $\Omega = \{z \in \mathrm{EP}(F) : Az \in \mathrm{EP}(G)\}$ denotes the equilibrium points of SEP (2) and (3). Some important implications of SEP are as follows: If $H_1 = H_2$, $C = Q$ and $A := I$ (the identity mapping), then inequalities (2) and (3) coincide with the classical equilibrium problem whose solution set is denoted as $\mathrm{EP}(F)$. Moreover, if $F(x^*, x) = \langle f(x^*), x - x^*\rangle$ and $G(x^*, x) = \langle g(x^*), x - x^*\rangle$ in (2) and (3), then we have the concept of SVIP. As a consequence, the existence and approximation results for these problems can easily be derived from the ones established for SEFFP. Hence, it shows the significance and range of applicability of SEFFP. Some interesting methods have been proposed and analyzed to find a feasible solution of SEFPP associated with different classes of nonlinear mappings in the framework of Hilbert spaces [9, 12, 17, 20, 21, 23, 24, 30, 31, 33].

Inspired and motivated by the aforementioned results, we aim to establish convergence results of the shrinking Cesáro means method satisfying an appropriate set of conditions devised for the control sequences of parameters. As a consequence, the proposed algorithm strongly converges to an element in the set of solutions of SEFPP associated with a finite family of total asymptotically nonexpansive mappings. We emphasize that the results accounted in this manuscript can be considered as an improvement and generalization of various existing exciting results in this field of study.

The remainder of the manuscript is furnished in the following manner. In Sect. 2, we recall some definitions, mathematical tools, and important results in the form of lemmas required in the sequel. In Sect. 3, we establish results concerning the convergence characteristics of shrinking Cesáro means method in Hilbert spaces. Section 4 deals with the results deduced from the main results of Sect. 3.

2 Preliminaries

This section is devoted to recalling some useful definitions, entailing mathematical tools, and helpful results in the form of lemmas required in the sequel.

We assume that C is a nonempty closed convex subset of a Hilbert space H_1. For each $x \in H_1$, we can find a projection denoted by $P_C x$, which is the unique nearest point in C such that

$$\|x - P_C x\| := \inf\bigl\{\|x - y\| : \text{for all } y \in C\bigr\}.$$

Such a mapping P_C is known as the nearest point projector or the metric projection of H_1 onto C. Now, for all $x, y \in C$, we concluded that the metric projection P_C:

(i) satisfies nonexpansiveness in Hilbert spaces;
(ii) satisfies firm nonexpansiveness in Hilbert spaces, that is,

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle.$$

Moreover, if $P_C x \in C$, then for all $x \in H_1$ and for all $y \in C$, we have

$$\langle x - P_C x, P_C x - y\rangle \geq 0. \tag{4}$$

It is remarked that (4) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{5}$$

We now recall some important classes of monotone operators required in the sequel. A nonlinear mapping $A : C \to H_1$ is: (i) monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$ and (ii) λ-inverse strongly monotone if for $\lambda > 0$ we have that $\langle Ax - Ay, x - y\rangle \geq \lambda\|Ax - Ay\|^2$ for all $x, y \in C$. Note that a λ-inverse strongly monotone operator satisfies monotonicity defined in (i) as well as Lipschitz continuity with the Lipschitz constant being $(\frac{1}{\lambda})$. Moreover, the relation between the concepts of metric projection operator and variational inequality problem can be expressed as follows:

$$x^* \in VI(C,f) \quad \Longleftrightarrow \quad x^* = P_C\bigl(x^* - \lambda Ax^*\bigr) \quad \text{for all } \lambda > 0.$$

It is remarked that the linear operator defined as $A = I - T$ is $(\frac{1}{2})$-inverse strongly monotone provided that T satisfies nonexpansiveness.

The following lemma collects some useful equations and inequalities in the context of a real Hilbert space.

Lemma 2.1 *Let H_1 be a real Hilbert space, then*:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H_1$;
(ii) $2\langle y, x + y\rangle \geq \|x + y\|^2 - \|x\|^2$ *for all* $x, y \in H_1$;
(iii) $2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2$ *for all* $x, y, u, v \in H_1$;
(iv) $\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2$ *for all* $x, y \in H_1$ *and* $\lambda \in \mathbb{R}$.

An important tool in metric fixed point theory is the well-known Opial condition which is defined as follows: let $\{x_n\}$ be a sequence in a Hilbert space H_1 satisfying $x_n \overset{n\to\infty}{\rightharpoonup} x$, then the following inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in H_1$ with $x \neq y$.

Another important tool in metric fixed point theory is the demiclosedness principal which states that a mapping $T : H_1 \to H_1$ is demiclosed at the origin provided that a sequence $\{x_n\}$ in H_1 satisfies $x_n \overset{n\to\infty}{\rightharpoonup} x$ as well as $\|x_n - Tx_n\| \overset{n\to\infty}{\longrightarrow} 0$, we have $x = Tx$.

The class of equilibrium problems, involving a bifunction $F : C \times C \to \mathbb{R}$, can be solved assuming the following essential conditions (cf. [3] and [11]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$ implies that F is monotone;

(A3) for each $x, z \in C$, the following relation

$$\lim_{t\to 0} F\big(tz + (1-t)x, y\big) \leq F(x,y)$$

implies that the function $x \mapsto F(x,y)$ is upper hemicontinuous for all $y \in C$;

(A4) the function $y \mapsto F(x,y)$ is convex and lower semicontinuous for all $x \in C$.

Lemma 2.2 ([11]) *Let C be a closed convex subset of a real Hilbert space H_1, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)–(A4). *For $r > 0$ and $x \in H_1$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \textit{for all } y \in C.$$

Moreover, define a mapping $T_r^F : H_1 \to C$ by

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C \right\}$$

for all $x \in H_1$. Then the following hold:

(i) T_r^F *is single-valued*;

(ii) T_r^F *is firmly nonexpansive, i.e., for every $x, y \in H$,*

$$\big\| T_r^F x - T_r^F y\big\|^2 \leq \big\langle T_r^F x - T_r^F y, x - y\big\rangle;$$

(iii) $F(T_r^F) = \mathrm{EP}(F)$;

(iv) $\mathrm{EP}(F)$ *is closed and convex.*

It is remarked that if $G : Q \times Q \to \mathbb{R}$ is a bifunction satisfying conditions (A1)–(A4), then for $s > 0$ and $w \in H_2$ we can define a mapping

$$T_s^G(w) = \left\{ d \in C : G(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0 \text{ for all } e \in Q \right\},$$

which is nonempty, single-valued, and firmly nonexpansive. Moreover, $\mathrm{EP}(G)$ is closed and convex, and $F(T_s^G) = \mathrm{EP}(G)$.

3 Main results

We now prove our main result of this section.

Theorem 3.1 *Let C, Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $\{f_i\}_{i=1}^N : C \times C \to \mathbb{R}$ and $\{g_i\}_{i=1}^N : Q \times Q \to \mathbb{R}$ be two finite families* *of bifunctions satisfying conditions* (A1)–(A4) *such that each* g_i *is upper semicontinuous for each* $i \in \{1,2,3,\dots,N\}$. *Let* $\{A_i\}_{i=1}^N : H_1 \to H_2$ *be a finite family of bounded linear operators, and let* $\{S_i\}_{i=1}^N : C \to C$ *be a finite family of uniformly continuous total asymptotically nonexpansive mappings satisfying the condition that*

$$\lim_{n\to\infty} \sup_{x\in K} \left\| S_i^{n+1}x - S_i^n x \right\| = 0, \quad 1 \le i \le N, \tag{6}$$

for any bounded subset K *of* C. *Assume that the solution set* $\mathbb{F} := [\bigcap_{i=1}^N F(S_i)] \cap \Omega \neq \emptyset$, *where* $\Omega = \{z \in C : z \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ *and* $A_i z \in \bigcap_{i=1}^N \mathrm{EP}(g_i)$ *for* $1 \le i \le N\}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned}
&x_1 \in C_1 = C,\\
&u_n = T_{r_n}^{f_{n(\mathrm{mod}\,N)}}\left(x_n - \gamma A_{n(\mathrm{mod}\,N)}^*\left(I - T_{s_n}^{g_{n(\mathrm{mod}\,N)}}\right)A_{n(\mathrm{mod}\,N)}x_n\right),\\
&y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{N}\sum_{i=1}^N S_i^n u_n,\\
&C_{n+1} = \left\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\right\},\\
&x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1,
\end{aligned} \tag{7}$$

where $\theta_n = (1-\alpha_n)\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ *with* $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. *Let* $\{r_n\}$, $\{s_n\}$ *be two positive real sequences, and let* $\{\alpha_n\}$ *be in* $(0,1)$ *such that* $\alpha_n \le a$. *Assume that if the following set of conditions holds*:

- (C1) $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \dots, L_N\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *where* A_i^* *is the adjoint of* A_i *for each* $i \in \{1,2,3,\dots,N\}$;
- (C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;
- (C3) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $\sum_{n=1}^\infty \mu_n < \infty$;
- (C4) *there exist constants* $M_i, M_i^* > 0$ *such that* $\xi_i(\lambda_i) \le M_i^* \lambda_i$ *for all* $\lambda_i \ge M_i$, $i = 1,2,3,\dots,N$, *then the sequence* $\{x_n\}$ *generated by* (7) *converges strongly to* $P_{\mathbb{F}}x_1$.

Proof For the sake of simplicity, we divide the proof into five steps.
Step 1. The sequence $\{x_n\}$ is well defined.

Proof of Step 1. We first show by mathematical induction that $\mathbb{F} \subset C_n$ for all $n \ge 1$. It is obvious from the assumption that $\mathbb{F} \subset C_1 = C$. Let $\mathbb{F} \subset C_k$ for some $k \ge 1$. We show that $\mathbb{F} \subset C_{k+1}$ for some $k \ge 1$. It follows from (7) that

$$\begin{aligned}
\|u_k - p\|^2 &= \left\| T_{r_k}^{f_{k(\mathrm{mod}\,N)}}\left(x_k - \gamma A_{k(\mathrm{mod}\,N)}^*\left(I - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}\right)A_{k(\mathrm{mod}\,N)}x_k\right) - T_{r_k}^{f_{k(\mathrm{mod}\,N)}}p\right\|^2\\
&\le \left\| x_k - \gamma A_{k(\mathrm{mod}\,N)}^*\left(I - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}\right)A_{k(\mathrm{mod}\,N)}x_k - p\right\|^2\\
&\le \|x_k - p\|^2 + \gamma^2 \left\| A_{k(\mathrm{mod}\,N)}^*\left(I - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}\right)A_{k(\mathrm{mod}\,N)}x_k\right\|^2\\
&\quad + 2\gamma\left\langle p - x_k, A_{k(\mathrm{mod}\,N)}^*\left(I - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}\right)A_{k(\mathrm{mod}\,N)}x_k\right\rangle\\
&\le \|x_k - p\|^2 + \gamma^2\left\langle A_{k(\mathrm{mod}\,N)}x_k - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}A_{k(\mathrm{mod}\,N)}x_k,\right.\\
&\quad \left. A_{k(\mathrm{mod}\,N)}A_{k(\mathrm{mod}\,N)}^*\left(I - T_{s_k}^{g_{k(\mathrm{mod}\,N)}}\right)A_{k(\mathrm{mod}\,N)}x_k\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\gamma\big\langle p - x_k, A^*_{k(\operatorname{mod} N)}\big(I - T_{s_k}^{g_{k(\operatorname{mod} N)}}\big)A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&\le \|x_k - p\|^2 \\
&\quad + L\gamma^2\big\langle A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k, A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&\quad + 2\gamma\big\langle p - x_k, A^*_{k(\operatorname{mod} N)}\big(I - T_{s_k}^{g_{k(\operatorname{mod} N)}}\big)A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&= \|x_k - p\|^2 + L\gamma^2\big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2 \\
&\quad + 2\gamma\big\langle p - x_k, A^*_{k(\operatorname{mod} N)}\big(I - T_{s_k}^{g_{k(\operatorname{mod} N)}}\big)A_{k(\operatorname{mod} N)}x_k\big\rangle. \qquad (8)
\end{aligned}
$$

Denote $\varLambda = 2\gamma\langle p - x_k, A^*_{k(\operatorname{mod} N)}(I - T_{s_k}^{g_{k(\operatorname{mod} N)}})A_{k(\operatorname{mod} N)}x_k\rangle$, we have

$$
\begin{aligned}
\varLambda &= 2\gamma\big\langle p - x_k, A^*_{k(\operatorname{mod} N)}\big(I - T_{s_k}^{g_{k(\operatorname{mod} N)}}\big)A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&= 2\gamma\big\langle A_{k(\operatorname{mod} N)}(p - x_k), A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&= 2\gamma\big\langle A_{k(\operatorname{mod} N)}(p - x_k) + \big(A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big) \\
&\quad - \big(A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big), A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&= 2\gamma\big\{\big\langle A_{k(\operatorname{mod} N)}p - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k, A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\rangle \\
&\quad - \big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2\big\} \\
&\le 2\gamma\bigg\{\frac{1}{2}\big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2 \\
&\quad - \big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2\bigg\} \\
&= -\gamma\big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2.
\end{aligned}
$$

Substituting the above simplified value of $\varLambda$ in (8), we have

$$
\|u_k - p\|^2 \le \|x_k - p\|^2 + \gamma(L\gamma - 1)\big\|A_{k(\operatorname{mod} N)}x_k - T_{s_k}^{g_{k(\operatorname{mod} N)}}A_{k(\operatorname{mod} N)}x_k\big\|^2. \qquad (9)
$$

From the definition of γ and condition (C1), we obtain

$$
\|u_k - p\|^2 \le \|x_k - p\|^2. \qquad (10)
$$

Let $S^k = \frac{1}{N}\sum_{i=1}^N S_i^k$, it then follows that

$$
\begin{aligned}
\big\|S^k x - S^k y\big\| &= \Bigg\|\frac{1}{N}\sum_{i=1}^N S_i^k x - \frac{1}{N}\sum_{i=1}^N S_i^k y\Bigg\| \\
&\le \frac{1}{N}\sum_{i=1}^N\big(\|x - y\| + \lambda_k\xi_k\big(\|x - y\|\big) + \mu_k\big) \\
&\le \|x - y\| + \lambda_k\xi_k\big(\|x - y\|\big) + \mu_k \quad \text{for all } x, y \in C. \qquad (11)
\end{aligned}
$$

Now, for any $p \in \mathbb{F}$, we have $S^k p = \frac{1}{N}\sum_{i=1}^{N} S_i^k p = p$. It follows from (8) and (11) that

$$\begin{aligned}
\|y_k - p\| &= \left\|\alpha_k u_k + (1-\alpha_k)S^k u_k - p\right\| \\
&= \alpha_k\|u_k - p\| + (1-\alpha_k)\left\|S^k u_k - p\right\| \\
&\leq \alpha_k\|u_k - p\| + (1-\alpha_k)\left\{\|u_k - p\| + \lambda_k\xi_k\left(\|u_k - p\|\right) + \mu_k\right\} \\
&\leq \|u_k - p\| + (1-\alpha_k)\left\{\lambda_k\xi_k(M_k) + \lambda_k M_k^*\|u_k - p\|) + \mu_k\right\} \\
&\leq \|x_k - p\| + (1-\alpha_k)\left\{\lambda_k\xi_k(M_k) + \lambda_k M_k^*\|x_k - p\|^2 + \mu_k\right\} \\
&\leq \|x_k - p\| + \theta_k, \qquad (12)
\end{aligned}$$

where $\theta_k = (1-\alpha_k)[(\lambda_k\xi_k(M_k) + \lambda_k D_k M_k^* + \mu_k)]$ with $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. Estimate (12) implies that $p \in C_{k+1}$ and hence $\mathbb{F} \subset C_n$ for all $n \geq 1$. Since

$$\left\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\right\} = \left\{z \in C : \|y_n\|^2 - \|x_n\|^2 \leq 2\langle y_n - x_n, z\rangle + \theta_n\right\},$$

it is closed and convex; therefore the sequence $\{x_n\}$ is well defined.

Step 2. The sequence $\{\|x_n - x_1\|\}$ is Cauchy.

Proof of Step 2. Note that $x_n = P_{C_n}x_1$, therefore we have

$$0 \leq \left\langle x_n - x_1, x^* - x_n\right\rangle \quad \text{for each } x^* \in C_n.$$

In particular,

$$0 \leq \langle x_n - x_1, p - x_n\rangle \quad \text{for each } p \in \mathbb{F}.$$

This further implies that

$$\begin{aligned}
0 &\leq \langle x_n - x_1, p - x_n\rangle \\
&= \langle x_n - x_1, p + x_1 - x_1 - x_n\rangle \\
&= \langle x_n - x_1, x_1 - x_n\rangle + \langle x_n - x_1, p - x_1\rangle \\
&= -\|x_n - x_1\|^2 + \|x_n - x_1\|\|p - x_1\|.
\end{aligned}$$

That is,

$$\|x_n - x_1\| \leq \|p - x_1\| \quad \text{for all } p \in \mathbb{F} \text{ and } n \geq 1.$$

Moreover, from $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$0 \leq \langle x_n - x_1, x_{n+1} - x_n\rangle$$

and

$$\begin{aligned}
0 &\leq \langle x_n - x_1, x_{n+1} - x_n\rangle \\
&= \langle x_n - x_1, x_{n+1} + x_1 - x_1 - x_n\rangle
\end{aligned}$$

$$= \langle x_n - x_1, x_1 - x_n \rangle + \langle x_n - x_1, x_{n+1} - x_1 \rangle$$
$$= -\|x_n - x_1\|^2 + \|x_n - x_1\| \|x_{n+1} - x_1\|.$$

This implies that

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\| \quad \text{for all } n \ge 1.$$

Hence, the sequence $\{\|x_n - x_1\|\}$ is bounded and nondecreasing; therefore we have

$$\lim_{n\to\infty} \|x_n - x_1\| \quad \text{exists.} \tag{13}$$

Note that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1 \rangle \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n + x_n - x_1 \rangle \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n \rangle \\ &\le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned}$$

It now follows from estimate (13) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{14}$$

Step 3. Show that:

(i) $\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - x_n\| = 0$,
(ii) $\lim_{n\to\infty} \|u_n - x_n\| = 0$,
(iii) $\lim_{n\to\infty} \|y_n - u_n\| = 0$,
(iv) $\lim_{n\to\infty} \|S_n u_n - u_n\| = 0$.

Proof of Step 3. By $x_{n+1} \in C_{n+1}$, we have $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \theta_n$. Using (14), we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0 \quad \text{for all } n \ge 1. \tag{15}$$

Since $\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, therefore using (14)–(15) we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0 \quad \text{for all } n \ge 1 \tag{16}$$

as $n \to \infty$.

Altogether, we deduce from (7), (9), and (12) that

$$\begin{aligned} &\gamma(1-\gamma L)\big\|A_{n(\mathrm{mod}\,N)}x_n - T_{s_n}^{g_{n(\mathrm{mod}\,N)}} A_{n(\mathrm{mod}\,N)}x_n\big\|^2 \\ &\quad \le \|x_n - p\|^2 - \|u_n - p\|^2 \\ &\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n \\ &\quad \le \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\| + \theta_n. \end{aligned}$$

From $\gamma(1-L\gamma)>0$ and (16), we obtain

$$\lim_{n\to\infty}\bigl\|A_{n(\operatorname{mod}N)}x_n-T_{s_n}^{g_{n(\operatorname{mod}N)}}A_{n(\operatorname{mod}N)}x_n\bigr\|^2=0\quad\text{for all } n\geq 1. \tag{17}$$

Next, we show that $\|u_n-x_n\|\to 0$ as $n\to\infty$. Since $p\in\mathbb{F}$, we have

$$\begin{aligned}
\|u_n-p\|^2 &= \bigl\|T_{r_n}^{f_{n(\operatorname{mod}N)}}\bigl(x_n-\gamma A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n\bigr)-T_{r_n}^{f_{n(\operatorname{mod}N)}}p\bigr\|^2\\
&\leq \bigl\langle u_n-p,x_n-\gamma A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n-p\bigr\rangle\\
&= \frac{1}{2}\Bigl\{\|u_n-p\|^2+\bigl\|x_n-\gamma A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n-p\bigr\|^2\\
&\quad-\bigl\|u_n-x_n+\gamma A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n\bigr\|^2\Bigr\}\\
&\leq \frac{1}{2}\Bigl\{\|u_n-p\|^2+\|x_n-p\|^2\\
&\quad-\bigl\|u_n-x_n+\gamma A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n\bigr\|^2\Bigr\}\\
&= \frac{1}{2}\Bigl\{\|u_n-p\|^2+\|x_n-p\|^2\\
&\quad-\Bigl(\|u_n-x_n\|^2+\gamma^2\bigl\|A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n\bigr\|^2\\
&\quad-2\gamma\bigl\langle u_n-x_n,A^*_{n(\operatorname{mod}N)}\bigl(I-T_{s_n}^{g_{n(\operatorname{mod}N)}}\bigr)A_{n(\operatorname{mod}N)}x_n\bigr\rangle\Bigr)\Bigr\}\\
&\leq \|x_n-p\|^2-\|u_n-x_n\|^2\\
&\quad+2\gamma\|u_n-x_n\|\bigl\|A_{n(\operatorname{mod}N)}x_n-T_{s_n}^{g_{n(\operatorname{mod}N)}}A_{n(\operatorname{mod}N)}x_n\bigr\|.
\end{aligned} \tag{18}$$

Consider the following variant of (12) together with (10):

$$\|y_n-p\|^2\leq\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2+\theta_n. \tag{19}$$

Altogether, it follows from (18) and (19) that

$$\begin{aligned}
(1-\alpha_n)\|u_n-x_n\|^2 &\leq \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|\\
&\quad+2\gamma\|u_n-x_n\|\bigl\|A_{n(\operatorname{mod}N)}x_n-T_{s_n}^{g_{n(\operatorname{mod}N)}}A_{n(\operatorname{mod}N)}x_n\bigr\|+\theta_n.
\end{aligned}$$

From (16) and (17), the above estimate implies that

$$\lim_{n\to\infty}\|u_n-x_n\|=0\quad\text{for all } n\geq 1. \tag{20}$$

Utilizing (16) and (20), we get

$$\begin{aligned}
\|y_n-u_n\| &\leq \|y_n-x_n\|+\|x_n-u_n\|\\
&\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{21}$$

Since $\|y_n-u_n\|=(1-\alpha_n)\|S^nu_n-u_n\|$ and $\alpha_n\leq a<1$, then from (21) it follows that

$$\lim_{n\to\infty}\bigl\|S^nu_n-u_n\bigr\|=0\quad\text{for all } n\geq 1. \tag{22}$$

From (20) and (22), we obtain

$$\begin{aligned} \|S^n u_n - x_n\| &\le \|S^n u_n - u_n\| + \|x_n - u_n\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{23}$$

Reasoning as above, we obtain

$$\lim_{n\to\infty} \|S^n u_n - y_n\| = 0 \quad \text{for all } n \ge 1. \tag{24}$$

It is evident from (20) and (23) that the following estimate implies that

$$\begin{aligned} \|S^n x_n - x_n\| &\le \|S^n x_n - S^n u_n\| + \|S^n u_n - x_n\| \\ &\le (1 + \lambda_n M_n^*)\|x_n - u_n\| + \lambda_n \xi_n(M_n) + \mu_n + \|S^n u_n - x_n\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{25}$$

Note that $\frac{1}{N}\|S_i^n u_n - u_n\|^2 \le \frac{1}{N}\sum_{i=1}^N \|S_i^n u_n - u_n\|^2$, therefore using (22) we have

$$\lim_{n\to\infty} \|S_i^n u_n - u_n\| = 0 \quad \text{for each } i = 1, 2, \dots, N.$$

Similarly, we also have that

$$\lim_{n\to\infty} \|S_i^n x_n - x_n\| = 0 \quad \text{for each } i = 1, 2, \dots, N. \tag{26}$$

Moreover, utilizing the uniform continuity of S_i and (26), we get

$$\begin{aligned} \|x_n - S_i x_n\| &\le \|x_n - S_i^n x_n\| + \|S_i^n x_n - S_i x_n\| \\ &\to 0 \quad \text{as } n \to \infty \text{ for each } i = 1, 2, \dots, N. \end{aligned}$$

Similarly, we also have that

$$\lim_{n\to\infty} \|u_n - S_i u_n\| = 0 \quad \text{for each } i = 1, 2, \dots, N.$$

Now, we show that $\omega(x_n) \subset \mathbb{F}$, where $\omega(x_n)$ is the set of all weak ω-limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, therefore $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, then there exists a subsequence $\{x_{Nn+i}\}$ of $\{x_n\}$ such that $x_{Nn+i} \rightharpoonup q$. Using the fact that $S_{Nn+i} = S_i$ for all $n \ge 1$ and the demiclosedness principle for each S_i, we have that $x \in F(S_i)$ for each $1 \le i \le N$. Next, we show that $q \in \Omega$, i.e., $q \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ and $A_i q \in \mathrm{EP}(g_i)$ for each $1 \le i \le N$. In order to show that $q \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$, that is, $q \in \mathrm{EP}(f_i)$ for each $1 \le i \le N$, we define a subsequence $\{n_j\}$ of index $\{n\}$ such that $n_j = Nj + i$ for all $n \ge 1$. As a consequence, we can write $f_{n_j} = f_i$ for $1 \le i \le N$.

From $u_{n_j} = T_{r_{n_j}}^{f_{i(\mathrm{mod}\,N)}}(I - \gamma A^*_{n_{j(\mathrm{mod}\,N)}}(I - T_{s_{n_j}}^{g_{n_j(\mathrm{mod}\,N)}})A_{n_{j(\mathrm{mod}\,N)}})x_{n_j}$, for all $n \ge 1$, we have

$$\begin{aligned} &f_{i(\mathrm{mod}\,N)}(u_{n_j}, y) \\ &\quad + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j} - \gamma A^*_{n_{j(\mathrm{mod}\,N)}}(I - T_{s_{n_j}}^{g_{n_j(\mathrm{mod}\,N)}})A_{n_{j(\mathrm{mod}\,N)}} x_{n_j}\rangle \ge 0 \end{aligned}$$

for all $y \in C$.

This implies that

$$f_{i(\bmod N)}(u_{n_j}, y) + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j}\rangle - \frac{1}{r_{n_j}}\bigl\langle y - u_{n_j}, \gamma A^*_{n_{j(\bmod N)}}\bigl(I - T^{g_{n_j(\bmod N)}}_{s_{n_j}}\bigr)A_{n_{j(\bmod N)}}x_{n_j}\bigr\rangle \geq 0.$$

From condition (A2), we have

$$\frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j}\rangle - \frac{1}{r_{n_j}}\bigl\langle y - u_{n_j}, \gamma A^*_{n_{j(\bmod N)}}\bigl(I - T^{g_{n_j(\bmod N)}}_{s_{n_j}}\bigr)A_{n_{j(\bmod N)}}x_{n_j}\bigr\rangle \geq f_{i(\bmod N)}(y, u_{n_j})$$

for all $y \in C$. Since $\liminf_{j\to\infty} r_{n_i} > 0$ (by (C2)), it follows from (17) and (20) that

$$f_{i(\bmod N)}(y, q) \leq 0 \quad \text{for all } y \in C \text{ and for } 1 \leq i \leq N.$$

Let $y_t = ty + (1-t)q$ for some $0 < t < 1$ and $y \in C$. Since $q \in C$, this implies that $y_t \in C$. Using conditions (A1) and (A4), the following estimate

$$0 = f_{i(\bmod N)}(y_t, y_t) \leq t f_{i(\bmod N)}(y_t, y) + (1-t) f_{i(\bmod N)}(y_t, q) \leq t f_{i(\bmod N)}(y_t, y)$$

implies that

$$f_{i(\bmod N)}(y_t, y) \geq 0 \quad \text{for } 1 \leq i \leq N.$$

Letting $t \to 0$, we have $f_{i(\bmod N)}(q, y) \geq 0$ for all $y \in C$. Thus, $q \in \mathrm{EP}(f_i)$ for $1 \leq i \leq N$. That is, $q \in \bigcap_{i=1}^{N} \mathrm{EP}(F_i)$. Reasoning as above, we show that $A_{i(\bmod N)}q \in \mathrm{EP}(g_i)$ for each $1 \leq i \leq N$. Since $x_{n_l} \longrightarrow q$ and $A_{n_l(\bmod N)}$ is a bounded linear operator, therefore $A_{n_{l(\bmod N)}}x_{n_l} \longrightarrow A_{n_{l(\bmod N)}}q$. Hence, it follows from (17) that

$$T^{g_{n_l(\bmod N)}}_{s_{n_l}} A_{n_{l(\bmod N)}}x_{n_l} \longrightarrow A_{n_{l(\bmod N)}}q \quad \text{as } l \to \infty.$$

Now, from Lemma 2.2, we have

$$g_{i(\bmod N)}\bigl(T^{g_{n_l(\bmod N)}}_{s_{n_l}} A_{n_l(\bmod N)}x_{n_l}, z\bigr) + \frac{1}{s_{n_l}}\bigl\langle z - T^{g_{n_l(\bmod N)}}_{s_{n_l}} A_{n_l(\bmod N)}x_{n_l}, T^{g_{n_l(\bmod N)}}_{s_{n_l}} A_{n_{l(\bmod N)}}x_{n_l} - A_{n_{l(\bmod N)}}x_{n_l}\bigr\rangle \geq 0$$

for all $z \in Q$.

Since g_i is upper hemicontinuous in the first argument for each $1 \leq i \leq N$, taking lim sup on both sides of the above estimate as $l \to \infty$ and utilizing (C2) and (17), we get

$$g_{i(\bmod N)}(A_{n_l(\bmod N)}x, z) \geq 0 \quad \text{for all } z \in Q \text{ and for each } 1 \leq i \leq N.$$

Hence $A_{i(\mathrm{mod}\,N)}q \in \mathrm{EP}(g_i)$ for each $1 \le i \le N$ and consequently $q \in \mathbb{F}$. It remains to show that $x_n \to q = P_{\mathbb{F}}x_1$. Let $x = P_{\mathbb{F}}x_1$, then from $\|x_n - x_1\| \le \|x - x_1\|$ we have

$$\begin{aligned}\|x - x_1\| &\le \|q - x_1\| \\ &\le \liminf_{j\to\infty} \|x_{n_j} - x_1\| \\ &\le \limsup_{j\to\infty} \|x_{n_j} - x_1\| \\ &\le \|x - x_1\|.\end{aligned}$$

This implies that

$$\lim_{j\to\infty} \|x_{n_j} - x_1\| = \|q - x_1\|.$$

Hence $x_{n_j} \to q = P_{\mathbb{F}}x_1$. From the arbitrariness of the subsequence $\{x_{n_j}\}$ of $\{x_n\}$, we conclude that $x_n \to x$ as $n \to \infty$. It is easy to see that $y_{n,i} \to x$ and $u_{n,i} \to x$. This completes the proof. □

We now give an example to justify the main result of this section.

Example 3.2 Let $H_1 = H_2 = \mathbb{R}$, $C = Q = [0, 10]$. Let $S_i : C \to C$ be defined by $S_i x = \frac{x}{i+1}$ for each $i = 1, 2, \dots, N$ with strictly increasing function $\xi : [0,\infty) \to [0,\infty)$ satisfying $\xi(0) = 0$ and $\lambda_n = \mu_n = \frac{1}{n^2} \xrightarrow{n\to\infty} 0$. Then observe that, for each fixed $i = 1, 2, \dots, N$,

$$\begin{aligned}&\bigl\|S_i^n x - S_i^n y\bigr\| - \|x - y\| - \lambda_n \xi_n\bigl(\|x - y\|\bigr) - \mu_n \\ &\quad\le \frac{1}{(i+1)^n}\|x - y\| - \|x - y\| - \lambda_n \xi_n\bigl(\|x - y\|\bigr) - \mu_n \\ &\quad\le \|x - y\| - \|x - y\| - \lambda_n \xi_n\bigl(\|x - y\|\bigr) - \mu_n \le 0.\end{aligned}$$

This shows that, for each fixed $i = 1, 2, \dots, N$, the mapping S_i is total asymptotically nonexpansive with $\bigcap_{i=1}^N F(S_i) = \{0\}$. Let $A : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = x$ for all $x \in H_1 = \mathbb{R}$. This implies that $A^* y = y$ for all $y \in H_2 = \mathbb{R}$. The two bifunctions f and g are defined by $f_i(u, v) = f(u, v) = 2u(v - u)$ for all $u, v \in C$ and $g_i(x, y) = g(x, y) = x(y - x)$ for all $x, y \in Q$, respectively. It is easy to check that f and g satisfy all the conditions in Theorem 3.1 (Main Result) with $\Omega = \{0\}$, and hence $\mathbb{F} = \{0\}$. Set $\beta_n = r_n = \frac{n}{100n+1}$ and $\gamma = \frac{1}{100}$. For each $r > 0$ and $x \in C$, we compute our iteration as follows.

Step 1. Find $z \in Q$ such that $g(z, y) + \frac{1}{r}\langle y - z, z - Ax\rangle \ge 0$ for all $y \in Q$. Since $Ax = x$, we have

$$\begin{aligned}g(z, y) + \frac{1}{r}\langle y - z, z - Ax\rangle \ge 0 \quad &\Leftrightarrow \quad z(y - z) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \\ &\Leftrightarrow \quad rz(y - z) + (y - z)(z - x) \ge 0, \\ &\Leftrightarrow \quad (y - z)\bigl((1 + r)z - x\bigr) \ge 0.\end{aligned}$$

It follows from Lemma 2.2(i) that $T_r^g Ax$ is single-valued; therefore we get $z = \frac{x}{1+r}$. This implies that $T_r^g Ax = \frac{x}{1+r}$.

Table 1 Numerical results of Example 3.2 with $\alpha_n = 0.5$ and initial guess $x_1 = 10$.

$\alpha_n = 0.5$ and $x_1 = 10$				
n	u_n	y_n	C_n	x_n
1	9.804864	4.950971	[0, 10.000000]	10.000000
2	7.328900	1.282254	[0, 7.475485]	7.475485
3	4.292865	0.396959	[0, 4.378870]	4.378870
⋮	⋮	⋮	⋮	⋮
10	0.044694	0.000849	[0, 0.045592]	0.045592
11	0.022763	0.000398	[0, 0.023220]	0.023220
⋮	⋮	⋮	⋮	⋮
25	0.000002	0.000000	[0, 0.000002]	0.000002
26	0.000001	0.000000	[0, 0.000001]	0.000001
27	0.000000	0.000000	[0, 0.000000]	0.000000

Step 2. Find $s \in C$ such that $s = w - \gamma A^*(I - T_r^G)Aw$. It follows from Step 1 that

$$\begin{aligned} s &= x - \gamma A^*\bigl(I - T_r^g\bigr)Ax \\ &= x - \frac{1}{100}A^*\left(x - \frac{x}{1+r}\right) \\ &= x - \frac{1}{100}\left(x - \frac{x}{1+r}\right) \\ &= \left(1 - \frac{1}{100}\right)x + \frac{1}{100}\left(\frac{x}{1+r}\right). \end{aligned}$$

Step 3. Find $u \in C$ such that $f(u,v) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0$ for all $v \in C$. It follows from Step 2 that

$$\begin{aligned} f(u,v) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0 \quad &\Leftrightarrow \quad 2u(v-u) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0, \\ &\Leftrightarrow \quad 2ru(v-u) + (v-u)(u-s) \geq 0, \\ &\Leftrightarrow \quad (v-u)\bigl((1+2r)u - s\bigr) \geq 0. \end{aligned}$$

Similarly, from Lemma 2.2(i) we obtain that $u = \frac{s}{1+2r} = (1 - \frac{1}{100})\frac{x}{1+2r} + \frac{1}{100}(\frac{x}{(1+r)(1+2r)})$.

Step 4. Find $y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{N}\sum_{i=1}^{N} S_i^n u_n$ where $u_n = (1 - \frac{1}{100})\frac{x_n}{1+2r_n} + \frac{1}{100}(\frac{x_n}{(1+r_n)(1+2r_n)})$.

Step 5. Find $C_{n+1} = \{z \in C : \|y_n - z\| \leq \|w_n - z\|\}$ where $C_1 = [0,10]$. Since $0 \leq y_1 \leq x_1 \leq 10$, therefore $C_2 = \{z \in C_1 : \|y_1 - z\| \leq \|x_1 - z\|\} = [0, \frac{y_1+x_1}{2}]$. Since $\frac{y_1+x_1}{2} \leq x_1$ and in particular $\frac{y_1+x_1}{2} \leq x_1$, therefore $x_2 = P_{C_2}x_1 = \frac{y_1+x_1}{2}$. In a similar fashion, we have $C_{n+1} = [0, \frac{y_n+x_n}{2}]$ and $x_{n+1} = P_{C_{n+1}}x_1 = \frac{y_n+x_n}{2}$.

Step 6. Compute the numerical results of $x_{n+1} = P_{C_{n+1}}x_1$.

Table 1 exhibits the performance of sequence $\{x_n\}$ defined in Theorem 3.1.

4 Applications

In this section, we deduce some results from Theorem 3.1. An immediate consequence of Theorem 3.1 is to establish the same result for a class of nonexpansive mappings.

Corollary 4.1 *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $f_i : C \times C \to \mathbb{R}$*

and $g_i : Q \times Q \to \mathbb{R}$ *be two finite families of bifunctions satisfying conditions* (A1)–(A4) *such that each* g_i *is upper semicontinuous for each* $i \in \{1,2,3,\dots,N\}$. *Let* $S_i : C \to C$ *be a finite family of nonexpansive mappings, and let* $A_i : H_1 \to H_2$ *be a finite family of bounded linear operators for each* $i \in \{1,2,3,\dots,N\}$. *Suppose that* $\mathbb{F} := F(S) \cap \Omega \neq \emptyset$, *where* $\Omega = \{z \in C : z \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ *and* $A_i z \in \bigcap_{i=1}^N \mathrm{EP}(g_i)$ *for* $1 \le i \le N\}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} &x_1 \in C_1 = C, \\ &u_n = T_{r_n}^{f_{n(\bmod N)}}\big(x_n - \gamma A^*_{n(\bmod N)}\big(I - T_{s_n}^{g_{n(\bmod N)}}\big)A_{n(\bmod N)}x_n\big), \\ &y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{N}\sum_{i=1}^{N} S^i u_n, \\ &C_{n+1} = \big\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\big\}, \\ &x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{aligned} \tag{27}$$

where $\theta_n = (1-\alpha_n)\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ *with* $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. *Let* $\{r_n\}$, $\{s_n\}$ *be two positive real sequences, and let* $\{\alpha_n\}$ *be in* $(0,1)$. *Assume that if the following set of conditions holds*:

- (C1) $0 \le k < a \le \alpha_n \le b < 1$ *and* $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \dots, L_N\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1,2,3,\dots,N\}$;
- (C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;
- (C3) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;
- (C4) *there exist constants* $M_i, M_i^* > 0$ *such that* $\xi_i(\lambda_i) \le M_i^* \lambda_i$ *for all* $\lambda_i \ge M_i$, $i = 1,2,3,\dots,N$, *then the sequence* $\{x_n\}$ *generated by* (27) *converges strongly to* $P_{\mathbb{F}} x_1$.

Corollary 4.2 *Let* H_1 *and* H_2 *be two real Hilbert spaces, and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $f_i : C \times C \to \mathbb{R}$ *and* $g_i : Q \times Q \to \mathbb{R}$ *be two finite families of bifunctions satisfying conditions* (A1)–(A4) *such that each* g_i *is upper semicontinuous for each* $i \in \{1,2,3,\dots,N\}$. *Let* $S : C \to C$ *be a nonexpansive mapping, and let* $A_i : H_1 \to H_2$ *be a finite family of bounded linear operators for each* $i \in \{1,2,3,\dots,N\}$. *Suppose that* $\mathbb{F} := F(S) \cap \Omega \neq \emptyset$, *where* $\Omega = \{z \in C : z \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ *and* $A_i z \in \bigcap_{i=1}^N \mathrm{EP}(g_i)$ *for* $1 \le i \le N\}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} &x_1 \in C_1 = C, \\ &u_n = T_{r_n}^{f_{n(\bmod N)}}\big(x_n - \gamma A^*_{n(\bmod N)}\big(I - T_{s_n}^{g_{n(\bmod N)}}\big)A_{n(\bmod N)}x_n\big), \\ &y_n = \alpha_n u_n + (1-\alpha_n) S u_n, \\ &C_{n+1} = \big\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\big\}, \\ &x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{aligned} \tag{28}$$

where $\theta_n = (1-\alpha_n)\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ *with* $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. *Let* $\{r_n\}$, $\{s_n\}$ *be two positive real sequences, and let* $\{\alpha_n\}$ *be in* $(0,1)$. *Assume that if the following set of conditions holds*:

(C1) $0 \leq k < a \leq \alpha_n \leq b < 1$ *and* $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \ldots, L_N\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \ldots, N\}$;

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;

(C3) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(C4) *there exist constants* $M_i, M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i$, $i = 1, 2, 3, \ldots, N$, *then the sequence* $\{x_n\}$ *generated by* (28) *converges strongly to* $P_{\mathbb{F}} x_1$.

Proof Set $S^i = S$ for $i \in \{1, 2, 3, \ldots, N\}$, then the desired result follows from Corollary 4.1 immediately. □

The following results suggest an iterative construction for a common solution of the classical equilibrium problem together with the fixed point problem.

Corollary 4.3 *Let* H_1 *and* H_2 *be two real Hilbert spaces, and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $f_i : C \times C \to \mathbb{R}$ *and* $g_i : Q \times Q \to \mathbb{R}$ *be two finite families of bifunctions satisfying conditions* (A1)–(A4) *such that each* g_i *is upper semicontinuous for each* $i \in \{1, 2, 3, \ldots, N\}$. *Let* $S_i : C \to C$ *be a finite family of uniformly continuous total asymptotically nonexpansive mappings, and let* $A_i : H_1 \to H_2$ *be a finite family of bounded linear operators for each* $i \in \{1, 2, 3, \ldots, N\}$. *Suppose that* $\mathbb{F} := [\bigcap_{i=1}^{N} F(S_i)] \cap \Omega \neq \emptyset$, *where* $\Omega = \{z \in C : z \in \bigcap_{i=1}^{N} \mathrm{EP}(f_i)$ *and* $A_i z \in \bigcap_{i=1}^{N} \mathrm{EP}(g_i)$ *for* $1 \leq i \leq N\}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} & x_1 \in C_1 = C, \\ & u_n = T_{r_n}^{f_{n(\mathrm{mod}\,N)}} \bigl(x_n - \gamma A_{n(\mathrm{mod}\,N)}^* \bigl(I - T_{s_n}^{g_{n(\mathrm{mod}\,N)}}\bigr) A_{n(\mathrm{mod}\,N)} x_n\bigr), \\ & y_n = \alpha_n u_n + (1-\alpha_n) \frac{1}{N} \sum_{i=1}^{N} S_i^n u_n, \\ & C_{n+1} = \bigl\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\bigr\}, \\ & x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{aligned} \tag{29}$$

where $\theta_n = (1-\alpha_n)\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ *with* $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. *Let* $\{r_n\}$, $\{s_n\}$ *be two positive real sequences, and let* $\{\alpha_n\}$ *be in* $(0,1)$. *Assume that if the following set of conditions holds*:

(C1) $0 \leq k < a \leq \alpha_n \leq b < 1$ *and* $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \ldots, L_N\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \ldots, N\}$;

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;

(C3) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(C4) *there exist constants* $M_i, M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i$, $i = 1, 2, 3, \ldots, N$, *then the sequence* $\{x_n\}$ *generated by* (29) *converges strongly to* $P_{\mathbb{F}} x_1$.

Proof Set $H_1 = H_2$, $C = Q$ and $A_i = I$ (the identity mapping) for $i = 1, 2, 3, \ldots, N$, then the desired result follows from Theorem 3.1 immediately. □

Corollary 4.4 *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $f_i : C \times C \to \mathbb{R}$ and $g_i : Q \times Q \to \mathbb{R}$ be two finite families of bifunctions satisfying conditions* (A1)–(A4) *such that each g_i is upper semicontinuous for each $i \in \{1, 2, 3, \dots, N\}$. Let $S_i : C \to C$ be a finite family of nonexpansive mappings, and let $A_i : H_1 \to H_2$ be a finite family of bounded linear operators for each $i \in \{1, 2, 3, \dots, N\}$. Suppose that $\mathbb{F} := [\bigcap_{i=1}^N F(S_i)] \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ and $A_i z \in \bigcap_{i=1}^N \mathrm{EP}(g_i)$ for $1 \le i \le N\}$. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{aligned}
&x_1 \in C_1 = C, \\
&u_n = T_{r_n}^{f_{n(\operatorname{mod} N)}} \bigl(x_n - \gamma A^*_{n(\operatorname{mod} N)} \bigl(I - T_{s_n}^{g_{n(\operatorname{mod} N)}}\bigr) A_{n(\operatorname{mod} N)} x_n\bigr), \\
&y_n = \alpha_n u_n + (1 - \alpha_n) \frac{1}{N} \sum_{i=1}^N S^i u_n, \\
&C_{n+1} = \bigl\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\bigr\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{aligned}
\tag{30}
$$

where $\theta_n = (1 - \alpha_n)\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ with $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. Let $\{r_n\}$, $\{s_n\}$ be two positive real sequences, and let $\{\alpha_n\}$ be in $(0, 1)$. Assume that if the following set of conditions holds:

- (C1) $0 \le k < a \le \alpha_n \le b < 1$ *and* $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \dots, L_N\}$ *and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each* $i \in \{1, 2, 3, \dots, N\}$;
- (C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;
- (C3) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $\sum_{n=1}^\infty \mu_n < \infty$;
- (C4) *there exist constants $M_i, M_i^* > 0$ such that $\xi_i(\lambda_i) \le M_i^* \lambda_i$ for all $\lambda_i \ge M_i$, $i = 1, 2, 3, \dots, N$, then the sequence $\{x_n\}$ generated by* (27) *converges strongly to $P_{\mathbb{F}} x_1$.*

Proof Set $H_1 = H_2$, $C = Q$, and $A_i = I$ (the identity mapping) for $i = 1, 2, 3, \dots, N$, then the desired result follows from Corollary 4.1 immediately. □

Corollary 4.5 *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $f_i : C \times C \to \mathbb{R}$ and $g_i : Q \times Q \to \mathbb{R}$ be two finite families of bifunctions satisfying conditions* (A1)–(A4) *such that each g_i is upper semicontinuous for each $i \in \{1, 2, 3, \dots, N\}$. Let $S : C \to C$ be a nonexpansive mapping, and let $A_i : H_1 \to H_2$ be a finite family of bounded linear operators for each $i \in \{1, 2, 3, \dots, N\}$. Suppose that $\mathbb{F} := F(S) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in \bigcap_{i=1}^N \mathrm{EP}(f_i)$ and $A_i z \in \bigcap_{i=1}^N \mathrm{EP}(g_i)$ for $1 \le i \le N\}$. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{aligned}
&x_1 \in C_1 = C, \\
&u_n = T_{r_n}^{f_{n(\operatorname{mod} N)}} \bigl(x_n - \gamma A^*_{n(\operatorname{mod} N)} \bigl(I - T_{s_n}^{g_{n(\operatorname{mod} N)}}\bigr) A_{n(\operatorname{mod} N)} x_n\bigr), \\
&y_n = \alpha_n u_n + (1 - \alpha_n) S u_n, \\
&C_{n+1} = \bigl\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\bigr\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{aligned}
\tag{31}
$$

where $\theta_n = (1-\alpha_n)\{\lambda_n\xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ *with* $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. *Let* $\{r_n\}$, $\{s_n\}$ *be two positive real sequences, and let* $\{\alpha_n\}$ *be in* $(0,1)$. *Assume that if the following set of conditions holds*:

(C1) $0 \le k < a \le \alpha_n \le b < 1$ *and* $\gamma \in (0, \frac{1}{L})$ *where* $L = \max\{L_1, L_2, \dots, L_N\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1,2,3,\dots,N\}$;

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;

(C3) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(C4) *there exist constants* $M_i, M_i^* > 0$ *such that* $\xi_i(\lambda_i) \le M_i^* \lambda_i$ *for all* $\lambda_i \ge M_i$, $i = 1,2,3,\dots,N$, *then the sequence* $\{x_n\}$ *generated by* (31) *converges strongly to* $P_{\mathbb{F}} x_1$.

Proof Set $S^i = S$ for $i \in \{1,2,3,\dots,N\}$, then the desired result follows from Corollary 4.4 immediately. □

Acknowledgements
The authors express their sincere thanks to the referee for his/her careful reading and suggestions that helped to improve this paper.

Funding
This project was supported by the deanship of scientific research at Prince Sattam Bin Abdulaziz University under the research project 2019/01/10960.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, 11991, Wadi Aldawasir, Kingdom of Saudi Arabia. [2]Department of Mathematics, COMSATS University Islamabad, 54000, Lahore, Pakistan. [3]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt. [4]Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [5]Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

Publisher's Note

Received: 19 February 2020 Accepted: 25 June 2020 Published online: 10 July 2020

References
1. Alber, Y.I., Chidume, C.E., Zegeye, H.: Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, 10673 (2006)
2. Baillon, J.B.: Un théoréme de type ergodique pour les contractions nonlinéairs dans un espaces de Hilbert. C. R. Acad. Sci. Paris, Ser. I **280**, 1511–1514 (1975)
3. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
4. Bruck, R.E.: A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces. Isr. J. Math. **32**, 107–116 (1979)
5. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
6. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
7. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
8. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **59**, 301–323 (2012)
9. Cholamjiak, W., Cholamjiak, P., Suantai, S.: Convergence of iterative schemes for solving fixed point of multi-valued nonself mappings and equilibrium problems. J. Nonlinear Sci. Appl. **8**, 1245–1256 (2015)

10. Combettes, P.L.: The convex feasibility problem in image recovery. Adv. Imaging Electron Phys. **95**, 155–453 (1996)
11. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
12. Deepho, J., Martínez-Moreno, J., Sitthithakerngkiet, K., Kumam, P.: Convergence analysis of hybrid projection with Cesáro mean method for the split equilibrium and general system of finite variational inequalities. J. Comput. Appl. Math. **318**, 658–673 (2017)
13. Halpern, B.: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
14. Haugazeau, Y.: Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes. Thèse, Thèse, Université de Paris, Paris, France (1968)
15. Hirano, N.: A proof of the mean ergodic theorem for nonexpansive mappings in Banach spaces. Proc. Am. Math. Soc. **78**, 361–365 (1980)
16. Hirano, N., Takahashi, W.: Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces. Kodai Math. J. **2**, 11–25 (1979)
17. Kesornprom, S., Pholasa, N., Cholamjiak, P.: On the convergence analysis of the gradient-CQ algorithms for the split feasibility problem. Numer. Algorithms **84**(3), 997–1017 (2020)
18. Khan, M.A.A.: Convergence analysis of a multi-step iteration for a finite family of asymptotically quasi-nonexpansive mappings. J. Inequal. Appl. **2013**, 423 (2013)
19. Khan, M.A.A.: Iteration schema for common fixed points of nonlinear mappings in spaces of nonpositive curvature. Adv. Pure Appl. Math. **5**(2), 65–74 (2014)
20. Khan, M.A.A.: Convergence characteristics of a shrinking projection algorithm in the sense of Mosco for split equilibrium problem and fixed point problem in Hilbert spaces. Linear Nonlinear Anal. **3**, 423–435 (2017)
21. Khan, M.A.A., Arafat, Y., Butt, A.R.: A shrinking projection approach to solve split equilibrium problems and fixed point problems in Hilbert spaces. UPB Sci. Bull., Ser. A **80**(1), 33–46 (2018)
22. Khan, M.A.A., Fukhar-ud-din, H., Kalsoom, A.: Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. Fixed Point Theory Appl. **2016**, 3 (2016)
23. Khan, S.A., Cholamjiak, W., Kazmi, K.R.: An inertial forward–backward splitting method for solving combination of equilibrium problems and inclusion problems. Comput. Appl. Math. **37**, 6283–6307 (2018)
24. Khan, S.A., Kazmi, K.R., Yambangwai, D., Cholamjiak, W.: A hybrid projective method for solving system of equilibrium problems with demicontractive mappings applicable in image restoration problems. Math. Methods Appl. Sci. **43**, 1–19 (2020)
25. Moudafi, A.: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275–283 (2011)
26. Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274–276 (1979)
27. Shioji, N., Takahashi, W.: Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces. J. Approx. Theory **97**, 53–64 (1999)
28. Shioji, N., Takahashi, W.: A strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces. Arch. Math. **72**, 354–359 (1999)
29. Song, Y.: Mann iteration of Cesáro means for asymptotically non-expansive mappings. Nonlinear Anal. **72**, 176–182 (2010)
30. Suantai, S., Cholamjiak, P., Cho, Y.J., Cholamjiak, W.: On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces. Fixed Point Theory Appl. **2016**, 35 (2016)
31. Suantai, S., Shehu, Y., Cholamjiak, P., Iyiola, O.S.: Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces. J. Fixed Point Theory Appl. **20**, 68 (2018)
32. Takahashi, W., Takeuchi, Y., Kubota, R.: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **341**, 276–286 (2008)
33. Wang, S.H., Chen, M.J.: Iterative algorithm for a split equilibrium problem and fixed problem for finite asymptotically nonexpansive mappings in Hilbert space. Filomat **31**(5), 1423–1434 (2017)