Alzabut et al. *Advances in Difference Equations* (2020) 2020:304
https://doi.org/10.1186/s13662-020-02766-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Solutions of two fractional q-integro-differential equations under sum and integral boundary value conditions on a time scale

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Abstract

In this manuscript, by using the Caputo and Riemann–Liouville type fractional q-derivatives, we consider two fractional q-integro-differential equations of the forms ${}^{c}\mathcal{D}_q^{\alpha}[x](t) + w_1(t, x(t), \varphi(x(t))) = 0$ and

$$ {}^{c}\mathcal{D}_q^{\alpha}[x](t) = w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^{c}\mathcal{D}_q^{\alpha}[x](t)\right) $$

for $t \in [0, l]$ under sum and integral boundary value conditions on a time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n\} \cup \{0\}$ for $n \in \mathbb{N}$ where $t_0 \in \mathbb{R}$ and q in $(0, 1)$. By employing the Banach contraction principle, sufficient conditions are established to ensure the existence of solutions for the addressed equations. Examples involving algorithms and illustrated graphs are presented to demonstrate the validity of our theoretical findings.

MSC: Primary 34A08; 34B16; secondary 39A13

Keywords: Sum boundary value conditions; Caputo q-derivative; Riemann–Liouville q-derivative; Integral boundary value conditions

1 Introduction

It has been recognized that fractional calculus provides a meaningful generalization for the classical integration and differentiation to any order. On the other hand, quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. It defines q-calculus where q stands for quantum. Despite the old history of these two theories, the investigation of their properties remains untouched until recent time. Fractional q-calculus, initially proposed by Jackson [1–3], was regarded as the fractional analogue of q-calculus. Soon afterwards, it was further promoted by Al-Salam in [4] and then continued by Agarwal in [5] where many outstanding theoretical results were given. Its emergence and development extended the application of interdisciplinarity and aroused widespread attention of scholars; see [6–28] and the references therein. The existence of solutions for q-fractional boundary value problems has been under consideration by many researchers; see for instance [29–40].

In [41], Ntouyas *et al.* studied the boundary value problem of first-order fractional differential equations given by

$$\begin{cases} {}^{c}D_{0^+}^{\beta_1}[f_1](x) = w_1(x, f_1(x), f_2(x)), \\ {}^{c}D_{0^+}^{\beta_2}[f_2](x) = w_2(x, f_1(x), f_2(x)), \end{cases} \quad t \in [0,1],$$

with Riemann–Liouville integral boundary conditions of different order $f_1(0) = c_1 I^{\alpha_1}[f_1](a_1)$ and $f_2(0) = c_2 I^{\alpha_2}[f_2](a_2)$ for $0 < a_1, a_2 < 1$, $\beta_i \in (0,1]$, $\alpha_i, c_i \in \mathbb{R}$ where $i = 1, 2$. In 2015, Zhang *et al.* through the spectral analysis and fixed point index theorem obtained the existence of positive solutions of the singular nonlinear fractional differential equation $-\mathcal{D}_t^{\alpha} u(t) = w(t, u(t), \mathcal{D}_t^{\beta} u(t))$ for $0 < t < 1$, with integral boundary value conditions $\mathcal{D}_t^{\beta} u(0) = 0$ and $\mathcal{D}_t^{\beta} u(1) = \int_0^1 \mathcal{D}_t^{\beta} u(r)\,\mathrm{d}N(r)$, where $\alpha \in (1,2]$, $\beta \in (0,1]$, $w(t,u,v)$ may be singular at both $t = 0, 1$ and $u = v = 0$, $\int_0^1 u(r)\,\mathrm{d}N(r)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $N : [0,1] \to \mathbb{R}$ is a function of bounded variation [42]. In 2016, Ahmad *et al.* investigated the existence of solutions for a q-antiperiodic boundary value problem of fractional q-difference inclusions given by

$${}^{c}\mathcal{D}_q^{\alpha}[k](t) \in F\bigl(t, k(t), \mathcal{D}_q[k](t), \mathcal{D}_q^2[k](t)\bigr)$$

for $t \in [0,1]$, $q \in (0,1)$, $2 < \alpha \le 3$, $0 < \beta \le 3$, and $k(0) + k(1) = 0$, $\mathcal{D}_q k(0) + \mathcal{D}_q k(1) = 0$, $\mathcal{D}_q^2 k(0) + \mathcal{D}_q^2 k(1) = 0$, where ${}^{c}\mathcal{D}_q^{\alpha}$ denotes Caputo fractional q-derivative of order α and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with $\mathcal{P}(\mathbb{R})$ a class of all subsets of $\mathbb{R}$ [15]. In 2019, Ren and Zhai discussed the existence of unique solution and multiple positive solutions for the fractional q-differential equation $\mathcal{D}_q^{\alpha} x(t) + w(t, x(t)) = 0$ for each $t \in [0,1]$ with nonlocal boundary conditions $x(0) = \mathcal{D}_q^{\alpha-2} x(0) = 0$ and $\mathcal{D}_q^{\alpha-1} x(1) = \mu[x] + \int_0^{\eta} \phi(r) \mathcal{D}_q^{\beta} x(t)\,\mathrm{d}_q r$, where $\mathcal{D}_q^{\alpha}$ is the standard Riemann–Liouville fractional q-derivative of order α, $2 < \alpha \le 3$, such that $\alpha - 1 - \beta > 0$, $q \in (0,1)$, $\phi \in L^1[0,1]$ is nonnegative, $\mu[x]$ is a linear functional given by $\mu[x] = \int_0^1 x(t)\,\mathrm{d}N(t)$ involving the Stieltjes integral with respect to the function $N : [0,1] \to \mathbb{R}$ such that $N(t)$ is right-continuous on $[0,1)$, left-continuous at $t = 1$ and, particularly, N is a nondecreasing function with $N(0) = 0$ and $\mathrm{d}N$ is positive Stieltjes measure [40]. The authors in [43] investigated a multi-term nonlinear fractional q-integro-differential equation

$${}^{c}D_q^{\alpha}[x](t) = w\bigl(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^{c}D_q^{\beta_1}[x](t), {}^{c}D_q^{\beta_2}[x](t), \ldots, {}^{c}D_q^{\beta_n}[x](t)\bigr)$$

under some boundary conditions. The existence of solutions for the multi-term nonlinear fractional q-integro-differential ${}^{c}D_q^{\alpha}[u](t)$ equations in two modes and inclusions of order $\alpha \in (n-1, n]$ with non-separated boundary and initial boundary conditions where natural number n is more than or equal to five was considered in [20]. Recently, some researchers discussed the existence of solutions for some singular fractional differential equations; see the papers [44–47].

Benefiting from the main ideas of the above said papers, we investigate the following two nonlinear fractional q-integro-differential equations in the spaces $\mathcal{A} = C(\bar{J} \times \mathbb{R}^2, \mathbb{R})$

and $\mathcal{B} = \{x : x, {}^c\mathcal{D}_q^\beta[x] \in C^2(\bar{J}, \mathbb{R}), \bar{J} = [0, l]\}$ with the norms defined by $\|x\| = \sup_{t\in\bar{J}} |x(t)|$ and

$$\|x\|_* = \sup_{t\in\bar{J}} |x(t)| + \sup_{t\in\bar{J}} \left|{}^c\mathcal{D}_q^\beta[x](t)\right|,$$

respectively.

(P1) First we investigate the nonlinear fractional q-integro-differential equation

$${}^c\mathcal{D}_q^\alpha[x](t) + w_1\big(t, x(t), \varphi\big(x(t)\big)\big) = 0 \tag{1}$$

for $t \in \bar{J}$ under sum and integral boundary value conditions

$$x'(a) = -\eta \int_0^1 x(r)\,\mathrm{d}r, \qquad x'(1) + x(0) = \sum_{i=1}^m c_i x'(b), \tag{2}$$

where $m \geq 1$, $1 \leq \alpha < 2$, $0 \leq a < b \leq 1$, $\eta \geq 0$, $c_i \geq 0$ for each $i = 1, 2, \ldots, m$ such that $2\Xi > -1$, here $\Xi = \sum_{i=1}^m c_i$, $\varphi(x(t)) = \int_0^t g(r)x(r)\,\mathrm{d}r$ and $w_1 : \bar{J} \times \mathcal{A}^2 \to \mathcal{A}$ is a continuous function.

(P2) Second we consider the nonlinear fractional q-integro-differential equation

$${}^c\mathcal{D}_q^\alpha[x](t) = w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^\zeta[x](t)\right) \tag{3}$$

for $t \in \bar{J}$ under the sum boundary conditions

$$x(0) = 0, \qquad x'(1) = \sum_{i=1}^m c_i x''(b), \tag{4}$$

where $1 \leq \alpha < 2$, $0 \leq \zeta < 1$, $0 < b < 1$, $m \geq 1$, $c_i \geq 0$ for all $i = 1, \ldots, m$ and $w_2 : \bar{J} \times \mathcal{B}^3 \to \mathcal{B}$ is a continuous function.

This paper is organized as follows: In Sect. 2, we state some useful definitions and lemmas on the fundamental concepts of q-fractional calculus and fixed point theory. In Sect. 3, some main theorems on the solutions of fractional q-integro-differential equations (1)–(2) and (3)–(4) are stated. Section 4 contains some illustrative examples to show the validity and applicability of our results. The paper concludes with some interesting observations.

2 Essential preliminaries

This section is devoted to some notations and essential preliminaries that are acting as necessary prerequisites for the results of the subsequent sections. Throughout this article, we apply the time scales calculus notation [9]. In fact, we consider the fractional q-calculus on the specific time scale $\mathbb{T} = \mathbb{R}$ where $\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n\}$ for nonnegative integer n, $t_0 \in \mathbb{R}$ and $q \in (0, 1)$. Let $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [2]. The power function $(x - y)_q^n$ with $n \in \mathbb{N}_0$ is defined by

$$(x - y)_q^{(n)} = \prod_{k=0}^{n-1} \big(x - yq^k\big)$$

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

```
function p = powerfunction(a, b, n, q)
%Power Gamma (a-b)^(n)
s=1;
if n==0
        p=1
else
for k=1:n-1
        s=s*(a-b*q^k)/(a- b*q^(alpha+k));
end;
p=a^alpha * s;
end;
end
```

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

```
function g = qGamma(q, x, n)
%q-Gamma Function
p=1;
for k=0:n
        p=p*(1-q^(k+1))/(1- q^(x+k));
end;
g=p/(1-q)^(x-1);
end
```

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

```
function g = Dq(q, x, n, fun)
if x==0
        g=limit ((fun(x)-fun(q*x))/((1-q)*x),x,0);
else
        g=(fun(x)-fun(q*x))/((1-q)*x);
end;
end
```

for $n \geq 1$ and $(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [6]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(x-y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty} \frac{x-yq^k}{x-yq^{\alpha+k}}.$$

If $y=0$, then it is clear that $x^{(\alpha)} = x^\alpha$ [8] (Algorithm 1). The q-gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$ [2]. Note that $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. Algorithm 2 shows a pseudo-code description of the technique for estimating q-gamma function of order n. The q-derivative of function f is defined by $\mathcal{D}_q[f](x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $\mathcal{D}_q[f](0) = \lim_{x\to 0} \mathcal{D}_q[f](x)$, which is shown in Algorithm 3 [6, 7]. Furthermore, the higher order q-derivative of a function f is defined by $\mathcal{D}_q^n[f](x) = \mathcal{D}_q[\mathcal{D}_q^{n-1}[f]](x)$ for $n \geq 1$, where $\mathcal{D}_q^0[f](x) = f(x)$ [6, 7]. Tables 1, 2, and 3 show the values $\Gamma_q(z)$ for some z and $q \in (0,1)$. The q-integral of a function f is defined on $[0,b]$ by

$$I_q f(x) = \int_0^x f(s)\,\mathrm{d}_q s = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right)$$

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$, and $n = 1, 2, \ldots, 15$ of Algorithm 2

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$, and $n = 1, 2, \ldots, 35$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

for $0 \le x \le b$, provided the series absolutely converges [6, 7]. If x in $[0, T]$, then

$$\int_x^T f(r)\, \mathrm{d}_q r = I_q[f](T) - I_q[f](x) = (1-q) \sum_{k=0}^{\infty} q^k \bigl[T f\bigl(T q^k\bigr) - x f\bigl(x q^k\bigr) \bigr],$$

whenever the series exists. The operator I_q^n is given by $I_q^0[h](x) = h(x)$ and $I_q^n[h](x) = I_q[I_q^{n-1}[h]](x)$ for $n \ge 1$ and $h \in C([0, T])$ [6, 7]. It has been proved that $D_q[I_q[h]](x) = h(x)$ and $I_q[D_q[h]](x) = h(x) - h(0)$ whenever h is continuous at $x = 0$ [6, 7]. The fractional Riemann–Liouville type q-integral of the function h on $J = (0, 1)$ for $\alpha \ge 0$ is defined by $\mathcal{I}_q^0[h](t) = h(t)$ and

$$\begin{aligned} \mathcal{I}_q^{\alpha}[h](t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} h(s)\, \mathrm{d}_q s \\ &= t^{\alpha} (1-q)^{\alpha} \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1} (1 - q^{\alpha+i})}{\prod_{i=1}^{k-1} (1 - q^{i+1})} h\bigl(t q^k\bigr) \end{aligned} \tag{5}$$

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and $n = 1, 2, \ldots, 40$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	<u>49.065390</u>	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	<u>11.257095</u>	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	<u>259.967394</u>
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

for $t \in J$ [11, 18]. One can use Algorithm 5 for calculating $\mathcal{I}_q^\alpha[h](t)$ according to Eq. (5). Also, the Caputo fractional q-derivative of a function h is defined by

$$\begin{aligned} {}^c\mathcal{D}_q^\alpha[h](t) &= \mathcal{I}_q^{[\alpha]-\alpha}\left[\mathcal{D}_q^{[\alpha]}[h]\right](t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t-qs)^{([\alpha]-\alpha-1)} \mathcal{D}_q^{[\alpha]}[h](s)\, \mathrm{d}_q s, \end{aligned} \tag{6}$$

where $t \in J$ and $\alpha > 0$ [18]. It has been proved that $\mathcal{I}_q^\beta[\mathcal{I}_q^\alpha[h]](x) = \mathcal{I}_q^{\alpha+\beta}[h](x)$ and $\mathcal{D}_q^\alpha[\mathcal{I}_q^\alpha[h]](x) = h(x)$, where $\alpha, \beta \geq 0$ [18]. Algorithm 5 shows pseudo-code $\mathcal{I}_q^\alpha[h](x)$.

We use $\|y\| = \max_{t\in\bar{J}} |y(t)|$ as the norm of $A = B = C^1(\bar{J})$. Clearly, $(A, \|.\|)$ and $(B, \|.\|)$ are Banach spaces. Also, the product space $(A \times B, \|(y,z)\|)$ is a Banach space where $\|(y,z)\| = \|y\| + \|z\|$. An operator $\mathcal{O} : A \to A$ is called completely continuous if restricted to any bounded set in A is compact.

Lemma 1 (Leray–Schauder alternative [48, p.4]) *Let $\mathcal{O} : \mathcal{Y} \to \mathcal{Y}$ be completely continuous and $\Omega(\mathcal{O}) = \{x \in \mathcal{Y} | x = \lambda\mathcal{O}(x)\}$, where $\lambda \in (0,1)$. Then either the set $\Omega(\mathcal{O})$ is unbounded or $\mathcal{O}$ has at least one fixed point.*

3 Main results

The main results are presented in this section. To facilitate exposition, we will provide our analysis in two separate folds.

3.1 The nonlinear sum and integral boundary value problem (1)–(2)

First, we provide our key lemma.

Lemma 2 *The function $x_0 \in \mathcal{A}$ is a solution for problem* (1) *under the sum and integral boundary value conditions* (2) *if and only if x_0 is a solution for the fractional q-integral* *equation*

$$x_0(t) = \int_0^1 G_q(t,r) w_1\bigl(r, x_0(r), \varphi\bigl(x_0(r)\bigr)\bigr)\, \mathrm{d}_q r,$$

where

$$\begin{aligned} G_q(t,r) = {} & -\frac{(t-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{[\eta(\frac{1}{2}-t)+1](1-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi(t-1)+\frac{1}{2})-\Xi](b-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi-1+t)](1-qr)^{(\alpha)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha+1)} \\ & + \frac{[\eta\Xi(\Xi-2)+(\Xi-1+t)](a-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \end{aligned}$$

whenever $r \leq a$ *and* $0 \leq r \leq t \leq 1$,

$$\begin{aligned} G_q(t,r) = {} & -\frac{(t-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{[\eta(\frac{1}{2}-t)+1](1-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi(t-1)+\frac{1}{2})-\Xi](b-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi-1+t)](1-r)^{(\alpha)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha+1)} \end{aligned}$$

whenever $r \leq b$ *and* $0 \leq a \leq r \leq t \leq 1$,

$$\begin{aligned} G_q(t,r) = {} & -\frac{(t-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{[\eta(\frac{1}{2}-t)+1](1-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi-1+t)](1-r)^{(\alpha)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha+1)} \end{aligned}$$

whenever $0 \leq a \leq b \leq r \leq t \leq 1$,

$$\begin{aligned} G_q(t,r) = {} & \frac{[\eta(\frac{1}{2}-t)+1](1-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi(t-1)+\frac{1}{2})-\Xi](b-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \\ & + \frac{[\eta(\Xi-1+t)](1-qr)^{(\alpha)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha+1)} \\ & + \frac{[\eta\Xi(\Xi-2)+(\Xi-1+t)](a-qr)^{(\alpha-2)}}{[\eta(\Xi-\frac{1}{2})+1]\Gamma_q(\alpha-1)} \end{aligned}$$

whenever $0 \le t \le r \le a \le b \le 1$,

$$G_q(t,r) = \frac{[\eta(\frac{1}{2} - t) + 1](1-qr)^{(\alpha-2)}}{[\eta(\Xi - \frac{1}{2}) + 1]\Gamma_q(\alpha-1)} + \frac{[\eta(\Xi(t-1) + \frac{1}{2}) - \Xi](b-qr)^{(\alpha-2)}}{[\eta(\Xi - \frac{1}{2}) + 1]\Gamma_q(\alpha-1)} + \frac{[\eta(\Xi - 1 + t)](1-qr)^{(\alpha)}}{[\eta(\Xi - \frac{1}{2}) + 1]\Gamma_q(\alpha+1)}$$

whenever $a \le r$ *and* $0 \le t \le r \le b \le 1$, *and*

$$G_q(t,r) = \frac{[\eta(\frac{1}{2} - t) + 1](1-qr)^{(\alpha-2)}}{[\eta(\Xi - \frac{1}{2}) + 1]\Gamma_q(\alpha-1)} + \frac{[\eta(\Xi - 1 + t)](1-qr)^{(\alpha)}}{[\eta(\Xi - \frac{1}{2}) + 1]\Gamma_q(\alpha+1)}$$

whenever $b \le r$ *and* $0 \le t \le r \le 1$.

Proof Let x_0 be a solution for Eq. (1)–(2). Take $v_0(t) = w_1(t, x_0(t), \varphi(x_0(t)))$. Choose $d_0, d_1 \in \mathbb{R}$ such that

$$x_0(t) = -\int_0^t \frac{(t-qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_0(r)\,\mathrm{d}_q r + d_0 + d_1 t. \tag{7}$$

Thus, we obtain $x_0'(t) = -\mathcal{I}_q^{\alpha-1}[v_0](t) + d_1$. At present, by using the boundary conditions (2), we conclude that $d_1 = \mathcal{I}_q^{\alpha-1}[v_0](a) - \eta \int_0^1 x_0(r)\,\mathrm{d}r$ and

$$d_0 = \mathcal{I}_q^{\alpha-1}[v_0](1) - \Xi \mathcal{I}_q^{\alpha-1}[v_0](b) + (\Xi - 1)\mathcal{I}_q^{\alpha-1}[v_0](a) + \eta(1-\Xi)\int_0^1 x_0(r)\,\mathrm{d}r.$$

Hence, by substituting d_0 in Eq. (7), we get

$$x_0(t) = -\mathcal{I}_q^{\alpha}[v_0](t) + \mathcal{I}_q^{\alpha-1}[v_0](1) - \Xi \mathcal{I}_q^{\alpha-1}[v_0](b) + (\Xi - 1)\mathcal{I}_q^{\alpha-1}[v_0](a) + \eta(1-\Xi)\int_0^1 x_0(r)\,\mathrm{d}r + t\mathcal{I}_q^{\alpha-1}[v_0](a) - \eta t \int_0^1 x_0(r)\,\mathrm{d}r. \tag{8}$$

Put $\delta = \int_0^1 x_0(r)\,\mathrm{d}r$. By computing the value of δ and substituting it in (8), we get

$$x_0(t) = -\mathcal{I}_q^{\alpha}[v_0](t) + \frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{3}{2}) + 1}\mathcal{I}_q^{\alpha-1}[v_0](1) + \frac{\eta(\Xi(t-1) + \frac{1}{2}) - \Xi}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha-1}[v_0](b) + \frac{\eta(\Xi - 1 + t)}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha+1}[v_0](1)$$

$$\begin{aligned}&+\frac{\eta\varXi(\varXi-2)+(\varXi-1+t)}{\eta(\varXi-\frac{1}{2})+1}\mathcal{I}_q^{\alpha-1}[v_0](a)\\&=\int_0^1 G_q(t,r)w_1\big(r,x_0(r),\varphi\big(x_0(r)\big)\big)\,\mathrm{d}_q r.\end{aligned}$$

Thus, x_0 is a solution for the fractional q-integral equation (7). It is obvious that x_0 is a solution for the fractional q-integro-differential equation (1) whenever x_0 is a solution for the fractional q-integral equation. This completes the proof. □

Theorem 3 *Let $g\in C(\overline{J},\mathbb{R})$ be a bounded function with upper bound $L>0$. Assume that for each $t\in\overline{J}$ there exist positive continuous functions $m_1(t)$ and $m_2(t)$ such that*

$$\begin{aligned}&\big\|w_1\big(t,x(t),\varphi\big(x(t)\big)\big)-w_1\big(t,y(t),\varphi\big(y(t)\big)\big)\big\|\\&\quad\le m_1(t)\|x-y\|+Lm_2(t)\|x-y\|\end{aligned}$$

for $x,y\in\mathcal{A}$. Also, put

$$\begin{aligned}M_0&=\max\Big\{\sup_{t\in\overline{J}}\big|\mathcal{I}_q^{\alpha}[m_1](t)\big|,\sup_{t\in\overline{J}}L\big|\mathcal{I}_q^{\alpha}[m_2](t)\big|\Big\},\\M_1&=\max\Big\{\sup_{t\in\overline{J}}\big|\mathcal{I}_q^{\alpha+1}[m_1](1)\big|,\sup_{t\in\overline{J}}L\big|\mathcal{I}_q^{\alpha+1}[m_2](1)\big|\Big\},\end{aligned}\tag{9}$$

and

$$M(s)=\max\Big\{\sup_{t\in\overline{J}}\big|\mathcal{I}_q^{\alpha-1}[m_1](s)\big|,\sup_{t\in\overline{J}}L\big|\mathcal{I}_q^{\alpha-1}[m_2](s)\big|\Big\}\tag{10}$$

for $s=1$, $s=a$, $s=b$, and

$$\begin{aligned}\Delta&=M_0+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}M_1+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}M(1)\\&\quad+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}M(a)+\frac{\frac{1}{2}\eta\varXi-\varXi}{\eta(\varXi-\frac{1}{2})+1}M(b).\end{aligned}\tag{11}$$

If $\Delta<1$, then the nonlinear fractional q-integro-differential equation (1)–(2) has a unique solution.

Proof We define the operator $\varTheta: C(\overline{J})\to C(\overline{J})$ by

$$\begin{aligned}(\varTheta y)(t)&=-\mathcal{I}_q^{\alpha}[w_1]\big(t,x(t),\varphi\big(x(t)\big)\big)\\&\quad+\frac{\eta(\frac{1}{2}-t)+1}{\eta(\varXi-\frac{1}{2})+1}\mathcal{I}_q^{\alpha-1}[w_1]\big(1,x(1),\varphi\big(x(1)\big)\big)\\&\quad+\frac{\eta(\varXi(t-1)+\frac{1}{2})-\varXi}{\eta(\varXi-\frac{1}{2})+1}\mathcal{I}_q^{\alpha-1}[w_1]\big(b,x(b),\varphi\big(x(b)\big)\big)\\&\quad+\frac{\eta(\varXi-1+t)}{\eta(\varXi-\frac{1}{2})+1}\mathcal{I}_q^{\alpha+1}[w_1]\big(1,x(1),\varphi\big(x(1)\big)\big)\end{aligned}$$

$$+\frac{\eta\varXi(\varXi-2)+(\varXi-1+t)}{\eta(\varXi-\frac{1}{2})+1}\mathcal{I}_q^{\alpha-1}[w_1]\bigl(a,x(a),\varphi\bigl(x(a)\bigr)\bigr).$$

Take $\ell=\sup_{t\in\bar{J}}|w_1(t,0,0)|$ and choose $r_0>0$ such that

$$\begin{aligned}r_0&\geq\frac{\ell}{1-k}\biggl[\frac{1}{\Gamma_q(\alpha+1)}+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha)}+\frac{\frac{1}{2}\eta-\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{b^{(\alpha-1)}}{\Gamma_q(\alpha)}\\&\quad+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha+2)}+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{\eta^{(\alpha-1)}}{\Gamma_q(\alpha)}\biggr].\end{aligned}$$

Put $B_{r_0}=\{x\in\mathcal{A}:\|x\|\leq r_0\}$. Let $x\in B_{r_0}$. Then we have

$$\begin{aligned}\bigl\|(\Theta x)(t)\bigr\|&\leq\mathcal{I}_q^{\alpha}\bigl(\bigl\|w_1\bigl(t,x(t),\varphi\bigl(x(t)\bigr)\bigr)-w_1(t,0,0)\bigr\|+\bigl\|w_1(t,0,0)\bigr\|\bigr)\\&\quad+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}\\&\quad\times\mathcal{I}_q^{\alpha-1}\bigl(\bigl\|w_1\bigl(1,x(1),\varphi\bigl(x(1)\bigr)\bigr)-w_1(1,0,0)\bigr\|+\bigl\|w_1(1,0,0)\bigr\|\bigr)\\&\quad+\frac{\frac{1}{2}\eta-\varXi}{\eta(\varXi-\frac{1}{2})+1}\\&\quad\times\mathcal{I}_q^{\alpha-1}\bigl(\bigl\|w_1\bigl(b,x(b),\varphi\bigl(x(b)\bigr)\bigr)-w_1(b,0,0)\bigr\|+\bigl\|w_1(b,0,0)\bigr\|\bigr)\\&\quad+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}\\&\quad\times\mathcal{I}_q^{\alpha+1}\bigl(\bigl\|w_1\bigl(1,x(1),\varphi\bigl(x(1)\bigr)\bigr)-w_1(1,0,0)\bigr\|+\bigl\|w_1(1,0,0)\bigr\|\bigr)\\&\quad+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}\\&\quad\times\mathcal{I}_q^{\alpha-1}\bigl(\bigl\|w_1\bigl(a,x(a),\varphi\bigl(x(a)\bigr)\bigr)-w_1(a,0,0)\bigr\|+\bigl\|w_1(a,0,0)\bigr\|\bigr)\\&\leq\biggl[M_0+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}M_1+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}M(1)\\&\quad+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}M(a)+\frac{\frac{1}{2}\eta\varXi-\varXi}{\eta(\varXi-\frac{1}{2})+1}M(b)\biggr]r_0\\&\quad+\biggl[\frac{1}{\Gamma_q(\alpha+1)}+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha)}+\frac{\frac{1}{2}\eta-\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{b}{\Gamma_q(\alpha)}\\&\quad+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha+2)}+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{a^{(\alpha-1)}}{\Gamma_q(\alpha)}\biggr]\ell\\&=\Delta r_0+\biggl[\frac{1}{\Gamma_q(\alpha+1)}+\frac{\frac{1}{2}\eta+1}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha)}\\&\quad+\frac{\frac{1}{2}\eta-\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{b^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{\eta\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{1}{\Gamma_q(\alpha+2)}\\&\quad+\frac{\eta\varXi(\varXi-2)+\varXi}{\eta(\varXi-\frac{1}{2})+1}\frac{\eta^{(\alpha-1)}}{\Gamma_q(\alpha)}\biggr]\ell\leq r_0.\end{aligned}$$

Hence, $\Theta(B_{r_0}) \subset B_{r_0}$. On the other hand, one can write

$$\begin{aligned}
&\|(\Theta x)(t) - (\Theta y)(t)\| \\
&\le \mathcal{I}_q^{\alpha}\big(m_1(t)\|x-y\| + m_2(t)\|\varphi(x)-\varphi(y)\|\big) \\
&\quad + \frac{\frac{1}{2}\eta + 1}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha-1}\big(m_1(1)\|x-y\| + m_2(1)\|\varphi(x)-\varphi(y)\|\big) \\
&\quad + \frac{\frac{1}{2}\eta - \Xi}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha-1}\big(m_1(b)\|x-y\| + m_2(b)\|\varphi(x)-\varphi(y)\|\big) \\
&\quad + \frac{\eta\Xi}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha+1}\big(m_1(1)\|x-y\| + m_2(1)\|\varphi(x)-\varphi(y)\|\big) \\
&\quad + \frac{\eta\Xi(\Xi-2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1}\mathcal{I}_q^{\alpha-1}\big(m_1(a)\|x-y\| + m_2(a)\|\varphi(x)-\varphi(y)\|\big) \\
&\le \bigg[M_0 + \frac{\eta\Xi}{\eta(\Xi - \frac{1}{2}) + 1}M_1 + \frac{\frac{1}{2}\eta + 1}{\eta(\Xi - \frac{1}{2}) + 1}M(1) \\
&\quad + \frac{\eta\Xi(\Xi-2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1}M(a) + \frac{\frac{1}{2}\eta\Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1}M(b)\bigg]\|x-y\| \\
&= \Delta\|x-y\|.
\end{aligned}$$

Since $\Delta < 1$, Θ is a contraction. Thus, by using the Banach contraction principle, Θ has a unique fixed point x_0 in $\mathcal{A}$. At present, by using Lemma 2, one can get that ${}^c\mathcal{D}_q^{\alpha}[x_0] \in \mathcal{A}$ and x_0 is the unique solution for the fractional q-integro-differential equation (1)–(2). □

3.2 The nonlinear boundary value problem (3)–(4)

Lemma 4 *Let $w_2 : \overline{J} \times \mathcal{B}^3 \to \mathcal{B}$ be a continuous function. An element $x_0 \in \mathcal{B}$ is a solution for the fractional q-integro-differential equation* (3) *under the sum boundary conditions* (4) *if and only if x_0 is a solution for the fractional integral equation*

$$y(t) = \mathcal{I}_q^{\alpha}[v](t) - t\mathcal{I}_q^{\alpha-1}[v](1) + t\Xi\mathcal{I}_q^{\alpha-2}[v](b).$$

Proof Put

$$v_0(t) = w_2\bigg(t, x_0(t), \int_0^t x_0(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^{\zeta}[x_0](t)\bigg).$$

Let x_0 be a solution for the fractional q-integro-differential equation (3). Choose $d_0, d_1 \in \mathbb{R}$ such that $x_0(t) = \mathcal{I}_q^{\alpha}[v_0](t) + d_0 + d_1 t$ for all $t \in \overline{J}$. Hence, $x_0'(t) = \mathcal{I}_q^{\alpha-1}[y_0](t) + d_1$ and $x_0''(t) = \mathcal{I}_q^{\alpha-2}[v_0](t)$. By using the sum boundary conditions (4), we get $d_0 = 0$ and $d_1 = -\mathcal{I}_q^{\alpha-1}[v_0](1) + \Xi\mathcal{I}_q^{\alpha-2}[v_0](b)$. By substituting d_0 and d_1, we obtain

$$x_0(t) = \mathcal{I}_q^{\alpha}[v_0](t) - t\mathcal{I}_q^{\alpha-1}[v_0](1) + t\Xi\mathcal{I}_q^{\alpha-2}[y_0](b).$$

Thus, x_0 is a solution for the fractional q-integral equation. It is obvious that x_0 is a solution for the fractional q-integro-differential equation (3) whenever x_0 is a solution for the fractional q-integral equation. This completes the proof. □

Theorem 5 *Suppose that $w_2 : \bar{J} \times \mathcal{B}^3 \to \mathcal{B}$ is a continuous map and there exist positive continuous functions m_1, m_2, and m_3 such that*

$$\begin{aligned}
&\left| w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^\zeta[x](t) \right) - w_2\left(t, y(t), \int_0^t y(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^\zeta[y](t) \right) \right| \\
&\quad \le m_1(t)|x-y| + m_2(t)\left| \int_0^t x(r)\,\mathrm{d}r - \int_0^t y(r)\,\mathrm{d}r \right| \\
&\qquad + m_3(t)\left| {}^c\mathcal{D}_q^\zeta[x](t) - {}^c\mathcal{D}_q^\zeta[y](t) \right|
\end{aligned}$$

for all $x, y \in \mathcal{B}$ and $t \in \bar{J}$. Let

$$\begin{aligned}
M_0 &= \max\left\{ \sup_{t\in\bar{J}} \left| \mathcal{I}_q^\alpha[m_1](t) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^\alpha[m_2](t) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^\alpha[m_3](t) \right| \right\}, \\
M(1) &= \max\left\{ \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-1}[m_1](1) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-1}[m_2](1) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-1}[m_3](1) \right| \right\}, \\
M(b) &= \max\left\{ \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-2}[m_1](b) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-2}[m_2](b) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-2}[m_3](b) \right| \right\},
\end{aligned} \tag{12}$$

and

$$M(t) = \max\left\{ \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-\zeta}[m_1](t) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-\zeta}[m_2](t) \right|, \sup_{t\in\bar{J}} \left| \mathcal{I}_q^{\alpha-\zeta}[m_3](t) \right| \right\}. \tag{13}$$

Put

$$\Delta = M_0 + M(t) + \frac{\Gamma_q(2-\zeta)+1}{\Gamma_q(2-\zeta)} M(1) + \frac{(1+\Gamma_q(2-\zeta))\Xi}{\Gamma_q(2-\zeta)} M(b). \tag{14}$$

If $\Delta < 1$, then the nonlinear fractional q-integro-differential equation (3)–(4) has a unique solution.

Proof Define the operator $\Theta : \mathcal{B} \to \mathcal{B}$ by

$$(\Theta x)(t) = \mathcal{I}_q^\alpha[v](t) - t\mathcal{I}_q^{\alpha-1}[v](1) + t\Xi\mathcal{I}_q^{\alpha-2}[v](b),$$

where

$$v(t) = w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^\zeta[x](t) \right).$$

Choose $r > 0$ such that

$$\begin{aligned}
r \ge \frac{\ell}{1-k}\Bigg[&\frac{1}{\Gamma_q(\alpha-\zeta+1)} + \frac{1+\alpha}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(2-\zeta)\Gamma_q(\alpha)} \\
&+ \frac{[\Gamma_q(2-\zeta)+1]b^{(\alpha-2)}\Xi}{\Gamma_q(2-\zeta)\Gamma_q(\alpha-1)} \Bigg],
\end{aligned}$$

where $\ell = \sup_{t\in\bar{J}} |w_2(t,0,0,0)|$. We show that $\Theta B_{r_0} \subset B_{r_0}$, where

$$B_{r_0} = \left\{ x \in \mathcal{B} : \|x\| \le r_0 \right\}.$$

Let $x \in B_{r_0}$. Then

$$\begin{aligned}
\left|(\Theta x)(t)\right| &\leq \mathcal{I}_q^{\alpha}\left(\left|w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^{\zeta}[x](t)\right) - w_2(t,0,0,0)\right|\right.\\
&\quad \left. + \left|w_2(t,0,0,0)\right|\right)\\
&\quad + \mathcal{I}_q^{\alpha-1}\left(\left|w_2\left(1, x(1), \int_0^1 x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^{\zeta}[x](1)\right) - w_2(1,0,0,0)\right|\right.\\
&\quad \left. + \left|w_2(1,0,0,0)\right|\right)\\
&\quad + \Xi\mathcal{I}_q^{\alpha-2}\left(\left|w_2\left(b, x(b), \int_0^b x(r)\,\mathrm{d}r, {}^c\mathcal{D}_q^{\zeta}[x](b)\right) - w_2(b,0,0,0)\right|\right.\\
&\quad \left. + \left|w_2(b,0,0,0)\right|\right)\\
&\leq M_0\left|x(t)\right| + \frac{\ell}{\Gamma_q(\alpha+1)} + M(1)\left|x(t)\right| + \frac{\ell}{\Gamma_q(\alpha)}\\
&\quad + \Xi\left(M(b)\left|x(t)\right|\right) + \frac{\ell\,\Xi\, b^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\\
&\leq \left(M_0 + M(1) + \Xi M(b)\right)\left|x(t)\right|\\
&\quad + \ell\left[\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha)} + \frac{b^{(\alpha-2)}\Xi}{\Gamma_q(\alpha-1)}\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\left|{}^c\mathcal{D}^{\zeta}[\Theta x](t)\right| &\leq M(t)\left|x(t)\right| + \frac{\ell}{\Gamma_q(\alpha-\zeta+1)} + \frac{1}{\Gamma_q(2-\zeta)}M(1)\left|x(t)\right|\\
&\quad + \frac{\ell}{\Gamma_q(2-\zeta)\Gamma_q(\alpha)} + \frac{\Xi}{\Gamma_q(2-\zeta)}\left(M(b)\left|x(t)\right|\right)\\
&\quad + \frac{\ell\,\Xi\, b^{\alpha-2}}{\Gamma_q(2-\zeta)\Gamma_q(\alpha-1)}\\
&\leq \left[M(t) + \frac{1}{\Gamma_q(2-\zeta)}M(1) + \frac{\Xi}{\Gamma_q(2-\zeta)}M(b)\right]\left|x(t)\right|\\
&\quad + \ell\left[\frac{1}{\Gamma_q(\alpha-\zeta+1)} + \frac{1}{\Gamma_q(2-\zeta)\Gamma_q(\alpha)} + \frac{\Xi\, b^{\alpha-2}}{\Gamma_q(2-\zeta)\Gamma_q(\alpha-1)}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|(\Theta x)(t)\right| &\leq \Delta r_0 + \ell\left[\frac{1}{\Gamma_q(\alpha-\zeta+1)} + \frac{1+\alpha}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(2-\alpha)\Gamma_q(\alpha)}\right.\\
&\quad \left. + \frac{\Xi\, b^{\alpha-2}[\Gamma_q(2-\zeta)+1]}{\Gamma_q(2-\zeta)\Gamma_q(\alpha-1)}\right] \leq r_0
\end{aligned}$$

and so $\Theta(B_{r_0}) \subseteq B_{r_0}$. Let $u, v \in X$ and $t \in J$. Then we have

$$\begin{aligned}\bigl|(\Theta x)(t) - (\Theta y)(t)\bigr| &\leq \mathcal{I}_q^{\alpha}\left(\sum_{i=1}^{3} m_i(t)\bigl|x(t) - y(t)\bigr|\right) \\ &\quad + \mathcal{I}_q^{\alpha-1}\left(\sum_{i=1}^{3} m_i(1)\bigl|x(1) - y(1)\bigr|\right) \\ &\quad + \Xi\mathcal{I}_q^{\alpha-2}\left(\sum_{i=1}^{3} m_i(b)\bigl|x(b) - y(b)\bigr|\right) \\ &\leq \bigl(M_0 + M(1) + \Xi M(b)\bigr)|x - y|.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\bigl|{}^c\mathcal{D}^{\zeta}[\Theta x](t) - {}^c\mathcal{D}^{\zeta}[\Theta y](t)\bigr| \\ &\quad\leq \left[M(t) + \frac{1}{\Gamma_q(2-\zeta)}I^{\alpha-1}M(1) + \frac{\Xi}{\Gamma_q(2-\zeta)}M(b)\right]|x - y|.\end{aligned}$$

Hence,

$$\begin{aligned}\bigl\|(\Theta x)(t) - (\Theta y)(t)\bigr\| &\leq \left[M(t) + M_0 + \frac{\Gamma_q(2-\zeta) + 1}{\Gamma_q(2-\zeta)}M(1) \right. \\ &\quad \left. + \frac{\Xi[1 + \Gamma_q(2-\zeta)]}{\Gamma_q(2-\zeta)}M(b)\right]\|x - y\| \\ &= \Delta\|x - y\|.\end{aligned}$$

Since $\Delta < 1$, Θ is a contraction and so, by using the Banach contraction principle, Θ has a unique fixed point. By using Lemma 4, it is clear that the unique fixed point of Θ is the unique solution for the nonlinear fractional integro-differential problem (3)–(4). □

4 Examples, numerical results, and algorithms

Herein, we give an example to show the validity of the main results. In this way, we give a computational technique for checking problems (1)–(2) and (3)–(4). We need to present a simplified analysis that is able to execute the values of the q-gamma function. For this purpose, we provided a pseudo-code description of the method for calculation of the q-gamma function of order n in Algorithms 2, 3, 4, and 5; for more details, follow these addresses https://en.wikipedia.org/wiki/Q-gamma_function and https://www.dm.uniba.it/members/garrappa/software. Tables 1, 2, and 3 show the values $\Gamma_q(z)$ for some z and $q \in (0, 1)$.

For problems for which the analytical solution is not known, we will use, as reference solution, the numerical approximation obtained with a tiny step h by the implicit trapezoidal PI rule, which, as we will see, usually shows an excellent accuracy [49]. All the experiments are carried out in MATLAB Ver. 8.5.0.197613 (R2015a) on a computer equipped with a CPU AMD Athlon(tm) II X2 245 at 2.90 GHz running under the operating system Windows 7.

Algorithm 4 The proposed method for calculated $\int_a^b f(r) d_q r$

```
function g = Iq(q, x, n, fun)
p=1;
for k=0:n
        p=p+ q^k*fun(x*q^k);
end;
g=x* (1-q) * p;
end
```

Algorithm 5 The proposed method for calculated $I_q^\alpha[x]$

```
function g = Iq_alpha(q, alpha,  x, n, fun)
p=0;
for k=0:n
    s1=1;
    for i=0:k-1
        s1=s1*(1-q^(alpha+i));
    end
    s2=1;
    for i=0:k-1
        s2=s2*(1-q^(i+1));
    end
    p=p + q^k*s1*eval(subs(fun, t*q^k))/s2;
end;
g=round((t^alpha)* ((1-q)^alpha)* p, 6);
end
```

Example 1 Consider the fractional q-integro-differential equation similar to problem (1) as follows:

$$ {}^{c}\mathcal{D}^{\frac{3}{2}}[x](t) + \frac{|x(t)|}{7(t^2+\frac{7}{4})^2(2+|x(t)|)} + \frac{t}{1600}\int_0^t e^{(-\frac{1}{3})} x(r)\,\mathrm{d}r = 0, \tag{15} $$

under sum and integral boundary value conditions $x'(\frac{1}{4}) = -\frac{1}{6}\int_0^1 x(r)\,\mathrm{d}r$ and

$$ x'(1) + x(0) = \sum_{i=1}^{5} c_i x'\left(\frac{3}{4}\right). $$

Note that $x'(\frac{3}{4}) = \frac{\partial}{\partial t}x(t)|_{\frac{3}{4}}$. Clearly, $\alpha = \frac{3}{2}$, $a = \frac{1}{4}$, $\eta = \frac{1}{6}$, $m = 5$, $b = \frac{3}{4}$. Let $c_1 = \frac{1}{8}$, $c_2 = \frac{-1}{5}$, $c_3 = \frac{3}{7}$, $c_4 = \frac{1}{3}$, and $c_5 = \frac{1}{6}$. Note that $\varXi = \sum_{i=1}^{5} c_i = \frac{239}{280}$ and so $2\varXi > -1$. We define the maps $w_1 : \overline{J} \times \mathcal{A}^2 \to \mathcal{A}$ and $g : \overline{J} \to [0,\infty)$ by

$$ w_1\big(t, x(t), \phi x(t)\big) = \frac{|x(t)|}{7(t^2+\frac{7}{4})^2(2+|x(t)|)} + \frac{t}{1600}\int_0^t e^{(-\frac{1}{3})} x(r)\,\mathrm{d}r $$

and $g(r) = \frac{1}{40}$ for all $t \in \overline{J}$, respectively. It is obvious that $g(t) \leq 0.025 = L$ for $t \in \overline{J}$. Now, we obtain

$$ \begin{aligned} &\big\| w_1\big(t, x(t), \varphi\big(x(t)\big)\big) - w_1\big(t, y(t), \varphi\big(y(t)\big)\big) \big\| \\ &\quad = \bigg\| \frac{|x(t)|}{7(t^2+\frac{7}{4})^2(2+|x(t)|)} + \frac{t}{1600}\int_0^t e^{(-\frac{1}{2})} x(r)\,\mathrm{d}r \\ &\qquad - \bigg[\frac{|y(t)|}{7(t^2+\frac{7}{4})^2(2+|y(t)|)} + \frac{t}{1600}\int_0^t e^{(-\frac{1}{2})} y(r)\,\mathrm{d}r \bigg] \bigg\| \end{aligned} $$

Table 4 Some numerical results of $\mathcal{I}_q^{\alpha}[m_1](t)$ in Example 1 for $t \in \overline{J}$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	$\mathcal{I}_q^{\alpha}[m_1](t)$			$\mathcal{I}_q^{\alpha+1}[m_1](1)$	$\mathcal{I}_q^{\alpha-1}[m_1](s)$		
	$t=0$	$t=1$	sup		$s=1$	$s=a$	$s=b$
$q=\frac{1}{8}$							
1	0	0.0516	0.0516	0.0454	0.0556	0.0404	0.0561
2	0	0.0527	0.0527	0.0465	0.0564	0.0408	0.0569
3	0	0.0529	<u>0.0529</u>	0.0466	0.0565	0.0409	0.0570
4	0	0.0529	0.0529	0.0466	0.0566	0.0409	0.057
$q=\frac{1}{2}$							
1	0	0.0347	0.0347	0.0196	0.0515	0.0362	0.0514
2	0	0.0446	0.0446	0.0265	0.0586	0.0399	0.0576
3	0	0.0499	0.0499	0.0305	0.062	0.0416	0.0605
4	0	0.0526	0.0526	0.0325	0.0636	0.0424	0.062
5	0	0.0539	0.0539	0.0336	0.0644	0.0428	0.0627
6	0	0.0546	0.0546	0.0341	0.0649	0.043	0.063
7	0	0.055	0.055	0.0344	0.0651	0.0431	0.0632
8	0	0.0552	<u>0.0552</u>	0.0345	0.0652	0.0432	0.0633
9	0	0.0552	0.0552	0.0346	0.0652	0.0432	0.0633
$q=\frac{6}{7}$							
1	0	0.0067	0.0067	0.0013	0.0293	0.0216	0.0298
2	0	0.011	0.011	0.0025	0.0363	0.0261	0.0366
3	0	0.0154	0.0154	0.004	0.0418	0.0294	0.0418
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	0	0.0551	0.0551	0.0257	0.0705	0.0446	0.0673
38	0	0.0551	0.0551	0.0257	0.0705	0.0447	0.0673
39	0	0.0552	0.0552	0.0257	0.0705	0.0447	0.0673
40	0	0.0552	0.0552	0.0257	0.0705	0.0447	0.0674
41	0	0.0552	0.0552	0.0258	0.0705	0.0447	0.0674
42	0	0.0552	0.0552	0.0258	0.0705	0.0447	0.0674
43	0	0.0553	<u>0.0553</u>	0.0258	0.0706	0.0447	0.0674

$$\begin{aligned}&\leq \left\| \frac{|x(t)|}{7(t^2+\frac{7}{4})^2} - \frac{|y(t)|}{7(t^2+\frac{7}{4})^2} \right\| \\ &\quad + \left\| \frac{t}{1600} \int_0^t e^{(-\frac{1}{2})}\bigl(x(r)-y(r)\bigr)\,\mathrm{d}r \right\| \\ &\leq \frac{1}{7(t^2+\frac{7}{4})^2}\|x-y\| + \frac{t}{1600}\|x-y\|.\end{aligned}$$

Thus

$$\bigl\| w_1\bigl(t,x(t),\varphi\bigl(x(t)\bigr)\bigr) - w_1\bigl(t,y(t),\varphi\bigl(y(t)\bigr)\bigr) \bigr\| \leq \left[\frac{1}{7(t^2+\frac{7}{4})^2} + \frac{t}{1600}\right]\|x-y\|$$

for all $t \in J$, $x, y \in \mathcal{A}$. We define the positive continuous maps $m_1(t) = \frac{1}{7(t^2+\frac{7}{4})^2}$ and $m_2(t) = \frac{t}{40}$. At present, by using Eqs. (9)–(10) and applying Algorithm 5, we calculate $\sup \mathcal{I}_q^{\alpha}[m_1](t)$, $\sup L\mathcal{I}_q^{\alpha}[m_2](t)$, $\sup \mathcal{I}_q^{\alpha+1}[m_1](t)$, $\sup L\mathcal{I}_q^{\alpha+1}[m_2](t)$, $\sup \mathcal{I}_q^{\alpha-1}[m_1](t)$, and $\sup L\mathcal{I}_q^{\alpha-1}[m_2](t)$ for $t \in (0,1)$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$. Tables 4 and 5 show these results. Also, Figures 1, 2 and 3 illustrate the numerical results of the tables. Therefore

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha}[m_1](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{3}{2}}\left(\frac{1}{7(t+\frac{7}{4})^2}\right) = 0.0529, 0.0552, 0.0553,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha+1}[m_1](1) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{3}{2}+1}\left(\frac{1}{7(1+\frac{7}{4})^2}\right) = 0.0466, 0.0346, 0.0258,$$

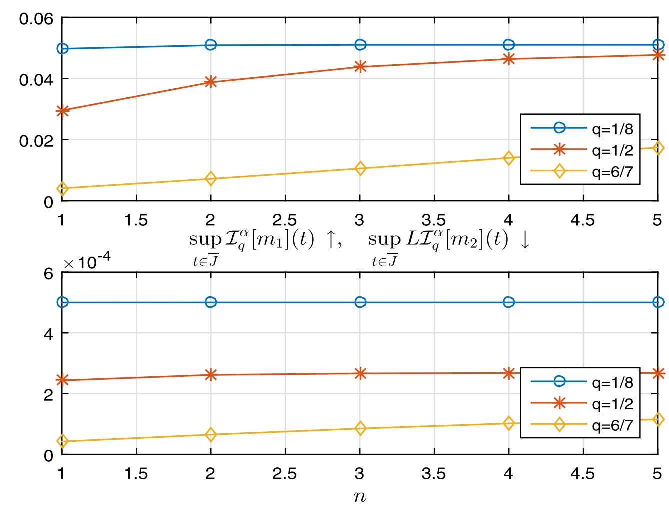

Figure 1 2D graph of $\mathcal{I}_q^{\alpha}[m_1](t)$ and $L\mathcal{I}_q^{\alpha}[m_2](t)$ for $t \in \overline{J}$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$ in Example 1

Table 5 Some numerical results of $\mathcal{I}_q^{\alpha}[m_2](t)$ in Example 1 for $t \in \overline{J}$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	$\mathcal{I}_q^{\alpha}[m_2](t)$			$\mathcal{I}_q^{\alpha+1}[m_2](1)$	$\mathcal{I}_q^{\alpha-1}[m_2](s)$		
	$t=0$	$t=1$	sup		$s=1$	$s=a$	$s=b$
$q=\frac{1}{8}$							
1	0	0.0005	0.0005	0.0005	0.0006	0.0001	0.0004
2	0	0.0005	0.0005	0.0005	0.0006	0.0001	0.0004
3	0	0.0005	0.0005	0.0005	0.0006	0.0001	0.0004
$q=\frac{1}{2}$							
1	0	0.0003	0.0003	0.0002	0.0005	0.0001	0.0003
2	0	0.0003	0.0003	0.0002	0.0005	0.0001	0.0003
3	0	0.0003	0.0003	0.0002	0.0005	0.0001	0.0003
$q=\frac{6}{7}$							
1	0	0.0001	0.0001	0	0.0003	0	0.0002
2	0	0.0001	0.0001	0	0.0004	0	0.0002
3	0	0.0001	0.0001	0	0.0004	0.0001	0.0003
4	0	0.0001	0.0001	0	0.0004	0.0001	0.0003
5	0	0.0002	0.0002	0	0.0004	0.0001	0.0003
6	0	0.0002	0.0002	0.0001	0.0005	0.0001	0.0003
7	0	0.0002	0.0002	0.0001	0.0005	0.0001	0.0003
8	0	0.0002	0.0002	0.0001	0.0005	0.0001	0.0003

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-1}[m_1](1) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{1}{7(1+\frac{7}{4})^2}\right) = 0.0566, 0.0652, 0.0706,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-1}[m_1](a) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{1}{7(\frac{1}{16}+\frac{7}{4})^2}\right) = 0.0409, 0.0432, 0.0447,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-1}[m_1](b) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{1}{7(\frac{9}{16}+\frac{7}{4})^2}\right) = 0.0570, 0.0633, 0.0674$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively, and

$$\sup_{t\in\overline{J}} L\mathcal{I}_q^{\alpha}[m_2](t) = \sup_{t\in\overline{J}} L\mathcal{I}_q^{\frac{3}{2}}\left(\frac{t}{40}\right) = 0.0005, 0.0003, 0.0002,$$

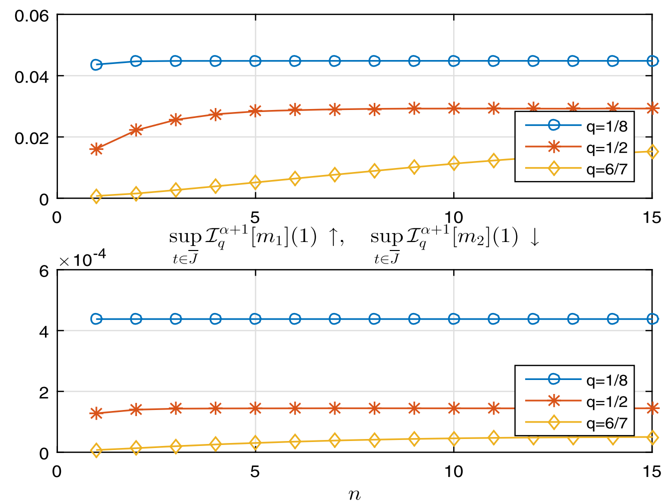

Figure 2 2D graph of $\mathcal{I}_q^{\alpha+1}[m_1](1)$ and $L\mathcal{I}_q^{\alpha+1}[m_2](1)$ for $t \in \overline{J}$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$ in Example 1

$$\sup_{t\in\overline{J}} L\mathcal{I}_q^{\alpha+1}[m_2](1) = \sup_{t\in\overline{J}} L\mathcal{I}_q^{\frac{3}{2}+1}\left(\frac{1}{40}\right) = 0.0005, 0.0002, 0.0001,$$

$$\sup_{t\in\overline{J}} L\mathcal{I}_q^{\alpha-1}[m_2](1) = \sup_{t\in\overline{J}} L\mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{1}{40}\right) = 0.0006, 0.0005, 0.0005,$$

$$\sup_{t\in\overline{J}} L\mathcal{I}_q^{\alpha-1}[m_2](a) = \sup_{t\in\overline{J}} L\mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{1}{160}\right) = 0.0001, 0.0001, 0.0001,$$

$$\sup_{t\in\overline{J}} L\mathcal{I}_q^{\alpha}[m_2](b) = \sup_{t\in\overline{J}} L\mathcal{I}_q^{\frac{3}{2}-1}\left(\frac{3}{160}\right) = 0.0004, 0.0003, 0.0003$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively. Hence, from Eqs. (9)–(10) and the above results in Tables 4 and 5, we obtain $M_0 = \max\{0.0529, 0.0005\} = 0.0529$, $M_1 = \max\{0.0466, 0.0005\} = 0.0466$, $M(1) = \max\{0.0566, 0.0006\} = 0.0566$,

$$M(a) = M\left(\frac{1}{4}\right) = \max\{0.0409, 0.0001\} = 0.0409,$$

$$M(b) = M\left(\frac{3}{4}\right) = \max\{0.0570, 0.0004\} = 0.0570$$

whenever $q = \frac{1}{8}$, $M_0 = \max\{0.0552, 0.0003\} = 0.0552$, $M_1 = \max\{0.0346, 0.0002\} = 0.0346$, $M(1) = \max\{0.0652, 0.0005\} = 0.0652$,

$$M(a) = M\left(\frac{1}{4}\right) = \max\{0.0432, 0.0001\} = 0.0432,$$

$$M(b) = M\left(\frac{3}{4}\right) = \max\{0.0633, 0.0003\} = 0.0633,$$

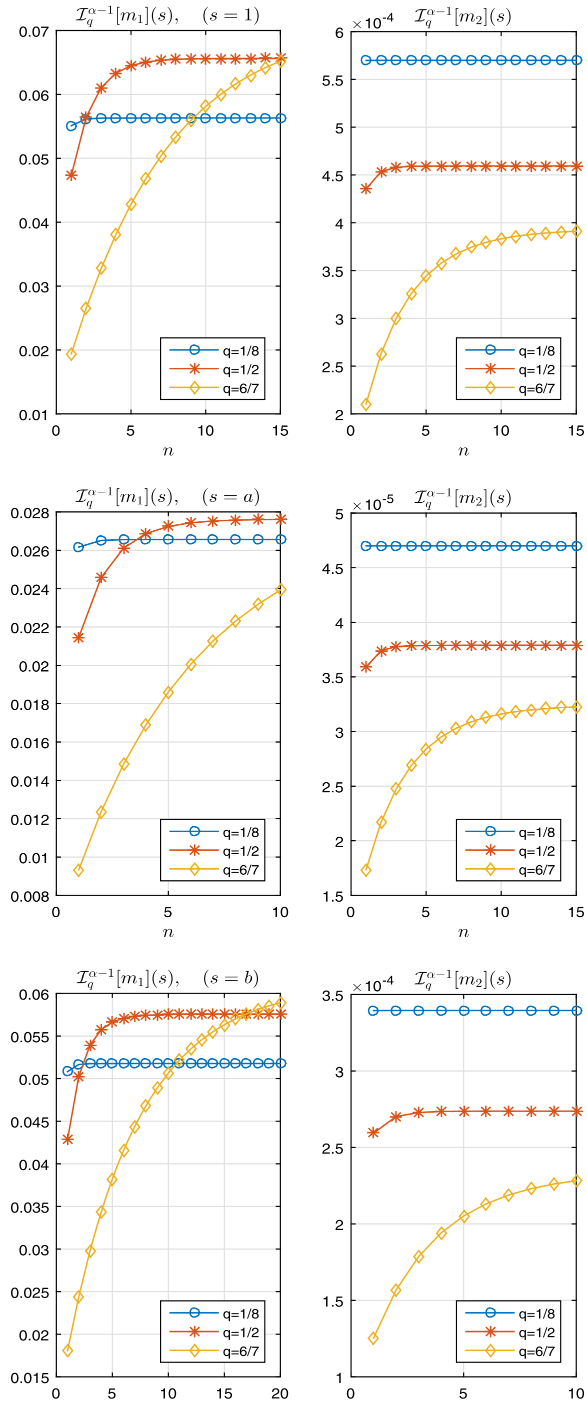

Figure 3 2D graph of $\mathcal{I}_q^{\alpha-1}[m_1](s)$ and $L\mathcal{I}_q^{\alpha-1}[m_2](s)$ for $t \in \overline{J}$ and $s = 1, a, b$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$ in Example 1

whenever $q = \frac{1}{2}$, $M_0 = \max\{0.0553, 0.0002\} = 0.0553$, $M_1 = \max\{0.0258, 0.0001\} = 0.0258$, $M(1) = \max\{0.0706, 0.0005\} = 0.0706$,

$$M(a) = M\left(\frac{1}{4}\right) = \max\{0.0447, 0.0001\} = 0.0447,$$

Table 6 Some numerical results for calculation of M_0, M_1, $M(1)$, $M(a)$, $M(b)$ and $\Delta < 1$ in Example 1 for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	M_0	M_1	$M(1)$	$M(a)$	$M(b)$	Δ
$q = \frac{1}{8}$						
1	0.0516	0.0454	0.0556	0.0404	0.0561	0.0994
2	0.0527	0.0465	0.0564	0.0408	0.0569	0.1013
3	0.0529	0.0466	0.0565	0.0409	0.0570	0.1015
4	0.0529	0.0466	0.0566	0.0409	0.0570	<u>0.1016</u>
5	0.0529	0.0466	0.0566	0.0409	0.0570	0.1016
$q = \frac{1}{2}$						
1	0.0347	0.0196	0.0515	0.0362	0.0514	0.0757
2	0.0446	0.0265	0.0586	0.0399	0.0576	0.0915
3	0.0499	0.0305	0.0620	0.0416	0.0605	0.0997
4	0.0526	0.0325	0.0636	0.0424	0.0620	0.1039
5	0.0539	0.0336	0.0644	0.0428	0.0627	0.1060
6	0.0546	0.0341	0.0649	0.0430	0.0630	0.1071
7	0.0550	0.0344	0.0651	0.0431	0.0632	0.1076
8	0.0552	0.0345	0.0652	0.0432	0.0633	0.1078
9	0.0552	0.0346	0.0652	0.0432	0.0633	0.1080
10	0.0553	0.0346	0.0652	0.0432	0.0634	0.1080
11	0.0553	0.0346	0.0652	0.0432	0.0634	<u>0.1081</u>
12	0.0553	0.0346	0.0653	0.0432	0.0634	0.1081
$q = \frac{6}{7}$						
1	0.0067	0.0013	0.0293	0.0216	0.0298	0.0288
2	0.0110	0.0025	0.0363	0.0261	0.0366	0.0384
3	0.0154	0.0040	0.0418	0.0294	0.0418	0.0470
⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	0.0551	0.0257	0.0705	0.0446	0.0673	0.1100
38	0.0551	0.0257	0.0705	0.0447	0.0673	0.1101
39	0.0552	0.0257	0.0705	0.0447	0.0673	0.1101
40	0.0552	0.0257	0.0705	0.0447	0.0674	0.1102
41	0.0552	0.0258	0.0705	0.0447	0.0674	0.1102
42	0.0552	0.0258	0.0705	0.0447	0.0674	0.1102
43	0.0553	0.0258	0.0706	0.0447	0.0674	<u>0.1103</u>
44	0.0553	0.0258	0.0706	0.0447	0.0674	0.1103

$$M(b) = M\left(\frac{3}{4}\right) = \max\{0.0674, 0.0003\} = 0.0674,$$

whenever $q = \frac{6}{7}$. Also, by using Eq. (11), we can calculate values of Δ. Table 6 shows these results. Thus, by using Eq. (11) we have

$$\begin{aligned}
\Delta &= M_0 + \frac{\eta \varXi}{\eta(\varXi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2}\eta + 1}{\eta(\varXi - \frac{1}{2}) + 1} M(1) \\
&\quad + \frac{\eta \varXi(\varXi - 2) + \varXi}{\eta(\varXi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2}\eta \varXi - \varXi}{\eta(\varXi - \frac{1}{2}) + 1} M(b) \\
&= 0.0529 + \frac{\frac{1}{6} \times \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0466 + \frac{\frac{1}{2} \times \frac{1}{6} + 1}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0566 \\
&\quad + \frac{\frac{1}{6} \times \frac{239}{280}(\frac{239}{280} - 2) + \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0409 + \frac{\frac{1}{2} \times \frac{1}{6} \times \frac{239}{280} - \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0570 \\
&= 0.1016 < 1
\end{aligned}$$

whenever $q = \frac{1}{8}$,

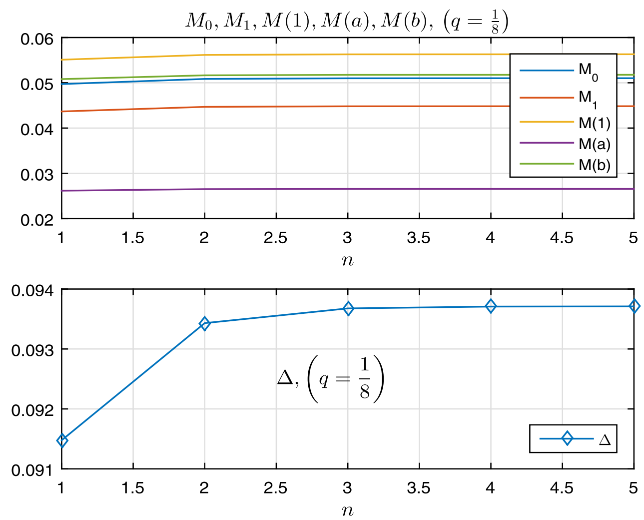

Figure 4 2D graphs M_0, M_1, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{1}{8}$ in Example 1

$$\begin{aligned}
\Delta &= M_0 + \frac{\eta \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2}\eta + 1}{\eta(\Xi - \frac{1}{2}) + 1} M(1) \\
&\quad + \frac{\eta \Xi(\Xi - 2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2}\eta \Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(b) \\
&= 0.0553 + \frac{\frac{1}{6} \times \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0346 + \frac{\frac{1}{2} \times \frac{1}{6} + 1}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0653 \\
&\quad + \frac{\frac{1}{6} \times \frac{239}{280}(\frac{239}{280} - 2) + \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0432 + \frac{\frac{1}{2} \times \frac{1}{6} \times \frac{239}{280} - \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0634 \\
&= 0.1081 < 1
\end{aligned}$$

whenever $q = \frac{1}{2}$, and

$$\begin{aligned}
\Delta &= M_0 + \frac{\eta \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2}\eta + 1}{\eta(\Xi - \frac{1}{2}) + 1} M(1) \\
&\quad + \frac{\eta \Xi(\Xi - 2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2}\eta \Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(b) \\
&= 0.0553 + \frac{\frac{1}{6} \times \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0258 + \frac{\frac{1}{2} \times \frac{1}{6} + 1}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0706 \\
&\quad + \frac{\frac{1}{6} \times \frac{239}{280}(\frac{239}{280} - 2) + \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0447 + \frac{\frac{1}{2} \times \frac{1}{6} \times \frac{239}{280} - \frac{239}{280}}{\frac{1}{6}(\frac{239}{280} - \frac{1}{2}) + 1} \times 0.0674 \\
&= 0.1103 < 1
\end{aligned}$$

whenever $q = \frac{6}{7}$. Figures 4, 5, and 6 show these results (Algorithm 6). Now, by using Theorem 3, the fractional q-integro-differential equation under sum and integral boundary value conditions (15) has a unique solution.

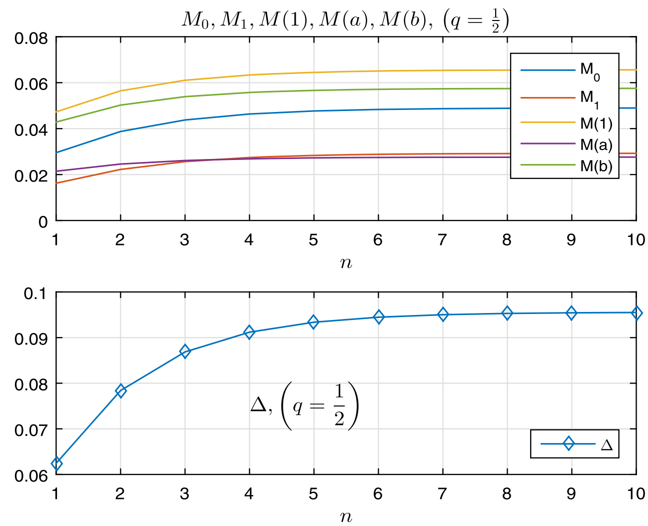

Figure 5 2D graphs M_0, M_1, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{1}{2}$ in Example 1

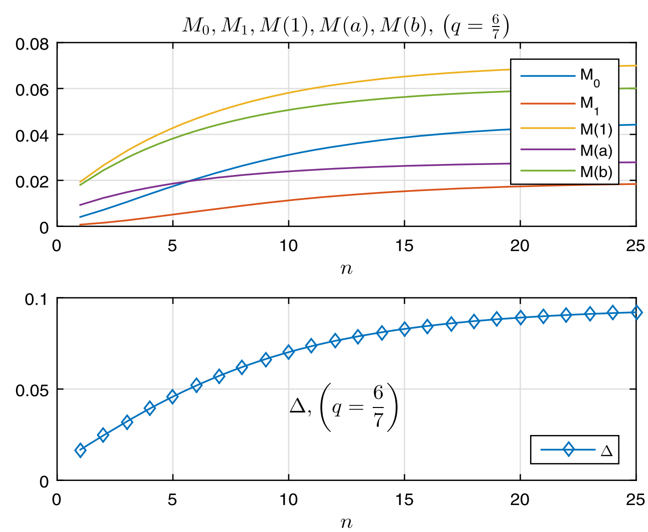

Figure 6 2D graphs M_0, M_1, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{1}{8}$ in Example 1

Example 2 Consider the fractional q-integro-differential equation similar to problem (3) as follows:

$$\begin{aligned} {}^{c}\mathcal{D}^{\frac{9}{5}}[x](t) &= \frac{|t|}{35(1+|t|)} + \frac{2}{35(4+\sqrt{t})}\big|x(t)\big| + \frac{3t}{70}\int_0^t \frac{x(r)}{\sqrt{r}+1}\,\mathrm{d}r \\ &\quad + \frac{3t}{35(t^3+2)}\,{}^{c}\mathcal{D}^{\frac{1}{8}}[x](t), \end{aligned} \tag{16}$$

under the sum boundary value conditions $x'(0) = 0$ and $x'(1) = \sum_{i=1}^{6} c_i x''(\frac{7}{9})$. Clearly, $\alpha = \frac{9}{5}$, $\zeta = \frac{1}{8}$, $b = \frac{7}{9}$, and $m = 6$. Let $c_1 = \frac{7}{12}$, $c_2 = \frac{9}{8}$, $c_3 = \frac{9}{5}$, $c_4 = \frac{2}{3}$, $c_5 = \frac{5}{6}$, $c_6 = \frac{11}{10}$, and so $\Xi =$

Algorithm 6 The MATLAB lines for calculation of all parameters in Example 1

```
format long
q=[1/8 1/2 6/7];
alpha=3/2; syms t;
t0=0; T=1; s=1; a=1/4; b=3/4;
Xi=239/280; eta=1/6;
m1=[sym(1/(7* (t^2 + 7/4)))];
m2=[sym(t/40)];
L=0.025; [xq, yq]=size(q);
column=1;
for j=1:yq
    for n=1:80
        A1(n,column)=n;
        A2(n,column)=n;
        A1(n,column+1)=Iq_alpha(q(j), alpha, t0, n, m1);
        A1(n,column+2)=Iq_alpha(q(j), alpha, T, n, m1);
        A1(n,column+3)=A1(n,column+1);
        if A1(n,column+2)>A1(n,column+1)
            A1(n,column+3)=A1(n,column+2);
        end;
        A2(n,column+1)=L*Iq_alpha(q(j), alpha, t0, n, m2);
        A2(n,column+2)=L*Iq_alpha(q(j), alpha, T, n, m2);
        A2(n,column+3)=A2(n,column+1);
        if A2(n,column+2)>A2(n,column+1)
            A2(n,column+3)=A2(n,column+2);
        end;
        A1(n,column+4)=Iq_alpha(q(j), alpha+1, T, n, m1);
        A2(n,column+4)=L*Iq_alpha(q(j), alpha+1, T, n, m2);
        s=T;
        A1(n,column+5)=Iq_alpha(q(j), alpha-1, s, n, m1);
        A2(n,column+5)=L*Iq_alpha(q(j), alpha-1, s, n, m2);
        s=a;
        A1(n,column+6)=Iq_alpha(q(j), alpha-1, s, n, m1);
        A2(n,column+6)=L*Iq_alpha(q(j), alpha-1, s, n, m2);
        s=b;
        A1(n,column+7)=Iq_alpha(q(j), alpha-1, s, n, m1);
        A2(n,column+7)=L*Iq_alpha(q(j), alpha-1, s, n, m2);
    end;
    column=column+8;
end;
column=1;
Acolumn=1;
for j=1:yq
    for n=1:80
        M(n,column)=n;
        M(n,column+1)=A1(n,Acolumn+3);
        if A2(n,Acolumn+3)> A1(n,Acolumn+3)
            M(n,column+1)=A2(n,Acolumn+3);
        end;
        M(n,column+2)=A1(n,Acolumn+4);
        if A2(n,Acolumn+4)> A1(n,Acolumn+4)
            M(n,column+2)=A2(n,Acolumn+4);
        end;
        M(n,column+3)=A1(n,Acolumn+5);
        if A2(n,Acolumn+5)> A1(n,Acolumn+5)
            M(n,column+3)=A2(n,Acolumn+5);
        end;
        M(n,column+4)=A1(n,Acolumn+6);
        if A2(n,Acolumn+6)> A1(n,Acolumn+6)
            M(n,column+4)=A2(n,Acolumn+6);
        end;
        M(n,column+5)=A1(n,Acolumn+7);
        if A2(n,Acolumn+7)> A1(n,Acolumn+7)
            M(n,column+5)=A2(n,Acolumn+7);
        end;
    end;
    column = column + 6;
    Acolumn =Acolumn+8;
end;
column=1;
Mcolumn=1;
for j=1:yq
    for n=1:80
        Delta(n, column)=n;
        Delta(n, column+1) = M(n, Mcolumn+1) + ...
            eta*Xi*M(n,Mcolumn+2)/(eta*(Xi -1/2) +1) + (eta/2 + ...
            1)*M(n,Mcolumn+3)/(eta* (Xi -1/2) +1) + (eta * Xi*(Xi ...
```

Algorithm 6 (Continued)

```
                -2) + Xi)*M(n,Mcolumn+4)/(eta* (Xi-1/2) +1) + (eta* ...
                Xi/2 - Xi)* M(n,Mcolumn+5)/ (eta* (Xi - 1/2) +1);
        end;
        column=column+2;
        Mcolumn=Mcolumn+6;
    end;
```

$\sum_{i=1}^{5} c_i = \frac{733}{120} = 6.1083$. We define the map $w_2 : \overline{J} \times \mathcal{B}^2 \to \mathcal{B}$ by

$$w_2\big(t, x(t), y(t), z(t)\big) = \frac{|t|}{35(1+|t|)} + \frac{2}{35(4+\sqrt{t})}\big|x(t)\big| + \frac{3t}{70}\int_0^t \frac{y(r)}{\sqrt{r}+1}\,\mathrm{d}r + \frac{3t}{35(t^3+2)}\,{}^{c}\mathcal{D}^{\frac{1}{8}}[z](t)$$

for all $t \in \overline{J}$. Now, we get

$$\begin{aligned}
&\left| w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^{c}\mathcal{D}_q^{\zeta}[x](t)\right) - w_2\left(t, y(t), \int_0^t y(r)\,\mathrm{d}r, {}^{c}\mathcal{D}_q^{\zeta}[y](t)\right)\right| \\
&= \left| \frac{|t|}{35(1+|t|)} + \frac{2}{35(4+\sqrt{t})}\big|x(t)\big| + \frac{3t}{70}\int_0^t \frac{x(r)}{\sqrt{r}+1}\,\mathrm{d}r \right. \\
&\quad + \frac{3t}{35(t^3+2)}\,{}^{c}\mathcal{D}^{\frac{1}{8}}[x](t) - \left[\frac{|t|}{35(1+|t|)} + \frac{2}{35(4+\sqrt{t})}\big|y(t)\big| \right. \\
&\quad \left.\left. + \frac{3t}{70}\int_0^t \frac{y(r)}{\sqrt{r}+1}\,\mathrm{d}r + \frac{3t}{35(t^3+2)}\,{}^{c}\mathcal{D}^{\frac{1}{8}}[y](t)\right]\right| \\
&\leq \frac{2}{35(4+\sqrt{t})}\Big|\big|x(t)\big| - \big|y(t)\big|\Big| + \frac{3t}{70}\left|\int_0^t \left(\frac{x(r)}{\sqrt{r}+1} - \frac{y(r)}{\sqrt{r}+1}\right)\mathrm{d}r\right| \\
&\quad + \frac{3t}{35(t^3+2)}\left|{}^{c}\mathcal{D}^{\frac{1}{8}}[x](t) - {}^{c}\mathcal{D}^{\frac{1}{8}}[y](t)\right| \\
&\leq \frac{2}{35(4+\sqrt{t})}\|x-y\| + \frac{3t}{70}\|x-y\| + \frac{3t}{35(t^3+2)}\|x-y\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left| w_2\left(t, x(t), \int_0^t x(r)\,\mathrm{d}r, {}^{c}\mathcal{D}_q^{\zeta}[x](t)\right) - w_2\left(t, y(t), \int_0^t y(r)\,\mathrm{d}r, {}^{c}\mathcal{D}_q^{\zeta}[y](t)\right)\right| \\
&\leq \left[\frac{2}{35(4+\sqrt{t})} + \frac{3t}{70} + \frac{3t}{35(t^3+2)}\right]\|x-y\|
\end{aligned}$$

for all $t \in J$, $x, y \in \mathcal{B}$. We define the positive continuous maps $m_1(t) = \frac{2}{35(4+\sqrt{t})}$, $m_2(t) = \frac{3t}{70}$, and $m_3(t) = \frac{3t}{35(t^3+2)}$. At present, by using Eqs. (12)–(13) and applying Algorithm 5, we calculate $\sup \mathcal{I}_q^{\alpha}[m_i](t)$, $\sup \mathcal{I}_q^{\alpha-1}[m_i](1)$, $\sup \mathcal{I}_q^{\alpha-2}[m_i](b)$, $\sup \mathcal{I}_q^{\alpha-\zeta}[m_i](t)$ for $i = 1, 2, 3$. Tables 7, 8, and 9 show these results. Therefore

$$\begin{aligned}
&\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha}[m_1](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}}\left(\frac{2}{35(4+\sqrt{t})}\right) = 0.0106, 0.0089, 0.0077, \\
&\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-11}[m_1](1) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-1}\left(\frac{2}{175}\right) = 0.0118, 0.0125, 0.0128,
\end{aligned}$$

Table 7 Some numerical results of $\mathcal{I}_q^{\alpha}[m_1](t)$, $\mathcal{I}_q^{\alpha-1}[m_1](1)$, $\mathcal{I}_q^{\alpha-2}[m_1](b)$, and $\mathcal{I}_q^{\alpha-\zeta}[m_1](t)$ in Example 2 for $t \in \overline{J}$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	$\sup \mathcal{I}_q^{\alpha}[m_1](t)$	$\sup \mathcal{I}_q^{\alpha-1}[m_1](1)$	$\sup \mathcal{I}_q^{\alpha-2}[m_1](b)$	$\sup \mathcal{I}_q^{\alpha-\zeta}[m_1](t)$
$q = \frac{1}{8}$				
1	0.0104	0.0116	0.0116	0.0106
2	<u>0.0106</u>	<u>0.0118</u>	<u>0.0115</u>	<u>0.0108</u>
3	0.0106	0.0118	0.0115	0.0108
4	0.0106	0.0118	0.0115	0.0108
$q = \frac{1}{2}$				
1	0.0058	0.0095	0.0119	0.0062
2	0.0073	0.011	0.0113	0.0077
3	0.0081	0.0118	0.011	0.0086
4	0.0085	0.0121	0.0109	0.009
5	0.0087	0.0123	0.0108	0.0092
6	0.0088	0.0124	0.0108	0.0093
7	<u>0.0089</u>	<u>0.0125</u>	0.0108	0.0094
8	0.0089	0.0125	<u>0.0107</u>	0.0094
9	0.0089	0.0125	0.0107	0.0094
10	0.0089	0.0125	0.0107	<u>0.0095</u>
11	0.0089	0.0125	0.0107	0.0095
$q = \frac{6}{7}$				
1	0.0009	0.0041	0.0147	0.001
2	0.0014	0.0055	0.0134	0.0017
3	0.002	0.0066	0.0126	0.0024
⋮	⋮	⋮	⋮	⋮
22	0.0074	0.0126	0.0105	0.0081
23	0.0075	0.0126	<u>0.0104</u>	0.0082
24	0.0075	0.0127	0.0104	0.0082
25	0.0076	0.0127	0.0104	0.0083
26	0.0076	0.0127	0.0104	0.0083
27	<u>0.0077</u>	<u>0.0128</u>	0.0104	0.0084
28	0.0077	0.0128	0.0104	0.0084
29	0.0077	0.0128	0.0104	0.0084
30	0.0077	0.0128	0.0104	0.0084
31	0.0077	0.0128	0.0104	<u>0.0085</u>
32	0.0078	0.0129	0.0104	0.0085

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-2}[m_1](b) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-2}\left(\frac{2}{35(4+\sqrt{\frac{7}{9}})}\right) = 0.0115, 0.0107, 0.0104,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-\zeta}[m_1](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-\frac{1}{8}}\left(\frac{2}{35(4+\sqrt{t})}\right) = 0.0108, 0.0095, 0.0085$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively,

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha}[m_2](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}}\left(\frac{3t}{70}\right) = 0.0343, 0.0184, 0.0109,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-11}[m_2](1) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-1}\left(\frac{3}{70}\right) = 0.0391, 0.0315, 0.0269,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-2}[m_2](b) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-2}\left(\frac{1}{30}\right) = 0.0357, 0.0367, 0.0374,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-\zeta}[m_2](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-\frac{1}{8}}\left(\frac{3t}{70}\right) = 0.0349, 0.0198, 0.0123$$

Table 8 Some numerical results of $\mathcal{I}_q^{\alpha}[m_2](t)$, $\mathcal{I}_q^{\alpha-1}[m_2](1)$, $\mathcal{I}_q^{\alpha-2}[m_2](b)$, and $\mathcal{I}_q^{\alpha-\zeta}[m_2](t)$ in Example 2 for $t \in \overline{J}$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	$\sup \mathcal{I}_q^{\alpha}[m_2](t)$	$\sup \mathcal{I}_q^{\alpha-1}[m_2](1)$	$\sup \mathcal{I}_q^{\alpha-2}[m_2](b)$	$\sup \mathcal{I}_q^{\alpha-\zeta}[m_2](t)$
$q = \frac{1}{8}$				
1	<u>0.0343</u>	<u>0.0391</u>	<u>0.0357</u>	<u>0.0349</u>
2	0.0343	0.0391	0.0357	0.0349
3	0.0343	0.0391	0.0357	0.0349
$q = \frac{1}{2}$				
1	0.0167	0.0299	0.0373	0.018
2	0.0179	0.0311	0.0368	0.0193
3	0.0183	0.0314	0.0368	0.0197
4	<u>0.0184</u>	<u>0.0315</u>	<u>0.0367</u>	<u>0.0198</u>
5	0.0184	0.0315	0.0367	0.0198
6	0.0184	0.0315	0.0367	0.0198
$q = \frac{6}{7}$				
1	0.0029	0.0144	0.0434	0.0036
2	0.0045	0.018	0.0407	0.0054
3	0.0058	0.0206	0.0394	0.0069
⋮	⋮	⋮	⋮	⋮
13	0.0107	0.0267	0.0375	0.0122
14	0.0108	0.0268	<u>0.0374</u>	0.0122
15	0.0108	0.0268	0.0374	<u>0.0123</u>
16	<u>0.0109</u>	<u>0.0269</u>	0.0374	0.0123
17	0.0109	0.0269	0.0374	0.0124
18	0.0109	0.0269	0.0374	0.0124

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively, and

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha}[m_3](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}}\left(\frac{3t}{35(t^3+2)}\right) = 0.0231, 0.0140, 0.0096,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-11}[m_3](1) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-1}\left(\frac{1}{35}\right) = 0.0262, 0.0230, 0.0215,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-2}[m_3](b) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-2}\left(\frac{7}{105((\frac{7}{9})^3+2)}\right) = 0.0288, 0.0292, 0.0029,$$

$$\sup_{t\in\overline{J}} \mathcal{I}_q^{\alpha-\zeta}[m_3](t) = \sup_{t\in\overline{J}} \mathcal{I}_q^{\frac{9}{5}-\frac{1}{8}}\left(\frac{3t}{35(t^3+2)}\right) = 0.0234, 0.00150, 0.00108$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively. Hence, from Eqs. (12)–(13) and the above results in Tables 7, 8, and 9, we obtain $M_0 = 0.0343$, $M(1) = 0.0391$, $M(b) = 0.0357$, $M(t) = 0.0349$ whenever $q = \frac{1}{8}$, $M_0 = 0.0184$, $M(1) = 0.0315$, $M(b) = 0.0367$, $M(t) = 0.0198$ whenever $q = \frac{1}{2}$, $M_0 = 0.0110$, $M(1) = 0.0270$, $M(b) = 0.0374$, $M(t) = 0.0125$ whenever $q = \frac{6}{7}$. Also, by using Eq. (14), we can calculate values of Δ. Table 10 shows these results. Thus, by using Eq. (14), we have

$$\begin{aligned}\Delta &= M_0 + M(t) + \frac{\Gamma_q(2-\zeta)+1}{\Gamma_q(2-\zeta)}M(1) + \frac{(1+\Gamma_q(2-\zeta))\times 6.1083}{\Gamma_q(2-\zeta)}M(b)\\ &= 0.0343 + 0.0349 + \frac{\Gamma_q(2-\frac{1}{8})+1}{\Gamma_q(2-\frac{1}{8})}\times 0.0391\\ &\quad + \frac{(1+\Gamma_q(2-\frac{1}{8}))\times 6.1083}{\Gamma_q(2-\frac{1}{8})}\times 0.0357 = 0.5859 < 1\end{aligned}$$

Table 9 Some numerical results of $\mathcal{I}_q^{\alpha}[m_3](t)$, $\mathcal{I}_q^{\alpha-1}[m_3](1)$, $\mathcal{I}_q^{\alpha-2}[m_3](b)$, and $\mathcal{I}_q^{\alpha-\zeta}[m_3](t)$ in Example 2 for $t \in \overline{J}$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	$\sup \mathcal{I}_q^{\alpha}[m_3](t)$	$\sup \mathcal{I}_q^{\alpha-1}[m_3](1)$	$\sup \mathcal{I}_q^{\alpha-2}[m_3](b)$	$\sup \mathcal{I}_q^{\alpha-\zeta}[m_3](t)$
$q = \frac{1}{8}$				
1	<u>0.0231</u>	<u>0.0262</u>	<u>0.0288</u>	<u>0.0234</u>
2	0.0231	0.0262	0.0288	0.0234
$q = \frac{1}{2}$				
1	0.0123	0.0213	0.0297	0.0133
2	0.0136	0.0226	0.0293	0.0146
3	0.0139	0.0229	<u>0.0292</u>	0.0149
4	<u>0.014</u>	<u>0.023</u>	0.0292	<u>0.015</u>
5	0.014	0.023	0.0292	0.015
6	0.014	0.023	0.0292	0.015
$q = \frac{6}{7}$				
1	0.0021	0.0101	0.0346	0.0026
2	0.0034	0.0131	0.0322	0.0041
3	0.0046	0.0154	0.031	0.0054
4	0.0057	0.017	0.0303	0.0066
⋮	⋮	⋮	⋮	⋮
13	0.0093	0.0213	0.029	0.0105
14	0.0094	0.0213	<u>0.029</u>	0.0106
15	0.0094	0.0214	0.029	0.0106
16	0.0095	0.0214	0.029	0.0107
17	0.0095	<u>0.0215</u>	0.029	0.0107
18	0.0095	0.0215	0.029	0.0107
19	0.0095	0.0215	0.029	<u>0.0108</u>
20	<u>0.0096</u>	0.0215	0.029	0.0108
21	0.0096	0.0215	0.029	0.0108

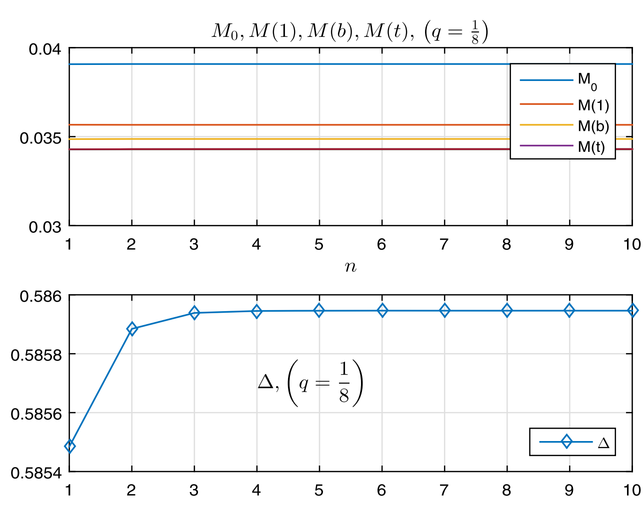

Figure 7 2D graphs of M_0, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{1}{8}$ in Example 2

whenever $q = \frac{1}{8}$,

$$\Delta = M_0 + M(t) + \frac{\Gamma_q(2-\zeta)+1}{\Gamma_q(2-\zeta)} M(1) + \frac{(1+\Gamma_q(2-\zeta)) \times 6.1083}{\Gamma_q(2-\zeta)} M(b)$$

Table 10 Some numerical results for calculation of M_0, $M(1)$, $M(b)$, $M(t)$, and $\Delta < 1$ in Example 2 for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

n	M_0	$M(1)$	$M(b)$	$M(t)$	Δ
$q = \frac{1}{8}$					
1	0.0343	0.0391	0.0357	0.0349	0.5855
2	0.0343	0.0391	0.0357	0.0349	0.5859
3	0.0343	0.0391	0.0357	0.0349	0.5859
4	0.0343	0.0391	0.0357	0.0349	0.5859
$q = \frac{1}{2}$					
1	0.0167	0.0299	0.0373	0.018	0.5275
2	0.0179	0.0311	0.0368	0.0193	0.5426
3	0.0183	0.0314	0.0368	0.0197	0.5504
4	0.0184	0.0315	0.0367	0.0198	0.5542
5	0.0184	0.0315	0.0367	0.0198	0.5561
6	0.0184	0.0315	0.0367	0.0198	0.5571
7	0.0184	0.0315	0.0367	0.0198	0.5576
8	0.0184	0.0315	0.0367	0.0198	0.5578
9	0.0184	0.0315	0.0367	0.0198	0.5579
10	0.0184	0.0315	0.0367	0.0198	0.5580
11	0.0184	0.0315	0.0367	0.0198	0.5580
12	0.0184	0.0315	0.0367	0.0198	0.5580
$q = \frac{6}{7}$					
1	0.0029	0.0144	0.0434	0.0036	0.4067
2	0.0045	0.018	0.0407	0.0054	0.4166
3	0.0058	0.0206	0.0394	0.0069	0.4317
⋮	⋮	⋮	⋮	⋮	⋮
13	0.0107	0.0267	0.0375	0.0122	0.5218
14	0.0108	0.0268	0.0374	0.0122	0.5253
15	0.0108	0.0268	0.0374	0.0123	0.5283
16	0.0109	0.0269	0.0374	0.0123	0.5308
17	0.0109	0.0269	0.0374	0.0124	0.533
18	0.0109	0.0269	0.0374	0.0124	0.5348
19	0.0109	0.0269	0.0374	0.0124	0.5364
20	0.011	0.0269	0.0374	0.0124	0.5378
21	0.011	0.0269	0.0374	0.0124	0.5389
22	0.011	0.027	0.0374	0.0125	0.5399
23	0.011	0.027	0.0374	0.0125	0.5408
24	0.011	0.027	0.0374	0.0125	0.5415
⋮	⋮	⋮	⋮	⋮	⋮
53	0.011	0.027	0.0374	0.0125	0.5457
54	0.011	0.027	0.0374	0.0125	0.5458
55	0.011	0.027	0.0374	0.0125	0.5458

$$
\begin{aligned}
&= 0.0184 + 0.0198 + \frac{\Gamma_q(2-\frac{1}{8})+1}{\Gamma_q(2-\frac{1}{8})} \times 0.0315 \\
&\quad + \frac{(1+\Gamma_q(2-\frac{1}{8})) \times 6.1083}{\Gamma_q(2-\frac{1}{8})} \times 0.0367 = 0.5580 < 1
\end{aligned}
$$

whenever $q = \frac{1}{2}$, and

$$
\begin{aligned}
\Delta &= M_0 + M(t) + \frac{\Gamma_q(2-\zeta)+1}{\Gamma_q(2-\zeta)} M(1) + \frac{(1+\Gamma_q(2-\zeta))\times}{\Gamma_q(2-\zeta)} M(b) \\
&= 0.0110 + 0.0125 + \frac{\Gamma_q(2-\frac{1}{8})+1}{\Gamma_q(2-\frac{1}{8})} \times 0.0270
\end{aligned}
$$

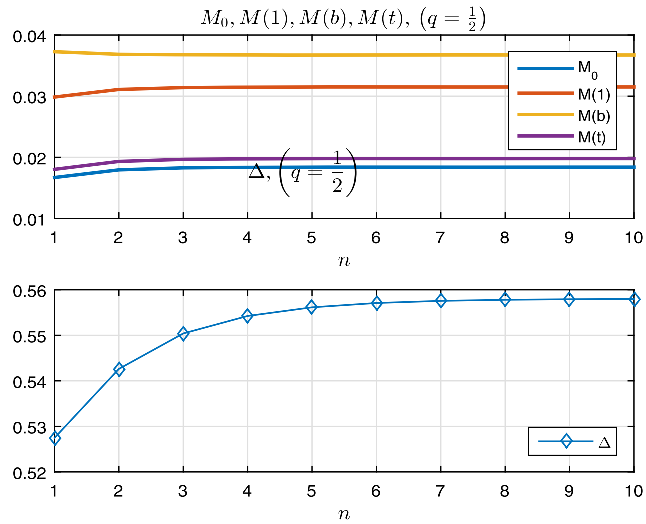

Figure 8 2D graphs of M_0, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{1}{2}$ in Example 2

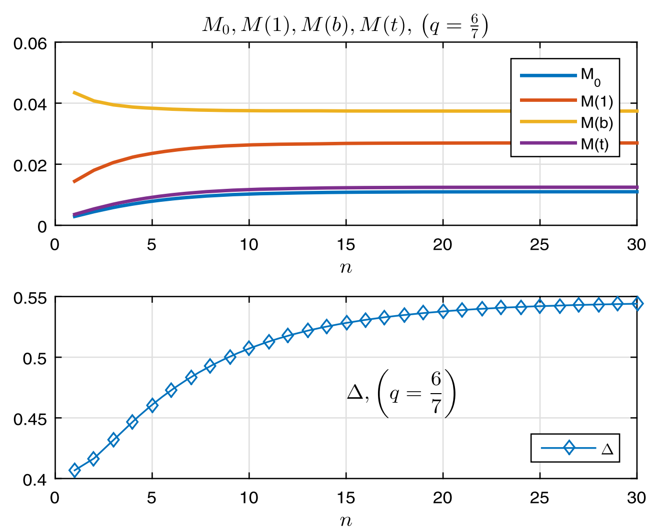

Figure 9 2D graphs of M_0, $M(1)$, $M(b)$, $M(t)$ on $t \in \overline{J}$ for $q = \frac{6}{7}$ in Example 2

$$+ \frac{(1 + \Gamma_q(2 - \frac{1}{8})) \times 6.1083}{\Gamma_q(2 - \frac{1}{8})} \times 0.0374 = 0.5458 < 1$$

whenever $q = \frac{6}{7}$. Figures 7, 8, and 9 show these results (Algorithm 7). Now, by using Theorem 5, the fractional q-integro-differential equation under sum boundary value conditions (16) has a unique solution.

5 Conclusion

The q-integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional q-calculus and its applications in various areas

Algorithm 7 The MATLAB lines for calculation of all parameters in Example 2

```
format long
q=[1/8 1/2 6/7];
alpha=9/5; zeta=1/8;
syms t;
t0=0; T=1; b=7/9;
ci=[7/12 9/8 9/5 2/3 5/6 11/10];
m1=[sym(2/(35*(4+ t^(1/2))))];
m2=[sym(3*t/70)];
m3=[sym(3*t/(35*(t^3 +2)))];
[xq, yq]=size(q);
[xci, yci]=size(ci);
Xi=0;
for j=1:yci
    Xi = Xi+ci(j);
end;
m=yci;
column=1;
for j=1:yq
    for n=1:80
        A1(n,column)=n;
        A2(n,column)=n;
        A3(n,column)=n;
        A1(n,column+1)=Iq_alpha(q(j), alpha, t0, n, m1);
        A1(n,column+2)=Iq_alpha(q(j), alpha, T, n, m1);
        A1(n,column+3)=A1(n,column+1);
        if A1(n,column+2)>A1(n,column+1)
            A1(n,column+3)=A1(n,column+2);
        end;
        A2(n,column+1)=Iq_alpha(q(j), alpha, t0, n, m2);
        A2(n,column+2)=Iq_alpha(q(j), alpha, T, n, m2);
        A2(n,column+3)=A2(n,column+1);
        if A2(n,column+2)>A2(n,column+1)
            A2(n,column+3)=A2(n,column+2);
        end;
        A3(n,column+1)=Iq_alpha(q(j), alpha, t0, n, m3);
        A3(n,column+2)=Iq_alpha(q(j), alpha, T, n, m3);
        A3(n,column+3)=A3(n,column+1);
        if A3(n,column+2)>A3(n,column+1)
            A3(n,column+3)=A3(n,column+2);
        end;
        A1(n,column+4)=Iq_alpha(q(j), alpha-1, T, n, m1);
        A2(n,column+4)=Iq_alpha(q(j), alpha-1, T, n, m2);
        A3(n,column+4)=Iq_alpha(q(j), alpha-1, T, n, m3);
        A1(n,column+5)=Iq_alpha(q(j), alpha-2, b, n, m1);
        A2(n,column+5)=Iq_alpha(q(j), alpha-2, b, n, m2);
        A3(n,column+5)=Iq_alpha(q(j), alpha-2, b, n, m3);
        A1(n,column+6)=Iq_alpha(q(j), alpha-zeta, t0, n, m1);
        A1(n,column+7)=Iq_alpha(q(j), alpha-zeta, T, n, m1);
        A1(n,column+8)=A1(n,column+6);
        if A1(n,column+7)>A1(n,column+6)
            A1(n,column+8)=A1(n,column+7);
        end;
        A2(n,column+6)=Iq_alpha(q(j), alpha-zeta, t0, n, m2);
        A2(n,column+7)=Iq_alpha(q(j), alpha-zeta, T, n, m2);
        A2(n,column+8)=A2(n,column+6);
        if A2(n,column+7)>A2(n,column+6)
            A2(n,column+8)=A2(n,column+7);
        end;
        A3(n,column+6)=Iq_alpha(q(j), alpha-zeta, t0, n, m3);
        A3(n,column+7)=Iq_alpha(q(j), alpha-zeta, T, n, m3);
        A3(n,column+8)=A3(n,column+6);
        if A3(n,column+7)>A3(n,column+6)
            A3(n,column+8)=A3(n,column+7);
        end;
    end;
    column=column+9;
end;
column=1;
Acolumn=1;
for j=1:yq
    for n=1:80
        M(n,column)=n;
        M1=A1(n,Acolumn+3);
        if A2(n,Acolumn+3)> M1
            M1=A2(n,Acolumn+3);
        end;
```

Algorithm 7 (Continued)

```
        if A3(n,Acolumn+3)> M1
            M1=A3(n,Acolumn+3);
        end;
        M(n,column+1)=M1;
        M2=A1(n,Acolumn+4);
        if A2(n,Acolumn+4)> M2
            M2=A2(n,Acolumn+4);
        end;
        if A3(n,Acolumn+4)> M2
            M2=A3(n,Acolumn+4);
        end;
        M(n,column+2)=M2;
        M3=A1(n,Acolumn+5);
        if A2(n,Acolumn+5)> M3
            M3=A2(n,Acolumn+5);
        end;
        if A3(n,Acolumn+5)> M3
            M3=A3(n,Acolumn+5);
        end;
        M(n,column+3)=M3;

        M4=A1(n,Acolumn+8);
        if A2(n,Acolumn+8)> M4
            M4=A2(n,Acolumn+8);
        end;
        if A3(n,Acolumn+8)> M4
            M4=A3(n,Acolumn+8);
        end;
        M(n,column+4)=M4;
    end;
    column = column + 5;
    Acolumn =Acolumn+9;
end;
column=1;
Mcolumn=1;
for j=1:yq
    for n=1:80
        Delta(n, column)=n;
        G=qGamma(q(j), 2-zeta, n);
        Delta(n, column+1) = M(n, Mcolumn+1) + M(n, Mcolumn+4) + ...
             (G+1)*M(n,Mcolumn+2)/G+ (1+G)*Xi*M(n,Mcolumn+3)/G;
    end;
    column=column+2;
    Mcolumn=Mcolumn+5;
end;
```

of science and technology. q-integro-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. The end of this article is to investigate a complicated case by utilizing an appropriate basic theory. In this manner, we prove the existence of a solution for two new q-integro-differential equations under sum and integral boundary conditions (1)–(2) and (3)–(4) on a time scale and show the perfect numerical effects for the problem which confirmed our results.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Publisher's Note

Received: 4 March 2020 Accepted: 10 June 2020 Published online: 19 June 2020

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