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On a fractional hybrid version of the Sturm–Liouville equation

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Abstract

It is well known that the Sturm–Liouville equation has many applications in different areas of science. Thus, it is important to review different versions of the well-known equation. The technique of α-admissible α-ψ-contractions was introduced by Samet et al. in (Nonlinear Anal. 75:2154–2165, 2012). Our aim in this work is to study a fractional hybrid version of the Sturm–Liouville equation by mixing the technique of Samet. In fact, by using the technique of α-admissible α-ψ-contractions, we investigate the existence of solutions for the fractional hybrid Sturm–Liouville equation by using the multi-point boundary value conditions. Also, we review the existence of solutions for a fractional hybrid version of the problem under the integral boundary value conditions. Finally, we provide two examples to illustrate our main results.

MSC: Primary 34A08; secondary 34A12

Keywords: α-ψ-contraction; Fractional hybrid version; Multi-point boundary value conditions; Sturm–Liouville equation

1 Introduction and preliminaries

What mathematics needs today is various applications to improve the standard of living of humanity. Although mathematics has had many uses in different fields so far, it can still have more beneficial effects in society. One of the most profitable ways to make mathematics more relevant in today's world is to produce modern software to reduce the consumption of minerals in chemical laboratories. Some chemistry experiments in software can be performed with high repeatability and by examining different pressure, temperature, and distinct conditions. It is a great advantage to do many experiments without the use of minerals. Computer software companies should pay particular attention to this issue.

It is logical that researchers concentrate on complicated fractional differential equations to increase their abilities for modeling of more real phenomena in the world. One of important methods in this way is working on different versions of well-known fractional differential equations. It is known that one of the famous ones is the Sturm–Liouville differential equation.

The Sturm–Liouville differential equation is an important differential equation in physics, applied mathematics, and other fields of engineering and science, and it has wide applications in quantum mechanics, classical mechanics, and wave phenomena (see, for example, [2] and [3] and the references therein). The existence of solutions and other properties for Sturm–Liouville boundary value problems have received considerable attention from many researchers during the last two decades (see, for example, [4–17]). Finally, a hybrid version of differential equations has a special appeal to everybody.

Nowadays, many researchers are currently studying various types of advanced mathematical modeling using fractional differential equations and its related inclusion version with more general boundary value conditions. Indeed, they try to model the processes so that it covers many general cases. In this situation, mathematicians would like to solve a wide range of these boundary value problems with advanced and complicate boundary conditions. Recently, many papers have been published on the existence of solutions for different fractional boundary value problems (see, for example, [18–34]). In the last few decades, fractional hybrid differential equations and inclusions with hybrid or non-hybrid boundary value conditions have received a great deal of interest and attention of many researchers (see, for example, [35–41]).

As you know, the Riemann–Liouville fractional integral of order $\alpha > 0$ for the function $u \in_L 1[0,T]$ is given by $I^\alpha u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds$, and the Caputo fractional derivative of order $n-1 \leq \alpha < n$ for the function u is given by $D^\alpha u(t) = I^{n-\alpha} \frac{d^n}{dt^n} u(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{d^n u(s)}{dt^n}\,ds$.

In 2011, Zhao et al. studied the fractional hybrid problem ${}^cD^\alpha(\frac{u(t)}{g(t,u(t))}) = f(t,u(t))$ with boundary value condition $u(0) = 0$, where $0 < \alpha < 1$, ${}^cD^\alpha$ denotes the Caputo fractional derivative, $g \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $f \in C(I \times \mathbb{R}, \mathbb{R})$ [41]. In 2019, El-Sayed et al. reviewed the fractional version of the Sturm–Liouville equation ${}^cD^\alpha(p(t)u'(t)) + q(t)u(t) = h(t)f(u(t))$ with multi-point boundary condition $u'(t) = 0$, $\sum_{i=1}^m \xi_i u(a_i) = \nu \sum_{j=1}^n \eta_j u(b_j)$, where α lies in $(0,1]$, ${}^cD^\alpha$ denotes the Caputo fractional derivative, $p \in C^1(I,\mathbb{R})$, $q(t)$ and $h(t)$ are absolutely continuous functions on $I = [0,T]$, $T < \infty$ with $p(t) \neq 0$ for all $t \in I$, $f : \mathbb{R} \to \mathbb{R}$ is defined and differentiable on the interval I, $0 \leq a_1 < a_2 < \cdots < a_m < c$, $d \leq b_1 < b_2 < \cdots < b_n < T$, $c < d$, and $\xi_1, \dots, \xi_m$, $\eta_1, \dots, \eta_n$, and ν are real constants [42].

Assume that $\alpha \in (0,1)$, ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $I = [0,T]$ with $T < \infty$, $p, \tilde{p} \in C^1(I,\mathbb{R})$, $\tilde{p}(t)$, $q(t)$, and $h(t)$ are absolutely continuous functions on I with $p(t) \neq 0$ for all $t \in I$, $f, \tilde{f} : \mathbb{R} \to \mathbb{R}$ are defined and differentiable on the interval I and $0 \leq a_1 < a_2 < \cdots < a_m < c$, $d \leq b_1 < b_2 < \cdots < b_n < T$, $c < d$, and $\xi_1, \dots, \xi_m$, $\eta_1, \dots, \eta_n$, and ν are real constants with $\sum_{i=1}^m \xi_i - \nu \sum_{j=1}^n \eta_j \neq 0$. Now, by mixing the main idea of the works, we investigate the general fractional hybrid version of the Sturm–Liouville equation with the hybrid multi-point boundary value condition

$$\begin{cases} {}^cD^\alpha(p(t)(\frac{u(t)}{g(t,u(t))})' - \tilde{p}(t)\tilde{f}(u(t))) + q(t)u(t) = h(t)f(u(t)), \quad (t \in I), \\ (\frac{u(t)}{g(t,u(t))})'_{t=0} = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(u(t)))_{t=0}, \\ \sum_{i=1}^m \xi_i(\frac{u(a_i)}{g(a_i,u(a_i))}) = \nu \sum_{j=1}^n \eta_j(\frac{u(b_j)}{g(b_j,u(b_j))}). \end{cases} \tag{1}$$

Moreover, we review the following problem under integral boundary value conditions:

$$\begin{cases} {}^cD^\alpha(p(t)(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t)))') + q(t)u(t) = h(t)f(u(t)), \quad (t \in I), \\ (\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t)))'_{t=0} = 0, \\ \int_a^c (\frac{u(\theta)}{g(\theta,u(\theta))} - \tilde{p}(\theta)\tilde{f}(u(\theta)))\,d\varpi(\theta) = \nu \int_d^e (\frac{u(\theta)}{g(\theta,u(\theta))} - \tilde{p}(\theta)\tilde{f}(u(\theta)))\,d\upsilon(\theta), \end{cases} \tag{2}$$

where $\varpi(\theta)$ and $\upsilon(\theta)$ are increasing functions, the integrals are in the Riemann–Stieltjes sense, and $0 \le a < c \le d < e \le T$.

We consider the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$ on the space $C(I,\mathbb{R})$ and $\|u\| = \int_0^T |u(s)|\,ds$ on $L_1[0,T]$. The Riemann–Liouville fractional integral of order α for a function f is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds$ $(\alpha > 0)$, and the Caputo derivative of order α for a function f is defined by ${}^cD^\alpha f(t) = I^{n-\alpha}\frac{d^n}{dt^n}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds$, where $n = [\alpha] + 1$ [43, 44]. Denote by Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. Let $T : X \to X$ be a self-map and $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is α-admissible whenever $\alpha(x,y) \ge 1$ implies $\alpha(Tx,Ty) \ge 1$ [1]. Let $\psi \in \Psi$ and $\alpha : X \times X \to [0,+\infty)$ be a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y))$ for all $x,y \in X$ [1]. We need the next result.

Lemma 1 ([1]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible α-ψ-contraction. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_n,x) \ge 1$ for all n whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_{n-1},x_n) \ge 1$ for all $n \ge 1$ and $x_n \to x$. Then T has a fixed point.*

2 Main results

Now, we are ready to state and prove our main results. For study of problem (1), we consider the following hypotheses.

- (D_1) The functions $f,\tilde{f} : \mathbb{R} \to \mathbb{R}$ are differentiable on the interval $[0,T]$ and $\frac{\partial f}{\partial u}$ and $\frac{\partial \tilde{f}}{\partial u}$ are bounded on $[0,T]$ with $\frac{\partial f}{\partial u} \le \mathcal{K}$ and $\frac{\partial \tilde{f}}{\partial u} \le \tilde{\mathcal{K}}$, respectively.
- (D_2) The function $p \in C^1(I,\mathbb{R})$ has this property that $p(t) \ne 0$ for all t and $\inf_{t\in I}|p(t)| = p$. Also, $\tilde{p}(t)$, $q(t)$, and $h(t)$ are absolutely continuous functions on I.
- (D_3) The function $g : I \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous in its two variables and there exists a function $\phi(t) \ge 0$ $(\forall t \in I)$ such that $|g(t,x) - g(t,y)| \le \phi(t)|x-y|$ for all (t,x,y) in $I \times \mathbb{R} \times \mathbb{R}$.
- (D_4) There exists a number $r > 0$ such that

$$\bigl(\|\phi\|r + g_0\bigr)(\mathcal{A}_1 r + \mathcal{A}_2) \le r \quad \text{and} \quad (2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1 < 1,$$

where $\mathcal{A}_1 = \frac{T}{p}(\widetilde{\mathcal{K}}\|\tilde{p}\| + \frac{T^\alpha(\|q\|+\mathcal{K}\|h\|)}{\Gamma(\alpha+2)})(|E|(\sum_{i=1}^m|\xi_i| + |\nu|\sum_{j=1}^n|\eta_j|) + 1)$,

$$\mathcal{A}_2 = \frac{T}{p}\left(\tilde{f}_0\|\tilde{p}\| + \frac{T^\alpha\|h\|f_0}{\Gamma(\alpha+2)}\right)\left(|E|\left(\sum_{i=1}^m|\xi_i| + |\nu|\sum_{j=1}^n|\eta_j|\right) + 1\right),$$

$f_0 = |f(0)|$, $\tilde{f}_0 = |\tilde{f}(0)|$, and $g_0 = \sup_{t\in I} g(t,0)$.

Lemma 2 *Assume that hypotheses (D_1)–(D_2) hold. Then problem (1) is equivalent to the integral equation*

$$\begin{aligned} u(t) = g\bigl(t,u(t)\bigr)\Biggl[&E\nu\sum_{j=1}^n \eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\bigl(u(s)\bigr)\,ds - E\sum_{i=1}^m \xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\bigl(u(s)\bigr)\,ds \\ &+ E\sum_{i=1}^m \xi_i\int_0^{a_i}\frac{1}{p(s)}I^\alpha\bigl(q(s)u(s)\bigr)\,ds - E\nu\sum_{j=1}^n \eta_j\int_0^{b_j}\frac{1}{p(s)}I^\alpha\bigl(q(s)u(s)\bigr)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ E\nu \sum_{j=1}^{n} \eta_j \int_0^{b_j} \frac{1}{p(s)} I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds \\
&- E \sum_{i=1}^{m} \xi_i \int_0^{a_i} \frac{1}{p(s)} I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds \\
&+ \int_0^t \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(u(s)\big)\,ds - \int_0^t \frac{1}{p(s)} I^{\alpha}\big(q(s)u(s)\big)\,ds \\
&+ \int_0^t \frac{1}{p(s)} I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds \Bigg],
\end{aligned} \tag{3}
$$

where $E = \frac{1}{\sum_{i=1}^{m} \xi_i - \nu \sum_{j=1}^{n} \eta_j}$. *Also, we have* $\frac{u}{g(t,u(t))} \in C^1(I, \mathbb{R})$ *and* $(\frac{u(t)}{g(t,u(t))})'' \in L_1(I, \mathbb{R})$. *Moreover, if* $(g(t, u(t)))' \in C(I, \mathbb{R})$, *then* $u \in C^1(I, \mathbb{R})$.

Proof Note that equation (1) can be written as

$$
I^{1-\alpha}\left(\frac{d}{dt}\left[p(t)\left(\frac{u(t)}{g(t,u(t))}\right)' - \tilde{p}(t)\tilde{f}\big(u(t)\big)\right]\right) = -q(t)u(t) + h(t)f\big(u(t)\big).
$$

Hence, $I^1(\frac{d}{dt}[p(t)(\frac{u(t)}{g(t,u(t))})' - \tilde{p}(t)\tilde{f}(u(t))]) = -I^{\alpha}(q(t)u(t)) + I^{\alpha}(h(t)f(u(t)))$, and so

$$
\begin{aligned}
&p(t)\left(\frac{u(t)}{g(t,u(t))}\right)' - \tilde{p}(t)\tilde{f}\big(u(t)\big) - p(0)\left(\frac{u(t)}{g(t,u(t))}\right)'_{t=0} + \tilde{p}(0)\tilde{f}\big(u(0)\big) \\
&\quad = -I^{\alpha}\big(q(t)u(t)\big) + I^{\alpha}\big(h(t)f\big(u(t)\big)\big).
\end{aligned}
$$

Since $(\frac{u(t)}{g(t,u(t))})'_{t=0} = (\frac{\tilde{p}(t)\tilde{f}(u(t))}{p(t)})_{t=0}$, we get

$$
p(t)\left(\frac{u(t)}{g(t,u(t))}\right)' = \tilde{p}(t)\tilde{f}\big(u(t)\big) - I^{\alpha}\big(q(t)u(t)\big) + I^{\alpha}\big(h(t)f\big(u(t)\big)\big),
$$

and so

$$
\left(\frac{u(t)}{g(t,u(t))}\right)' = \frac{\tilde{p}(t)}{p(t)}\tilde{f}\big(u(t)\big) - \frac{1}{p(t)} I^{\alpha}\big(q(t)u(t)\big) + \frac{1}{p(t)} I^{\alpha}\big(h(t)f\big(u(t)\big)\big). \tag{4}
$$

Thus, we obtain

$$
\begin{aligned}
\frac{u(t)}{g(t,u(t))} - \ell = &\int_0^t \frac{\tilde{p}(s)}{p(s)} \tilde{f}\big(u(s)\big)\,ds - \int_0^t \frac{1}{p(s)} I^{\alpha}\big(q(s)u(s)\big)\,ds \\
&+ \int_0^t \frac{1}{p(s)} I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds,
\end{aligned} \tag{5}
$$

where $\ell = \frac{u(0)}{g(0,u(0))}$. For simplicity, put $A(t) = \int_0^t \frac{\tilde{p}(s)}{p(s)}\tilde{f}(u(s))\,ds$, $B(t) = \int_0^t \frac{1}{p(s)} I^{\alpha}(q(s)u(s))\,ds$, and $C(t) = \int_0^t \frac{1}{p(s)} I^{\alpha}(h(s)f(u(s)))\,ds$. Then we get

$$
\sum_{i=1}^{m} \xi_i \left(\frac{u(a_i)}{g(a_i,u(a_i))}\right) - \sum_{i=1}^{m} \xi_i \ell = \sum_{i=1}^{m} \xi_i A(a_i) - \sum_{i=1}^{m} \xi_i B(a_i) + \sum_{i=1}^{m} \xi_i C(a_i) \tag{6}
$$

and

$$\nu\sum_{j=1}^{n}\eta_j\left(\frac{u(b_j)}{g(b_j,u(b_j))}\right)-\nu\sum_{j=1}^{n}\eta_j\ell = \nu\sum_{j=1}^{n}\eta_j A(b_j)-\nu\sum_{j=1}^{n}\eta_j B(b_j)+\nu\sum_{j=1}^{n}\eta_j C(b_j). \tag{7}$$

By subtracting (6) from (7) and applying

$$\sum_{i=1}^{m}\xi_i\left(\frac{u(a_i)}{g(a_i,u(a_i))}\right)=\nu\sum_{j=1}^{n}\eta_j\left(\frac{u(b_j)}{g(b_j,u(b_j))}\right),$$

$$\ell = E\nu\sum_{j=1}^{n}\eta_j A(b_j)-E\sum_{i=1}^{m}\xi_i A(a_i)+E\sum_{i=1}^{m}\xi_i B(a_i)-E\nu\sum_{j=1}^{n}\eta_j B(b_j) + E\nu\sum_{j=1}^{n}\eta_j C(b_j)-E\sum_{i=1}^{m}\xi_i C(a_i),$$

where $E=\frac{1}{\sum_{i=1}^{m}\xi_i-\nu\sum_{j=1}^{n}\eta_j}$. By substituting the value of ℓ in (5), we conclude that

$$u(t)=g\bigl(t,u(t)\bigr)\Biggl[E\nu\sum_{j=1}^{n}\eta_j A(b_j)-E\sum_{i=1}^{m}\xi_i A(a_i)+E\sum_{i=1}^{m}\xi_i B(a_i)-E\nu\sum_{j=1}^{n}\eta_j B(b_j) + E\nu\sum_{j=1}^{n}\eta_j C(b_j)-E\sum_{i=1}^{m}\xi_i C(a_i)+\int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\bigl(u(s)\bigr)\,ds-\int_0^t\frac{1}{p(s)}I^{\alpha}\bigl(q(s)u(s)\bigr)\,ds + \int_0^t\frac{1}{p(s)}I^{\alpha}\bigl(h(s)f\bigl(u(s)\bigr)\bigr)\,ds\Biggr].$$

For the next part, by using (4) we have

$$\left(\frac{u(t)}{g(t,u(t))}\right)'=\frac{\tilde{p}(t)}{p(t)}\tilde{f}\bigl(u(t)\bigr)-\frac{1}{p(t)}I^{\alpha}\bigl(q(t)u(t)\bigr)+\frac{1}{p(t)}I^{\alpha}\bigl(h(t)f\bigl(u(t)\bigr)\bigr)\in C(I,\mathbb{R}), \tag{8}$$

and so $\frac{d}{dt}(p(t)(\frac{u(t)}{g(t,u(t))})'-\tilde{p}(t)\tilde{f}(u(t)))=-\frac{d}{dt}I^{\alpha}(q(t)u(t))+\frac{d}{dt}I^{\alpha}(h(t)f(u(t)))$. Hence,

$$I^{1-\alpha}\frac{d}{dt}\left(p(t)\left(\frac{u(t)}{g(t,u(t))}\right)'-\tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)=-I^{1-\alpha}\frac{d}{dt}I^{\alpha}\bigl(q(t)u(t)\bigr) + I^{1-\alpha}\frac{d}{dt}I^{\alpha}\bigl(h(t)f\bigl(u(t)\bigr)\bigr),$$

and so

$$^{c}D^{\alpha}\left(p(t)\left(\frac{u(t)}{g(t,u(t))}\right)'-\tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)=-I^{1-\alpha}I^{\alpha}\frac{d}{dt}\bigl(q(t)u(t)\bigr) + I^{1-\alpha}I^{\alpha}\frac{d}{dt}\bigl(h(t)f\bigl(u(t)\bigr)\bigr) - I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}q(0)u(0)+I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}h(0)f\bigl(u(0)\bigr).$$

Thus, we obtain

$$
\begin{aligned}
{}^{c}D^{\alpha}\left(p(t)\left(\frac{u(t)}{g(t,u(t))}\right)' - \tilde{p}(t)\tilde{f}\big(u(t)\big)\right) &= -I^{1}\frac{d}{dt}\big(q(t)u(t)\big) + I^{1}\frac{d}{dt}\big(h(t)f\big(u(t)\big)\big) \\
&\quad - q(0)u(0) + h(0)f\big(u(0)\big) \\
&= -q(t)u(t) + h(t)f\big(u(t)\big).
\end{aligned}
$$

By using (4), we get $(\frac{u(t)}{g(t,u(t))})'_{t=0} = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(u(t)))_{t=0}$. Also, by using simple computations and (3), we obtain $\sum_{i=1}^{m}\xi_i(\frac{u(a_i)}{g(a_i,u(a_i))}) = \nu\sum_{j=1}^{n}\eta_j(\frac{u(b_j)}{g(b_j,u(b_j))})$. Now, we show that $(\frac{u(t)}{g(t,u(t))})'' \in L_1[0,1]$. From (4) and (iii), we have

$$
\begin{aligned}
\left(\frac{u(t)}{g(t,u(t))}\right)'' &= \frac{d}{dt}\left[\frac{\tilde{p}(t)}{p(t)}\tilde{f}\big(u(t)\big) + \frac{1}{p(t)}I^{\alpha}\big(-q(t)u(t) + h(t)f\big(u(t)\big)\big)\right] \\
&= \left(\frac{\tilde{p}'(t)}{p(t)} - \frac{\tilde{p}(t)p'(t)}{p^2(t)}\right)\tilde{f}\big(u(t)\big) + \frac{\tilde{p}(t)}{p(t)}\frac{\partial\tilde{f}(u(t))}{\partial t}u'(t) \\
&\quad - \frac{p'(t)}{p^2(t)}I^{\alpha}\big(-q(t)u(t) + h(t)f\big(u(t)\big)\big) \\
&\quad + \frac{1}{p(t)}I^{\alpha}\frac{d}{dt}\big(-q(t)u(t) + h(t)f\big(u(t)\big)\big) \\
&\quad + \frac{1}{p(t)}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\big(q(0)u(0) + h(0)f\big(u(0)\big)\big).
\end{aligned}
$$

Now, we can write

$$
\begin{aligned}
\left|\left(\frac{u(t)}{g(t,u(t))}\right)''\right| &\le \left(\frac{|\tilde{p}'(t)|}{|p(t)|} + \frac{|\tilde{p}(t)||p'(t)|}{|p^2(t)|}\right)\big|\tilde{f}\big(u(t)\big)\big| + \frac{|\tilde{p}(t)|}{|p(t)|}\left|\frac{\partial\tilde{f}(u(t))}{\partial t}\right|\big|u'(t)\big| \\
&\quad + \frac{|p'(t)|}{|p^2(t)|}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(\big|q(s)\big|\big|u(s)\big| + \big|h(s)\big|\big|f\big(u(s)\big)\big|\big)\,ds \\
&\quad + \frac{1}{|p(t)|}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigg(\big|q'(s)\big|\big|u(s)\big| + \big|q(s)\big|\big|u'(s)\big| + \big|h'(s)\big|\big|f\big(u(s)\big)\big| \\
&\quad + \big|h(s)\big|\left|\frac{\partial f(u(s))}{\partial u}\right|\big|u'(s)\big|\bigg)\,ds \\
&\quad + \frac{1}{|p(t)|}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\big(\big|q(0)\big|\big|u(0)\big| + \big|h(0)\big|\big|f\big(u(0)\big)\big|\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^T\left|\left(\frac{u(t)}{g(t,u(t))}\right)''\right|dt \\
&\quad\le \int_0^T\left[\left(\frac{|\tilde{p}'(t)|}{|p(t)|} + \frac{|\tilde{p}(t)||p'(t)|}{|p^2(t)|}\right)\big|\tilde{f}\big(u(t)\big)\big| + \frac{|\tilde{p}(t)|}{|p(t)|}\left|\frac{\partial\tilde{f}(u(t))}{\partial t}\right|\big|u'(t)\big|\right]dt \\
&\qquad + \int_0^T\left[\frac{|p'(t)|}{|p^2(t)|}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(\big|q(s)\big|\big|u(s)\big| + \big|h(s)\big|\big|f\big(u(s)\big)\big|\big)\,ds\right]dt \\
&\qquad + \int_0^T\left[\frac{1}{|p(t)|}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigg(\big|q'(s)\big|\big|u(s)\big| + \big|q(s)\big|\big|u'(s)\big|\right.
\end{aligned}
$$

$$+ |h'(s)|f(u(s))| + |h(s)|\left|\frac{\partial f(u(s))}{\partial u}\right||u'(s)|\Big) \, ds\Big] \, dt$$
$$+ \int_0^T \left[\frac{1}{|p(t)|}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\big(|q(0)||u(0)| + |h(0)||f(u(0))|\big)\right] dt. \tag{9}$$

Note that

$$\begin{aligned}
&\int_0^T \left[\left(\frac{|\tilde{p}'(t)|}{|p(t)|} + \frac{|\tilde{p}(t)||p'(t)|}{|p^2(t)|}\right)|\tilde{f}(u(t))| + \frac{|\tilde{p}(t)|}{|p(t)|}\left|\frac{\partial \tilde{f}(u(t))}{\partial t}\right||u'(t)|\right] dt \\
&\quad \le \int_0^T \left[\left(\frac{\|\tilde{p}'\|}{p} + \frac{\|\tilde{p}\|\|p'\|}{p^2}\right)\|\tilde{f}\| + \widetilde{\mathcal{K}}\frac{\|\tilde{p}\|}{p}\|u'\|\right] dt \\
&\quad = T\left[\left(\frac{p\|\tilde{p}'\| + \|\tilde{p}\|\|p'\|}{p^2}\right)\|\tilde{f}\| + \widetilde{\mathcal{K}}\frac{\|\tilde{p}\|}{p}\|u'\|\right].
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&\int_0^T \frac{|p'(t)|}{|p^2(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(|q(s)||u(s)| + |h(s)||f(u(s))|\big) \, ds \, dt \\
&\quad = \int_0^T \big(|q(s)||u(s)| + |h(s)||f(u(s))|\big) \, ds \int_s^T \frac{|p'(t)|}{|p^2(t)|}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, dt \\
&\quad \le \big(\|q(s)\|\|u(s)\| + \|h(s)\|\|f(u(s))\|\big)\frac{\|p'\|T^{\alpha+1}}{p^2\Gamma(\alpha+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \frac{1}{|p(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(|q'(s)||u(s)| + |q(s)||u'(s)| + |h'(s)||f(u(s))| \\
&\qquad + |h(s)|\left|\frac{\partial f(u(s))}{\partial u}\right||u'(s)|\Big) \, ds \, dt \\
&\quad \le \big(\|q'\|_{L_1}\|u\| + \|q\|\|u'\|T + \|h'\|_{L_1}\|f\| + \mathcal{K}\|h\|\|u'\|T\big)\frac{T^\alpha}{p\Gamma(\alpha+1)}.
\end{aligned}$$

Furthermore, $\int_0^T \frac{1}{|p(t)|}\frac{t^{\alpha-1}}{\Gamma(\alpha)}(|q(0)||u(0)| + |h(0)||f(u(0))|) \, dt \le \frac{T^\alpha}{p\Gamma(\alpha+1)}(|q(0)||u(0)| + |h(0)||f(u(0))|)$. By using (9), we can deduce that

$$\begin{aligned}
&\int_0^T \left|\left(\frac{u(t)}{g(t,u(t))}\right)''\right| dt \\
&\quad \le T\left[\left(\frac{p\|\tilde{p}'\| + \|\tilde{p}\|\|p'\|}{p^2}\right)\|\tilde{f}\| + \widetilde{\mathcal{K}}\frac{\|\tilde{p}\|}{p}\|u'\|\right] \\
&\qquad + \big(\|q(s)\|\|u(s)\| + \|h(s)\|\|f(u(s))\|\big)\frac{\|p'\|T^{\alpha+1}}{p^2\Gamma(\alpha+1)} \\
&\qquad + \big(\|q'\|_{L_1}\|u\| + \|q\|\|u'\|T + \|h'\|_{L_1}\|f\| + \mathcal{K}\|h\|\|u'\|T\big)\frac{T^\alpha}{p\Gamma(\alpha+1)} \\
&\qquad + \frac{T^\alpha}{p\Gamma(\alpha+1)}\big(|q(0)||u(0)| + |h(0)||f(u(0))|\big).
\end{aligned}$$

That is, $(\frac{u(t)}{g(t,u(t))})'' \in L_1[0,1]$.

Finally, assume that $\zeta(t) = (\frac{u(t)}{g(t,u(t))})'$. From (8) we know that $\zeta(t) \in C(I,\mathbb{R})$. Let $(g(t, u(t)))' \in C(I,\mathbb{R})$. Then

$$\begin{aligned}\zeta(t) &= \left(\frac{u(t)}{g(t,u(t))}\right)' = \frac{u'(t)g(t,u(t)) - (g(t,u(t)))'u(t)}{(g(t,u(t)))^2}\\ &= \frac{u'(t)}{g(t,u(t))} - \frac{(g(t,u(t)))'u(t)}{(g(t,u(t)))^2},\end{aligned}$$

which implies $u'(t) = g(t,u(t))[\zeta(t) + \frac{(g(t,u(t)))'u(t)}{(g(t,u(t)))^2}] \in C(I,\mathbb{R})$. This completes the proof. □

Now we are ready to state and prove our main result.

Theorem 3 *Assume that hypotheses* (D_1)*–*(D_4) *hold. Then the fractional hybrid Sturm–Liouville problem* (1) *has a solution* $u \in X = C(I,\mathbb{R})$*. Moreover, if* $(g(t,u(t)))' \in C(I,\mathbb{R})$*, then* $u \in C^1(I,\mathbb{R})$*.*

Proof By using Lemma(6), problem (1) is equivalent to the integral equation (3). Define the map $\Theta : X \to X$ by $\Theta u(t) = g(t,u(t))Hu(t)$, where

$$\begin{aligned}Hu(t) = {}& E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds\\ &+ E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds - E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds\\ &+ E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds\\ &- E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds\\ &+ \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds + \int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds.\end{aligned}$$

By using (D_4), there exists $r > 0$ such that

$$\big(\|\phi\|r + g_0\big)(\mathcal{A}_1 r + \mathcal{A}_2) \le r \quad \text{and} \quad (2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1 < 1.$$

Consider the closed ball B_r, where $B_r = \{u \in X : \|u\| \le r\}$. Clearly, B_r is a closed and bounded subset of X. Define the map $\alpha : X \times X \to [0,\infty)$ by $\alpha(u,v) = 1$ whenever $u, v \in B_r$ and $\alpha(u,v) = 0$ otherwise. Note that

$$\begin{aligned}\big|\tilde{f}\big(u(s)\big)\big| &= \big|\tilde{f}\big(u(s)\big) - \tilde{f}(0) + \tilde{f}(0)\big| \le \big|\tilde{f}\big(u(s)\big) - \tilde{f}(0)\big| + \big|\tilde{f}(0)\big|\\ &\le \widetilde{\mathcal{K}}\big|u(s)\big| + \big|\tilde{f}(0)\big| \le \widetilde{\mathcal{K}}\|u\| + \tilde{f}_0,\end{aligned}$$

$|g(s,u(s))| \le \|\phi\|\|u\| + g_0$, and $|f(u(s))| \le \mathcal{K}\|u\| + f_0$. We prove that the operator Θ satisfies the conditions of Lemma 1. We prove it in some steps.

Step 1: In this step, we prove $\|\Theta u\| \leq r$ whenever $u \in B_r$.
Let $u \in B_r$. Then we have

$$\begin{aligned}
&|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(u(s)\big)\big|\,ds \\
&\quad\leq \frac{|E||\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|u\|+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j|b_j}{p} \\
&\quad\leq \frac{|E||\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}r+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j|T}{p} \\
&\quad= \frac{T|E||\nu|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{j=1}^{n}|\eta_j|}{p}r + \frac{T|E||\nu|\|\tilde{p}\|\tilde{f}_0\sum_{j=1}^{n}|\eta_j|}{p}
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(u(s)\big)\big|\,ds &\leq \frac{T|E|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{i=1}^{m}|\xi_i|}{p}r \\
&\quad+ \frac{T|E|\|\tilde{p}\|\tilde{f}_0\sum_{i=1}^{m}|\xi_i|}{p}.
\end{aligned} \tag{11}$$

Since $I^\alpha(1) = \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau = \frac{s^\alpha}{\Gamma(\alpha+1)}$, we get

$$\begin{aligned}
&|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{1}{|p(s)|}I^\alpha\big(|q(s)||u(s)|\big)\,ds \\
&\quad\leq \frac{|E|\|q\|\|u\|}{p}\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\right)ds \\
&\quad\leq \frac{T^{\alpha+1}|E|\|q\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}r
\end{aligned} \tag{12}$$

and

$$|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{1}{|p(s)|}I^\alpha\big(|q(s)||u(s)|\big)\,ds \leq \frac{T^{\alpha+1}|E||\nu|\|q\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}r. \tag{13}$$

Moreover, we have

$$\begin{aligned}
|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{1}{|p(s)|}I^\alpha\big(|h(s)|\big|f\big(u(s)\big)\big|\big)\,ds &\leq \frac{T^{\alpha+1}\mathcal{K}|E||\nu|\|h\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}r \\
&\quad+ \frac{T^{\alpha+1}|E||\nu|\|h\|f_0\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)},
\end{aligned} \tag{14}$$

$$\begin{aligned}
|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{1}{|p(s)|}I^\alpha\big(|h(s)|\big|f\big(u(s)\big)\big|\big)\,ds &\leq \frac{T^{\alpha+1}\mathcal{K}|E|\|h\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}r \\
&\quad+ \frac{T^{\alpha+1}|E|\|h\|f_0\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)},
\end{aligned} \tag{15}$$

$$\int_0^t \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}(u(s))\right| ds \le \frac{T\widetilde{\mathcal{K}}\|\tilde{p}\|}{p} r + \frac{T\tilde{f}_0\|\tilde{p}\|}{p}, \tag{16}$$

$$\int_0^t \frac{1}{|p(s)|} I^\alpha\left(|q(s)||u(s)|\right) ds \le \frac{T^{\alpha+1}\|q\|}{p\Gamma(\alpha+2)} r, \tag{17}$$

and

$$\int_0^t \frac{1}{|p(s)|} I^\alpha\left(|h(s)|\left|f(u(s))\right|\right) ds \le \frac{T^{\alpha+1}\mathcal{K}\|h\|}{p\Gamma(\alpha+2)} r + \frac{T^{\alpha+1}\|h\|f_0}{p\Gamma(\alpha+2)}. \tag{18}$$

Since

$$\begin{aligned}
&\left|Hu(t)\right| \\
&\quad\le |E||\nu|\sum_{j=1}^n |\eta_j| \int_0^{b_j} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}(u(s))\right| ds + |E|\sum_{i=1}^m |\xi_i| \int_0^{a_i} \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}(u(s))\right| ds \\
&\qquad + |E|\sum_{i=1}^m |\xi_i| \int_0^{a_i} \frac{1}{|p(s)|} I^\alpha\left(|q(s)||u(s)|\right) ds \\
&\qquad + |E||\nu|\sum_{j=1}^n |\eta_j| \int_0^{b_j} \frac{1}{|p(s)|} I^\alpha\left(|q(s)||u(s)|\right) ds \\
&\qquad + |E||\nu|\sum_{j=1}^n |\eta_j| \int_0^{b_j} \frac{1}{|p(s)|} I^\alpha\left(|h(s)|\left|f(u(s))\right|\right) ds \\
&\qquad + |E|\sum_{i=1}^m |\xi_i| \int_0^{a_i} \frac{1}{|p(s)|} I^\alpha\left(|h(s)|\left|f(u(s))\right|\right) ds \\
&\qquad + \int_0^t \frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}(u(s))\right| ds + \int_0^t \frac{1}{|p(s)|} I^\alpha\left(|q(s)||u(s)|\right) ds \\
&\qquad + \int_0^t \frac{1}{|p(s)|} I^\alpha\left(|h(s)|\left|f(u(s))\right|\right) ds,
\end{aligned}$$

by using (10)–(18), we find $|Hu(t)| \le \mathcal{A}_1 r + \mathcal{A}_2$, where

$$\begin{aligned}
\mathcal{A}_1 &= \frac{T|E||\nu|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{j=1}^n |\eta_j|}{p} + \frac{T|E|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{i=1}^m |\xi_i|}{p} \\
&\quad + \frac{T^{\alpha+1}|E|\|q\|\sum_{i=1}^m |\xi_i|}{p\Gamma(\alpha+2)} \\
&\quad + \frac{T^{\alpha+1}|E||\nu|\|q\|\sum_{j=1}^n |\eta_j|}{p\Gamma(\alpha+2)} + \frac{T^{\alpha+1}\mathcal{K}E||\nu|\|h\|\sum_{j=1}^n |\eta_j|}{p\Gamma(\alpha+2)} \\
&\quad + \frac{T^{\alpha+1}\mathcal{K}E||h\|\sum_{i=1}^m |\xi_i|}{p\Gamma(\alpha+2)} \\
&\quad + \frac{T\widetilde{\mathcal{K}}\|\tilde{p}\|}{p} + \frac{T^{\alpha+1}\|q\|}{p\Gamma(\alpha+2)} + \frac{T^{\alpha+1}\mathcal{K}\|h\|}{p\Gamma(\alpha+2)} \\
&= \frac{T\|\tilde{p}\|\widetilde{\mathcal{K}}}{p}\left(|E|\left(\sum_{i=1}^m |\xi_i| + |\nu|\sum_{j=1}^n |\eta_j|\right) + 1\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{T^{\alpha+1}}{p\Gamma(\alpha+2)}\left[\left(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right) + 1\right)\left(\|q\| + \mathcal{K}\|h\|\right)\right] \\
&= \frac{T}{p}\left(\widetilde{\mathcal{K}}\|\tilde{p}\| + \frac{T^{\alpha}(\|q\| + \mathcal{K}\|h\|)}{\Gamma(\alpha+2)}\right)\left(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right) + 1\right)
\end{aligned}
$$

and $\mathcal{A}_2 = \frac{T}{p}(\tilde{f}_0\|\tilde{p}\| + \frac{T^{\alpha}\|h\|f_0}{\Gamma(\alpha+2)})(|E|(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1)$. Thus,

$$
\left|\Theta u(t)\right| = \left|g\left(t,u(t)\right)\right|\left|Hu(t)\right| \leq \left(\|\phi\|r + g_0\right)(\mathcal{A}_1 r + \mathcal{A}_2) \leq r.
$$

Hence, $\|\Theta u\| \leq r$ and so $\Theta B_r \subseteq B_r$.

Step 2: Let $u, \nu \in B_r$. By using a similar method to that in step 1, we get

$$
\begin{aligned}
&|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\left(u(s)\right) - \tilde{f}\left(\nu(s)\right)\right|ds \\
&\quad\leq \frac{T|E||\nu|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{j=1}^{n}|\eta_j|}{p}\|u-\nu\|, \\
&|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\left(u(s)\right) - \tilde{f}\left(\nu(s)\right)\right|ds \\
&\quad\leq \frac{T|E|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{i=1}^{m}|\xi_i|}{p}\|u-\nu\|, \\
&|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{1}{|p(s)|}I^{\alpha}\left(\left|q(s)\right|\left|u(s)-\nu(s)\right|\right)ds \leq \frac{T^{\alpha+1}|E|\|q\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\|u-\nu\|, \\
&|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{1}{|p(s)|}I^{\alpha}\left(\left|q(s)\right|\left|u(s)-u(s)\right|\right)ds \\
&\quad\leq \frac{T^{\alpha+1}|E||\nu|\|q\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\|u-\nu\|, \\
&|E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{1}{|p(s)|}I^{\alpha}\left(\left|h(s)\right|\left|f\left(u(s)\right) - f\left(\nu(s)\right)\right|\right)ds \\
&\quad\leq \frac{T^{\alpha+1}\mathcal{K}|E||\nu|\|h\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\|u-\nu\|, \\
&|E|\sum_{i=1}^{m}|\xi_i|\int_0^{a_i}\frac{1}{|p(s)|}I^{\alpha}\left(\left|h(s)\right|\left|f\left(u(s)\right) - f\left(\nu(s)\right)\right|\right)ds \\
&\quad\leq \frac{T^{\alpha+1}\mathcal{K}|E|\|h\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\|u-\nu\|, \\
&\int_0^{t}\frac{|\tilde{p}(s)|}{|p(s)|}\left|\tilde{f}\left(u(s)\right) - \tilde{f}\left(\nu(s)\right)\right|ds \leq \frac{T\widetilde{\mathcal{K}}\|\tilde{p}\|}{p}\|u-\nu\|, \\
&\int_0^{t}\frac{1}{|p(s)|}I^{\alpha}\left(\left|q(s)\right|\left|u(s)-\nu(s)\right|\right)ds \leq \frac{T^{\alpha+1}\|q\|}{p\Gamma(\alpha+2)}\|u-\nu\|,
\end{aligned}
$$

and $\int_0^t \frac{1}{|p(s)|} I^\alpha(|h(s)||f(u(s)) - f(v(s))|)\,ds \le \frac{T^{\alpha+1}\mathcal{K}\|h\|}{p\Gamma(\alpha+2)}\|u-v\|$. Thus,

$$\begin{aligned}
&\bigl|Hu(t) - Hv(t)\bigr| \\
&\quad\le |E||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{|\tilde{p}(s)|}{|p(s)|}\bigl|\tilde{f}(u(s)) - \tilde{f}(v(s))\bigr|\,ds \\
&\qquad + |E| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{|\tilde{p}(s)|}{|p(s)|}\bigl|\tilde{f}(u(s)) - \tilde{f}(v(s))\bigr|\,ds \\
&\qquad + |E| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{1}{|p(s)|} I^\alpha\bigl(|q(s)||u(s) - v(s)|\bigr)\,ds \\
&\qquad + |E||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{1}{|p(s)|} I^\alpha\bigl(|q(s)||u(s) - v(s)|\bigr)\,ds \\
&\qquad + |E||\nu| \sum_{j=1}^{n} |\eta_j| \int_0^{b_j} \frac{1}{|p(s)|} I^\alpha\bigl(|h(s)||f(u(s)) - f(v(s))|\bigr)\,ds \\
&\qquad + |E| \sum_{i=1}^{m} |\xi_i| \int_0^{a_i} \frac{1}{|p(s)|} I^\alpha\bigl(|h(s)||f(u(s)) - f(v(s))|\bigr)\,ds \\
&\qquad + \int_0^t \frac{|\tilde{p}(s)|}{|p(s)|}\bigl|\tilde{f}(u(s)) - \tilde{f}(v(s))\bigr|\,ds + \int_0^t \frac{1}{|p(s)|} I^\alpha\bigl(|q(s)||u(s) - u(s)|\bigr)\,ds \\
&\qquad + \int_0^t \frac{1}{|p(s)|} I^\alpha\bigl(|h(s)||f(u(s)) - f(v(s))|\bigr)\,ds \\
&\quad\le \mathcal{A}_1\|u - v\|.
\end{aligned}$$

Hence, $|Hu(t) - Hv(t)| \le \mathcal{A}_1\|u-v\|$. This implies that

$$\begin{aligned}
\bigl|\Theta u(t) - \Theta v(t)\bigr| &= \bigl|g(t,u(t))Hu(t) - g(t,v(t))Hv(t)\bigr| \\
&= \bigl|g(t,u(t))Hu(t) - g(t,u(t))Hv(t) + g(t,u(t))Hv(t) - g(t,v(t))Hv(t)\bigr| \\
&= \bigl|g(t,u(t))[Hu(t) - Hv(t)] + Hv(t)[g(t,u(t)) - g(t,v(t))]\bigr| \\
&\le \bigl|g(t,u(t))\bigr|\bigl|Hu(t) - Hv(t)\bigr| + \bigl|Hv(t)\bigr|\bigl|g(t,u(t)) - g(t,v(t))\bigr| \\
&\le (\|\phi\|r + g_0)\mathcal{A}_1\|u-v\| + (\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\|\|u-v\| \\
&= \bigl((2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1\bigr)\|u-v\|,
\end{aligned}$$

and so $\|\Theta u - \Theta v\| \le ((2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1)\|u-v\|$ for all $u, v \in B_r$. Now, consider the map $\psi(t) = ((2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1)t$. Then $\psi \in \Psi$ and $\|\Theta u - \Theta v\| \le \psi(\|u-v\|)$ for all $u, v \in B_r$. Thus, $\alpha(u,v)\|\Theta u - \Theta v\| \le \psi(\|u-v\|)$ for all $u, v \in C(I,\mathbb{R})$, that is, Θ is an α-ψ-contraction. Now, we show that Θ is an α-admissible map. Let $\alpha(u,v) \ge 1$. Then $u, v \in B_r$. By using the first step, $\Theta u, \Theta v \in B_r$ and so $\alpha(\Theta u, \Theta v) \ge 1$. Assume that $\{u_n\}$ is a sequence in $C(I,\mathbb{R})$ such that $\alpha(u_{n-1}, u_n) \ge 1$ for all $n \ge 1$ and $u_n \to u \in C(I,\mathbb{R})$. Then $\{u_n\}$ is a sequence in B_r. Since B_r is closed, $u \in B_r$ and so $\alpha(u_n, u) \ge 1$ for all n. Let $u_0 \in B_r \subset X$. Since $\Theta B_r \subset B_r$, $\Theta u_0 \in B_r$ and so $\alpha(u_0, \Theta u_0) \ge 1$. Now, by using Lemma 1, Θ has a fixed point in $C(I,\mathbb{R})$ which is a solution for problem (1). □

Example 1 Consider the fractional hybrid Sturm–Liouville differential equation

$$D^{\frac{4}{5}}\left(600\sqrt{1+t^2}\left(\frac{u(t)}{g(t,u(t))}\right)' - \frac{e^{-t}}{100}\left(\frac{1}{3}\sin u(t)+1\right)\right) + e^{-\sqrt{t}}u(t) = e^{-t}\cos t \tan^{-1}\bigl(u(t)+1\bigr) \tag{19}$$

with boundary value conditions

$$\begin{cases} (\frac{u(t)}{g(t,u(t))})'_{t=0} = \frac{1}{60{,}000}(\frac{1}{3}u(0)+1), \\ \sum_{i=1}^{2}\frac{1}{2000i}(\frac{u(\frac{1}{4^i})}{g(\frac{1}{4^i},u(\frac{1}{4^i}))}) = \frac{1}{111}\sum_{j=1}^{3}\frac{1}{10^j}(\frac{u(\frac{1}{3^j})}{g(\frac{1}{3^j},u(\frac{1}{3^j}))}), \end{cases} \tag{20}$$

where $g(t,u(t)) = \frac{|\sin t|}{2\pi}\frac{|u(t)|}{1+|u(t)|} + \frac{|\cos t|}{2}e^{-2\pi t}$. Put $\alpha = \frac{4}{5}$, $T = 1$, $r = 0.1$, $\xi_1 = \frac{1}{2000}$, $\xi_2 = \frac{1}{4000}$, $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{100}$, $\eta_3 = \frac{1}{1000}$, $p(t) = 600\sqrt{1+t^2}$, $\tilde{p}(t) = \frac{e^{-t}}{100}$, $q(t) = e^{-\sqrt{t}}$, $h(t) = e^{-t}\cos t$, $f(u(t)) = \tan^{-1}(u(t)+1)$, and $\tilde{f}(u(t)) = \frac{1}{3}\sin u(t) + 1$. Then we have $|\frac{\partial f(u)}{\partial u}| \le 1 = \mathcal{K}$, $f_0 = \frac{\pi}{4}$, $|\frac{\partial \tilde{f}(u)}{\partial u}| \le \frac{1}{3} = \tilde{\mathcal{K}}$, $\tilde{f}_0 = 1$, $p = 600$, $\|\tilde{p}\| = \frac{1}{100}$, $\|q\| = 1$, $\|h\| = 1$. Also,

$$\bigl|g\bigl(t,u(t)\bigr) - g\bigl(t,v(t)\bigr)\bigr| = \frac{|\sin t|}{2\pi}\frac{||u(t)|-|v(t)||}{(1+|u(t)|)(1+|v(t)|)} \le \frac{|\sin t|}{2\pi}\bigl|u(t)-v(t)\bigr|.$$

Note that $\|\phi\| = \frac{1}{2\pi}$ and $g_0 = \frac{1}{2}$, $\sum_{i=1}^{2}\frac{1}{2000i} - \frac{1}{111}\sum_{j=1}^{3}\frac{1}{(10)^j} = \frac{3}{4000} - \frac{1}{1000} = -\frac{1}{4000} \ne 0$ and $E = -4000$. Then $|E|(\sum_{i=1}^{2}|\xi_i| + |\nu|\sum_{j=1}^{3}|\eta_j|) + 1 = 4000(\frac{3}{4000} + \frac{1}{111}\frac{111}{1000}) + 1 = 8$, and so

$$\begin{aligned}
\mathcal{A}_1 &= \frac{T}{p}\left(\tilde{\mathcal{K}}\|\tilde{p}\| + \frac{T^\alpha(\|q\| + \mathcal{K}\|h\|)}{\Gamma(\alpha+2)}\right)\left(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right)+1\right) \\
&= \frac{8}{600}\left(\frac{1}{300} + \frac{2}{\Gamma(\frac{14}{5})}\right) \approx 0.0159506855, \\
\mathcal{A}_2 &= \frac{T}{p}\left(\tilde{f}_0\|\tilde{p}\| + \frac{T^\alpha\|h\|f_0}{\Gamma(\alpha+2)}\right)(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right) \\
&= \frac{8}{600}\left(\frac{1}{100} + \frac{\pi}{4\Gamma(\frac{14}{5})}\right) \approx 0.0063796996, \\
\bigl(\|\phi\|r + g_0\bigr)(\mathcal{A}_1 r + \mathcal{A}_2) &\approx \left(\frac{0.1}{2\pi} + \frac{1}{2}\right)(0.0159506855 \times 0.1 + 0.0063796996) \\
&\approx 0.0041143065 \le 0.1 = r,
\end{aligned}$$

and

$$\begin{aligned}
&(2\mathcal{A}_1 r + \mathcal{A}_2)\|\phi\| + g_0\mathcal{A}_1 \\
&\quad\approx (2 \times 0.0159506855 \times 0.1 + 0.0063796996) \times \frac{1}{2\pi} + \frac{1}{2} \times 0.0159506855 \\
&\quad\approx 0.0094984296 < 1.
\end{aligned}$$

Now, by using Theorem 3, problem (19)–(20) has a solution.

We will need the following corollary in the next section.

In Theorem 3, put $g(t,x) = 1$ for all $t \in I$ and $x \in \mathbb{R}$. Then conditions $(D_1) - (D_2)$ reduce to the following conditions:

(M_1) The functions $f, \tilde{f} : \mathbb{R} \to \mathbb{R}$ are differentiable on $[0,T]$, $\frac{\partial f}{\partial u}$ and $\frac{\partial \tilde{f}}{\partial u}$ are bounded on $[0,T]$ with $\frac{\partial f}{\partial u} \le \mathcal{K}$ and $\frac{\partial \tilde{f}}{\partial u} \le \tilde{\mathcal{K}}$.

(M_2) The function $p \in C^1(I,\mathbb{R})$ with $p(t) \neq 0$ for all $t \in I$, $\inf_{t\in I} |p(t)| = p$ and $\tilde{p} \in C^1(I,\mathbb{R})$. Also, $\tilde{p}(t)$, $q(t)$, and $h(t)$ are absolutely continuous functions on I.

Corollary 1 *Assume that hypotheses (M_1)–(M_2) hold and there exists a number $r > 0$ such that $\frac{\mathcal{C}_2}{1-\mathcal{C}_1} \le r$, where $E = \frac{1}{\sum_{i=1}^m \xi_i - \nu \sum_{j=1}^n \eta_j}$,*

$$\mathcal{C}_1 = \frac{T}{p}\left(\widetilde{\mathcal{K}}\|\tilde{p}\| + \frac{T^\alpha(\|q\| + \mathcal{K}\|h\|)}{\Gamma(\alpha+2)}\right)\left(|E|\left(\sum_{i=1}^m |\xi_i| + |\nu| \sum_{j=1}^n |\eta_j|\right) + 1\right) < 1,$$

$\mathcal{C}_2 = \frac{T}{p}(\tilde{f}_0\|\tilde{p}\| + \frac{T^\alpha \|h\| f_0}{\Gamma(\alpha+2)})(|E|(\sum_{i=1}^m |\xi_i| + |\nu| \sum_{j=1}^n |\eta_j|) + 1)$, $f_0 = |f(0)|$, and $\tilde{f}_0 = |\tilde{f}(0)|$. Then the fractional hybrid Sturm–Liouville differential equation

$$^cD^\alpha\big(p(t)u'(t) - \tilde{p}(t)\tilde{f}\big(u(t)\big)\big) + q(t)u(t) = h(t)f\big(u(t)\big) \tag{21}$$

with hybrid multi-point boundary condition

$$\begin{cases} u'(0) = \left(\frac{\tilde{p}(t)}{p(t)}\tilde{f}(u(t))\right)_{t=0}, \\ \sum_{i=1}^m \xi_i u(a_i) = \nu \sum_{j=1}^n \eta_j u(b_j) \end{cases} \tag{22}$$

has a solution $u \in C^1(I,\mathbb{R})$ if and only if u solves the integral equation

$$\begin{aligned} u(t) = {} & E\nu \sum_{j=1}^n \eta_j \int_0^{b_j} \frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - E\sum_{i=1}^m \xi_i \int_0^{a_i} \frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds \\ & + E\sum_{i=1}^m \xi_i \int_0^{a_i} \frac{1}{p(s)} I^\alpha\big(q(s)u(s)\big)\,ds - E\nu \sum_{j=1}^n \eta_j \int_0^{b_j} \frac{1}{p(s)} I^\alpha\big(q(s)u(s)\big)\,ds \\ & + E\nu \sum_{j=1}^n \eta_j \int_0^{b_j} \frac{1}{p(s)} I^\alpha\big(h(s)f\big(u(s)\big)\big)\,ds - E\sum_{i=1}^m \xi_i \int_0^{a_i} \frac{1}{p(s)} I^\alpha\big(h(s)f\big(u(s)\big)\big)\,ds \\ & + \int_0^t \frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \int_0^t \frac{1}{p(s)} I^\alpha\big(q(s)u(s)\big)\,ds + \int_0^t \frac{1}{p(s)} I^\alpha\big(h(s)f\big(u(s)\big)\big)\,ds. \end{aligned}$$

Proof Note that problem (21)–(22) is a special case of problem (1) with $g(t,x) = 1$ for all $t \in I$ and $x \in \mathbb{R}$. Now, by using Theorem 3, we can conclude that problem (21)–(22) has a solution $u \in C^1(I,\mathbb{R})$. □

In Theorem 3, put $\tilde{p}(t) = 0$ for all $t \in I$ and $\tilde{f}(x) = 0$ for all $x \in \mathbb{R}$. Then $(D_1) - (D_3)$ reduce to the following conditions:

(L_1) The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $[0,T]$ and $\frac{\partial f}{\partial u}$ is bounded on $[0,T]$ with $\frac{\partial f}{\partial u} \le \mathcal{K}$.

(L_2) The function $p \in C^1(I,\mathbb{R})$ with $p(t) \neq 0$ for all $t \in I$, $\inf_{t\in I} |p(t)| = p$. Also, $q(t)$ and $h(t)$ are absolutely continuous functions on I.

(L_3) The function $g : I \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous in its two variables and there exists a function $\phi(t) \geq 0$ ($\forall t \in I$) such that $|g(t,x) - g(t,y)| \leq \phi(t)|x - y|$ for all $(t,x,y) \in I \times \mathbb{R} \times \mathbb{R}$.

In this case, we obtain the next result.

Corollary 2 *Assume that hypotheses* (L_1)–(L_3) *hold and there exists a number* $r > 0$ *such that* $(\|\phi\| r + g_0)(\mathcal{B}_1 r + \mathcal{B}_2) \leq r$ *and* $(2\mathcal{B}_1 r + \mathcal{B}_2)\|\phi\| + g_0\mathcal{B}_1 < 1$, *where*

$$\mathcal{B}_1 = \frac{T^{\alpha+1}(\|q\| + \mathcal{K}\|h\|)}{p\Gamma(\alpha+1)}\left(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right) + 1\right),$$

$$\mathcal{B}_2 = \frac{T^{\alpha+1}\|h\| f_0}{p\Gamma(\alpha+2)}\left(|E|\left(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|\right) + 1\right),$$

$E = \frac{1}{\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j}$, $f_0 = |f(0)|$, *and* $g_0 = \sup_{t\in I} g(t,0)$. *Then the hybrid Sturm–Liouville problem*

$$\begin{cases} {}^cD^\alpha(p(t)(\frac{u(t)}{g(t,u(t))})') + q(t)u(t) = h(t)f(u(t)), \\ (\frac{u(t)}{g(t,u(t))})'_{t=0} = 0, \\ \sum_{i=1}^{m}\xi_i(\frac{u(a_i)}{g(a_i,u(a_i))}) = \nu\sum_{j=1}^{n}\eta_j(\frac{u(b_j)}{g(b_j,u(b_j))}), \end{cases} \tag{23}$$

has a solution $u \in C(I,\mathbb{R})$. *Moreover, if* $(g(t,u(t)))' \in C(I,\mathbb{R})$, *then* $u \in C^1(I,\mathbb{R})$.

Proof By a method similar to that in the proof of Corollary 1, we can conclude that problem 23 has a solution $u \in C(I,\mathbb{R})$ (also, $u \in C^1(I,\mathbb{R})$ whenever $(g(t,u(t)))' \in C(I,\mathbb{R})$). □

3 Continuous dependence

In this section we are going to investigate continuous dependence(on the coefficient ξ_i and η_j of the hybrid multi-point condition) of the solution of the fractional hybrid Sturm–Liouville differential equation (21) with the hybrid multi-point boundary condition (22). Note that the main theorem of this section is a hybrid version of Theorem 3.2 in [42].

Definition 4 (see [42]) The solution of the fractional hybrid Sturm–Liouville differential equation (21) is continuously dependent on the data ξ_i and η_j if, for every $\epsilon > 0$, there exist $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ such that, for any two solutions $u(t)$ and $\tilde{u}(t)$ of (21) with the initial data (22) and

$$\begin{cases} \tilde{u}'(0) = (\frac{\tilde{p}(t)}{p(t)}\tilde{f}(\tilde{u}(t)))_{t=0}, \\ \sum_{i=1}^{m}\tilde{\xi}_i\tilde{u}(a_i) = \nu\sum_{j=1}^{n}\tilde{\eta}_j\tilde{u}(b_j), \end{cases} \tag{24}$$

respectively, one has $\sum_{i=1}^{m}|\xi_i - \tilde{\xi}_i| < \delta_1$ and $\sum_{j=1}^{n}|\eta_j - \tilde{\eta}_j| < \delta_2$, then $\|u - \tilde{u}\| < \epsilon$ for all $t \in I$.

Theorem 5 *Assume that the assertions of Corollary* (1) *are satisfied. Then the solution of the fractional hybrid Sturm–Liouville problem* (21)–(22) *is continuously dependent on the coefficients* ξ_i *and* η_j *of the hybrid multi-point boundary condition.*

Proof Let u be solution of the fractional hybrid Sturm–Liouville problem (21)–(22), and let

$$\begin{aligned}\tilde{u}(t) &= \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds - \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds \\ &\quad + \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds \\ &\quad + \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds - \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds \\ &\quad + \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds - \int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds + \int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\end{aligned}$$

be a solution of the fractional hybrid Sturm–Liouville differential equation (21) with hybrid multi-point boundary condition (24). Hence

$$\begin{aligned}&\big|u(t)-\tilde{u}(t)\big| \\ &\quad\le \left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right| \\ &\qquad + \left|-E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds + \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right| \\ &\qquad + \left|E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds - \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds\right| \\ &\qquad + \left|-E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds + \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds\right| \\ &\qquad + \left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\right| \\ &\qquad + \left|-E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds + \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\right| \\ &\qquad + \left|\int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right| \\ &\qquad + \bigg|-\int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds + \int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds \\ &\qquad + \int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds - \int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\bigg|.\end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned}&\left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right| \\ &\quad = \bigg|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\end{aligned}$$

$$+ E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds$$
$$\left. + \tilde{E}\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right|,$$

and so

$$\begin{aligned}
&\left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right|\\
&\quad\le |E||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(u(s)\big)-\tilde{f}\big(\tilde{u}(s)\big)\big|\,ds\\
&\qquad + |E-\tilde{E}||\nu|\sum_{j=1}^{n}|\eta_j|\int_0^{b_j}\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(\tilde{u}(s)\big)\big|\,ds\\
&\qquad + |\tilde{E}||\nu|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|\int_0^{b_j}\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(\tilde{u}(s)\big)\big|\,ds\\
&\quad\le \frac{T|E||\nu|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{j=1}^{n}|\eta_j|}{p}\|u-\tilde{u}\|\\
&\qquad + |E||\tilde{E}|\left(\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i| + |\nu|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|\right)\frac{T|\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j|}{p}\\
&\qquad + \frac{T|\tilde{E}||\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|}{p}.
\end{aligned}$$

Since $\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|<\delta_1$ and $\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|<\delta_2$, we get

$$\begin{aligned}
&\left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds - \tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right|\\
&\quad\le \frac{T|E||\nu|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{j=1}^{n}|\eta_j|}{p}\|u-\tilde{u}\|\\
&\qquad + |E||\tilde{E}|\big(\delta_1+|\nu|\delta_2\big)\frac{T|\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j|}{p}\\
&\qquad + \frac{T|\tilde{E}||\nu|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\delta_2}{p}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|-E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds + \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right|\\
&\quad\le \frac{T|E|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{i=1}^{m}|\xi_i|}{p}\|u-\tilde{u}\|
\end{aligned}$$

$$
\begin{aligned}
&+|E||\tilde{E}|\left(\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|+|\nu|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|\right)\frac{T\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{i=1}^{m}|\xi_i|}{p} \\
&+\frac{T|\tilde{E}|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|}{p} \\
\leq{}&\frac{T|E|\|\tilde{p}\|\widetilde{\mathcal{K}}\sum_{i=1}^{m}|\xi_i|}{p}\|u-\tilde{u}\|+|E||\tilde{E}|\big(\delta_1+|\nu|\delta_2\big)\frac{T\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{i=1}^{m}|\xi_i|}{p} \\
&+\frac{T|\tilde{E}|\|\tilde{p}\|(\widetilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\delta_1}{p}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left|E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds\right| \\
&\quad\leq\frac{T^{\alpha+1}|E|\|q\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\|u-\tilde{u}\| \\
&\qquad+\frac{T^{\alpha+1}\|q\||E||\tilde{E}|\|\tilde{u}\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\big(\delta_1+|\nu|\delta_2\big)+\frac{T^{\alpha+1}\|q\||\tilde{E}|\|\tilde{u}\|\delta_1}{p\Gamma(\alpha+2)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left|-E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds+\tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds\right| \\
&\quad\leq\frac{T^{\alpha+1}|E||\nu|\|q\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\|u-\tilde{u}\| \\
&\qquad+\frac{T^{\alpha+1}\|q\||\nu||E||\tilde{E}|\|\tilde{u}\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\big(\delta_1+|\nu|\delta_2\big)+\frac{T^{\alpha+1}\|q\||\nu||\tilde{E}|\|\tilde{u}\|\delta_2}{p\Gamma(\alpha+2)}.
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\left|E\nu\sum_{j=1}^{n}\eta_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds-\tilde{E}\nu\sum_{j=1}^{n}\tilde{\eta}_j\int_0^{b_j}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\right| \\
&\quad\leq\frac{T^{\alpha+1}|E||\nu|\|h\|\mathcal{K}\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\|u-\tilde{u}\| \\
&\qquad+\frac{T^{\alpha+1}\|h\||\nu||E||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\big(\delta_1+|\nu|\delta_2\big) \\
&\qquad+\frac{T^{\alpha+1}\|h\||\nu||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\delta_2}{p\Gamma(\alpha+2)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left|-E\sum_{i=1}^{m}\xi_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds+\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i\int_0^{a_i}\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\right| \\
&\quad\leq\frac{T^{\alpha+1}|E|\|h\|\mathcal{K}\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\|u-\tilde{u}\|
\end{aligned}
$$

$$+\frac{T^{\alpha+1}\|h\||E||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\big(\delta_1+|\nu|\delta_2\big)$$
$$+\frac{T^{\alpha+1}\|h\||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\delta_1}{p\Gamma(\alpha+2)}.$$

Note that

$$\begin{aligned}\left|\int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(u(s)\big)\,ds-\int_0^t\frac{\tilde{p}(s)}{p(s)}\tilde{f}\big(\tilde{u}(s)\big)\,ds\right| &= \int_0^t\frac{|\tilde{p}(s)|}{|p(s)|}\big|\tilde{f}\big(u(s)\big)-\tilde{f}\big(\tilde{u}(s)\big)\big|\,ds\\ &\le\frac{\tilde{\mathcal{K}}\|\tilde{p}\|}{p}\|u-\tilde{u}\|.\end{aligned}$$

Also,

$$\begin{aligned}&\bigg|-\int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds+\int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)\tilde{u}(s)\big)\,ds\\ &\quad+\int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds-\int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(\tilde{u}(s)\big)\big)\,ds\bigg|\\ &\le\frac{T^{\alpha+1}(\mathcal{K}\|h\|+\|q\|)}{p\Gamma(\alpha+2)}\|u-\tilde{u}\|.\end{aligned}$$

So from (25) we have

$$\begin{aligned}&\big|u(t)-\tilde{u}(t)\big|\\ &\quad\le\mathcal{C}_1\|u-\tilde{u}\|+\Delta_1\big(\delta_1+|\nu|\delta_2\big)+\Delta_2\big(\delta_1+|\nu|\delta_2\big)\\ &\quad=\mathcal{C}_1\|u-\tilde{u}\|+(\Delta_1+\Delta_2)\big(\delta_1+|\nu|\delta_2\big),\end{aligned}$$

where

$$\begin{aligned}\Delta_1 &= |E||\tilde{E}|\frac{T|\nu|\|\tilde{p}\|(\tilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{j=1}^{n}|\eta_j|}{p}+|E||\tilde{E}|\frac{T\|\tilde{p}\|(\tilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)\sum_{i=1}^{m}|\xi_i|}{p}\\ &\quad+\frac{T^{\alpha+1}\|q\||E||\tilde{E}|\|\tilde{u}\|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}+\frac{T^{\alpha+1}\|q\||\nu||E||\tilde{E}|\|\tilde{u}\|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\\ &\quad+\frac{T^{\alpha+1}\|h\||\nu||E||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+2)}\\ &\quad+\frac{T^{\alpha+1}\|h\||E||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+2)}\end{aligned}$$

and

$$\Delta_2=\frac{T|\tilde{E}|\|\tilde{p}\|(\tilde{\mathcal{K}}\|\tilde{u}\|+\tilde{f}_0)}{p}+\frac{T^{\alpha+1}\|q\||\tilde{E}|\|\tilde{u}\|}{p\Gamma(\alpha+2)}+\frac{T^{\alpha+1}\|h\||\tilde{E}|(\mathcal{K}\|\tilde{u}\|+f_0)}{p\Gamma(\alpha+2)}.$$

Hence,

$$\|u-\tilde{u}\|\le\epsilon=(1-\mathcal{C}_1)^{-1}(\Delta_1+\Delta_2)\big(\delta_1+|\nu|\delta_2\big).$$

So we proved that for every $\epsilon > 0$ there exist $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ such that $\sum_{i=1}^{m} |\xi_i - \tilde{\xi}_i| < \delta_1$ and $\sum_{j=1}^{n} |\eta_j - \tilde{\eta}_j| < \delta_2$, then $\|u - \tilde{u}\| < \epsilon$. □

4 Fractional hybrid Sturm–Liouville equation with integral boundary value conditions

In this section, we investigate the fractional hybrid Sturm–Liouville equation with integral boundary value conditions.

Lemma 6 *Assume that hypotheses (D_1)–(D_2) hold. Then problem (2) is equivalent to the integral equation*

$$\begin{aligned} u(t) = g\bigl(t,u(t)\bigr)\biggl[&\int_a^c \int_0^\theta \int_0^s \frac{\mathcal{E}(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\ &- \int_a^c \int_0^\theta \int_0^s \frac{\mathcal{E}(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\ &- \int_d^e \int_0^\theta \int_0^s \frac{\nu\mathcal{E}(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\ &+ \int_d^e \int_0^\theta \int_0^s \frac{\nu\mathcal{E}(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\ &- \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\biggr], \end{aligned} \tag{26}$$

where $\mathcal{E} = \frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}$, $\varpi(c) - \varpi(a) \neq \nu(\upsilon(e) - \upsilon(d))$, $\varpi(\theta)$ and $\upsilon(\theta)$ are increasing functions, the integrals are in the Riemann–Stieltjes sense, and $0 \leq a < c \leq d < e \leq T$. Also, $(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t))) \in C^1(I,\mathbb{R})$ and $(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t)))'' \in L_1[0,1]$. Moreover, if $(g(t,u(t)))' \in C(I,\mathbb{R})$, then $u \in C^1(I,\mathbb{R})$.

Proof Note that equation (2) can be written as

$$I^{1-\alpha}\left(\frac{d}{dt}\left[p(t)\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)'\right]\right) = -q(t)u(t) + h(t)f\bigl(u(t)\bigr).$$

Hence, $I^1(\frac{d}{dt}[p(t)(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t)))']) = -I^\alpha(q(t)u(t)) + I^\alpha(h(t)f(u(t)))$, and so

$$\begin{aligned} &p(t)\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)' - p(0)\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)'_{t=0} \\ &\quad = -I^\alpha\bigl(q(t)u(t)\bigr) + I^\alpha\bigl(h(t)f\bigl(u(t)\bigr)\bigr). \end{aligned}$$

Since $(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t)))'_{t=0} = 0$, we get

$$p(t)\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)' = -I^\alpha\bigl(q(t)u(t)\bigr) + I^\alpha\bigl(h(t)f\bigl(u(t)\bigr)\bigr),$$

and so

$$\left(\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}\big(u(t)\big)\right)'=-\frac{1}{p(t)}I^{\alpha}\big(q(t)u(t)\big)+\frac{1}{p(t)}I^{\alpha}\big(h(t)f\big(u(t)\big)\big). \tag{27}$$

Thus, we obtain

$$\begin{aligned}&\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}\big(u(t)\big)-\ell\\&\quad=-\int_0^t\frac{1}{p(s)}I^{\alpha}\big(q(s)u(s)\big)\,ds+\int_0^t\frac{1}{p(s)}I^{\alpha}\big(h(s)f\big(u(s)\big)\big)\,ds,\end{aligned} \tag{28}$$

where $\ell=\frac{u(0)}{g(0,u(0))}-\tilde{p}(0)\tilde{f}(u(0))$. For simplicity, put $A(t)=\int_0^t\frac{1}{p(s)}I^{\alpha}(q(s)u(s))\,ds$ and $B(t)=\int_0^t\frac{1}{p(s)}I^{\alpha}(h(s)f(u(s)))\,ds$. Then we get

$$\begin{aligned}&\int_a^c\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\varpi(\theta)-\ell\int_a^c d\varpi(\theta)\\&\quad=-\int_a^c A(\theta)\,d\varpi(\theta)+\int_a^c B(\theta)\,d\varpi(\theta)\end{aligned}$$

and

$$\begin{aligned}&\nu\int_d^e\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\upsilon(\theta)-\ell\nu\int_d^e d\upsilon(\theta)\\&\quad=-\nu\int_d^e A(\theta)\,d\upsilon(\theta)+\nu\int_d^e B(\theta)\,d\upsilon(\theta).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_a^c\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\varpi(\theta)-\ell\big(\varpi(c)-\varpi(a)\big)\\&\quad=-\int_a^c A(\theta)\,d\varpi(\theta)+\int_a^c B(\theta)\,d\varpi(\theta)\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\nu\int_d^e\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\upsilon(\theta)-\ell\nu\big(\upsilon(e)-\upsilon(d)\big)\\&\quad=-\nu\int_d^e A(\theta)\,d\upsilon(\theta)+\nu\int_d^e B(\theta)\,d\upsilon(\theta).\end{aligned} \tag{30}$$

By subtracting (29) from (30) and applying

$$\int_a^c\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\varpi(\theta)=\nu\int_d^e\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\big(u(\theta)\big)\right)d\upsilon(\theta),$$

we get $\ell = \mathcal{E}\int_a^c A(\theta)\,d\varpi(\theta) - \mathcal{E}\int_a^c B(\theta)\,d\varpi(\theta) - \nu\mathcal{E}\int_d^e A(\theta)\,d\upsilon(\theta) + \nu\mathcal{E}\int_d^e B(\theta)\,d\upsilon(\theta)$, where $\mathcal{E} = \frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}$. By substituting the value of ℓ in (28), we conclude that

$$\begin{aligned} u(t) = g\bigl(t,u(t)\bigr)\biggl[&\mathcal{E}\int_a^c\int_0^\theta \frac{1}{p(s)}I^\alpha\bigl(q(s)u(s)\bigr)\,ds\,d\varpi(\theta) \\ &- \mathcal{E}\int_a^c\int_0^\theta \frac{1}{p(s)}I^\alpha\bigl(h(s)f\bigl(u(s)\bigr)\bigr)\,ds\,d\varpi(\theta) \\ &- \nu\mathcal{E}\int_d^e\int_0^\theta \frac{1}{p(s)}I^\alpha\bigl(q(s)u(s)\bigr)\,ds\,d\upsilon(\theta) + \nu\mathcal{E}\int_d^e\int_0^\theta \frac{1}{p(s)}I^\alpha\bigl(h(s)f\bigl(u(s)\bigr)\bigr)\,ds\,d\upsilon(\theta) \\ &+ \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr) - \int_0^t \frac{1}{p(s)}I^\alpha\bigl(q(s)u(s)\bigr)\,ds + \int_0^t \frac{1}{p(s)}I^\alpha\bigl(h(s)f\bigl(u(s)\bigr)\bigr)\,ds\biggr], \end{aligned}$$

where $\mathcal{E} = \frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}$. Note that

$$\frac{1}{p(s)}I^\alpha\bigl(z(s)\bigr) = \frac{1}{p(s)}\int_0^s \frac{(s-\tau)^{\alpha-1}z(\tau)}{\Gamma(\alpha)}\,d\tau = \int_0^s \frac{(s-\tau)^{\alpha-1}z(\tau)}{p(s)\Gamma(\alpha)}\,d\tau,$$

where z is a function. Hence, we can write

$$\begin{aligned} u(t) = g\bigl(t,u(t)\bigr)\biggl[&\mathcal{E}\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\ &- \mathcal{E}\int_a^c\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\ &- \nu\mathcal{E}\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\ &+ \nu\mathcal{E}\int_d^e\int_0^\theta\int_0^s \frac{(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\ &- \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\ &+ \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\biggr], \end{aligned}$$

where $\mathcal{E} = \frac{1}{\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))}$. For the next part, by using (27) we have

$$\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)' = -\frac{1}{p(t)}I^\alpha\bigl(q(t)u(t)\bigr) + \frac{1}{p(t)}I^\alpha\bigl(h(t)f\bigl(u(t)\bigr)\bigr) \in C(I,\mathbb{R}),$$

and so $\frac{d}{dt}(p(t)(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}(u(t))')) = -\frac{d}{dt}I^\alpha(q(t)u(t)) + \frac{d}{dt}I^\alpha(h(t)f(u(t)))$. Hence,

$$\begin{aligned} &I^{1-\alpha}\frac{d}{dt}\left(p(t)\left(\frac{u(t)}{g(t,u(t))} - \tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)'\right) \\ &\quad= -I^{1-\alpha}\frac{d}{dt}I^\alpha\bigl(q(t)u(t)\bigr) + I^{1-\alpha}\frac{d}{dt}I^\alpha\bigl(h(t)f\bigl(u(t)\bigr)\bigr), \end{aligned}$$

and so

$$\begin{aligned}&{}^{c}D^{\alpha}\left(p(t)\left(\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)'\right)\\&\quad=-I^{1-\alpha}I^{\alpha}\frac{d}{dt}\bigl(q(t)u(t)\bigr)+I^{1-\alpha}I^{\alpha}\frac{d}{dt}\bigl(h(t)f\bigl(u(t)\bigr)\bigr)\\&\qquad-I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}q(0)u(0)+I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}h(0)f\bigl(u(0)\bigr).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}{}^{c}D^{\alpha}\left(p(t)\left(\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)'\right)&=-I^{1}\frac{d}{dt}\bigl(q(t)u(t)\bigr)+I^{1}\frac{d}{dt}\bigl(h(t)f\bigl(u(t)\bigr)\bigr)\\&\quad-q(0)u(0)+h(0)f\bigl(u(0)\bigr)\\&=-q(t)u(t)+h(t)f\bigl(u(t)\bigr).\end{aligned}$$

By using (27), we get $(\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}(u(t)))'_{t=0}=0$. Also, by using simple computations and (26), we obtain

$$\int_a^c\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\bigl(u(\theta)\bigr)\right)d\varpi(\theta)=\nu\int_d^e\left(\frac{u(\theta)}{g(\theta,u(\theta))}-\tilde{p}(\theta)\tilde{f}\bigl(u(\theta)\bigr)\right)d\upsilon(\theta).$$

By a similar method to that in the proof of Lemma 2, we can conclude that

$$u\in C^1(I,\mathbb{R})\quad\text{and}\quad\left(\frac{u(t)}{g(t,u(t))}-\tilde{p}(t)\tilde{f}\bigl(u(t)\bigr)\right)''\in L_1[0,1].$$

This completes the proof. □

Put $\mathcal{A}_1^*=\frac{T^{\alpha+1}(\|q\|+\mathcal{K}\|h\|)}{p\Gamma(\alpha+2)}(\frac{\varpi(c)-\varpi(a)+|\nu|(\upsilon(e)-\upsilon(d))}{|\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))|}+1)+\widetilde{\mathcal{K}}\|\tilde{p}\|$ and

$$\mathcal{A}_2^*=\frac{T^{\alpha+1}\|h\|f_0}{p\Gamma(\alpha+2)}\left(\frac{\varpi(c)-\varpi(a)+|\nu|(\upsilon(e)-\upsilon(d))}{|\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))|}+1\right)+\tilde{f}_0\|\tilde{p}\|,$$

where $\varpi(c)-\varpi(a)\neq\nu(\upsilon(e)-\upsilon(d))$, $\varpi(\theta)$ and $\upsilon(\theta)$ are increasing functions, the integrals are in the Riemann–Stieltjes sense, and $0\le a<c\le d<e\le T$.

Theorem 7 *Assume that hypotheses (D_1)–(D_3) hold and there exists a number $r>0$ such that $(\|\phi\|r+g_0)(\mathcal{A}_1^*r+\mathcal{A}_2^*)\le r$ and $(2\mathcal{A}_1^*r+\mathcal{A}_2^*)\|\phi\|+g_0\mathcal{A}_1^*<1$. Then the fractional hybrid Sturm–Liouville problem (2) has a solution $u\in X=C(I,\mathbb{R})$. Moreover, if $(g(t,u(t)))'\in C(I,\mathbb{R})$, then $u\in C^1(I,\mathbb{R})$.*

Proof By using Lemma(6), problem (2) is equivalent to the integral equation (26). Define the map $\Theta:X\to X$ by $\Theta u(t)=g(t,u(t))Hu(t)$, where

$$\begin{aligned}Hu(t)&=\int_a^c\int_0^\theta\int_0^s\frac{\mathcal{E}(s-\tau)^{\alpha-1}q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)\\&\quad-\int_a^c\int_0^\theta\int_0^s\frac{\mathcal{E}(s-\tau)^{\alpha-1}h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta)\end{aligned}$$

$$
\begin{aligned}
&- \int_d^e \int_0^\theta \int_0^s \frac{\nu\mathcal{E}(s-\tau)^{\alpha-1} q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&+ \int_d^e \int_0^\theta \int_0^s \frac{\nu\mathcal{E}(s-\tau)^{\alpha-1} h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&- \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} q(\tau)u(\tau)}{p(s)\Gamma(\alpha)}\,d\tau\,ds \\
&+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1} h(\tau)f(u(\tau))}{p(s)\Gamma(\alpha)}\,d\tau\,ds + \tilde{p}(t)\tilde{f}\big(u(t)\big).
\end{aligned}
$$

By using the hypothesis, there exists $r > 0$ such that

$$
\big(\|\phi\| r + g_0\big)\big(\mathcal{A}_1^* r + \mathcal{A}_2^*\big) \le r \quad \text{and} \quad \big(2\mathcal{A}_1^* r + \mathcal{A}_2^*\big)\|\phi\| + g_0\mathcal{A}_1 < 1.
$$

Consider the ball $B_r = \{u \in X : \|u\| \le r\}$. Clearly, B_r is a closed and bounded subset of X. Define the map $\alpha : X \times X \to [0,\infty)$ by $\alpha(u,v) = 1$ whenever $u, v \in B_r$ and $\alpha(u,v) = 0$ otherwise. Note that $|\tilde{f}(u(s))| \le \widetilde{\mathcal{K}}\|u\| + \tilde{f}_0$, $|g(s,u(s))| \le \|\phi\|\|u\| + g_0$ and $|f(u(s))| \le \mathcal{K}\|u\| + f_0$. We prove that the operator Θ satisfies all the conditions of Lemma 1. We prove it in some steps.

Step 1: In this step, we prove $\|\Theta u\| \le r$ whenever $u \in B_r$.

Let $u \in B_r$. Then we have

$$
\begin{aligned}
&\int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\quad \le \frac{|\mathcal{E}|\|q\| r}{p} \int_a^c \int_0^\theta \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|q\|(\varpi(c)-\varpi(a))}{p\Gamma(\alpha+2)} r, \\
&\int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|h\|\mathcal{K}(\varpi(c)-\varpi(a))}{p\Gamma(\alpha+2)} r \\
&\qquad + \frac{T^{\alpha+1}|\mathcal{E}|\|h\| f_0(\varpi(c)-\varpi(a))}{p\Gamma(\alpha+2)}, \\
&\int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \le \frac{T^{\alpha+1}|\mathcal{E}|\|q\||\nu|(\upsilon(e)-\upsilon(d))}{p\Gamma(\alpha+2)} r, \\
&\int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|h\|\mathcal{K}|\nu|(\upsilon(e)-\upsilon(d))}{p\Gamma(\alpha+2)} r \\
&\qquad + \frac{T^{\alpha+1}|\mathcal{E}|\|h\| f_0|\nu|(\upsilon(e)-\upsilon(d))}{p\Gamma(\alpha+2)}, \\
&\int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \le \frac{T^{\alpha+1}\|q\|}{p\Gamma(\alpha+2)} r, \\
&\int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \le \frac{T^{\alpha+1}\mathcal{K}\|h\|}{p\Gamma(\alpha+2)} r + \frac{T^{\alpha+1}\|h\| f_0}{p\Gamma(\alpha+2)},
\end{aligned}
$$

and $|\tilde{p}(t)\tilde{f}(u(t))| \le \widetilde{\mathcal{K}}\|\tilde{p}\|r + \tilde{f}_0\|\tilde{p}\|$. Since

$$\begin{aligned}
\left|Hu(t)\right| \le{}& \int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&+ \int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&+ \int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&+ \int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \\
&+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds + \left|\tilde{p}(t)\tilde{f}\big(u(t)\big)\right|,
\end{aligned}$$

we get $|Hu(t)| \le \mathcal{A}_1^* r + \mathcal{A}_2^*$, where

$$\mathcal{A}_1^* = \frac{T^{\alpha+1}(\|q\| + \mathcal{K}\|h\|)}{p\Gamma(\alpha+2)}\big(|\mathcal{E}|\big(\varpi(c) - \varpi(a) + |\nu|\big(\upsilon(e) - \upsilon(d)\big)\big) + 1\big) + \widetilde{\mathcal{K}}\|\tilde{p}\|$$

and $\mathcal{A}_2^* = \frac{T^{\alpha+1}\|h\|f_0}{p\Gamma(\alpha+2)}(|\mathcal{E}|(\varpi(c) - \varpi(a) + |\nu|(\upsilon(e) - \upsilon(d))) + 1) + \tilde{f}_0\|\tilde{p}\|$. Thus,

$$\left|\Theta u(t)\right| = \left|g\big(t, u(t)\big)\right|\left|Hu(t)\right| \le \big(\|\phi\|r + g_0\big)\big(\mathcal{A}_1^* r + \mathcal{A}_2^*\big) \le r.$$

Hence, $\|\Theta u\| \le r$ and so $\Theta B_r \subseteq B_r$.

Step 2: Let $u, v \in B_r$. By using a method similar to that in step 1, we get

$$\begin{aligned}
&\int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau) - v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|q\|(\varpi(c) - \varpi(a))}{p\Gamma(\alpha+2)}\|u - v\|, \\
&\int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau)) - f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|h\|\mathcal{K}(\varpi(c) - \varpi(a))}{p\Gamma(\alpha+2)}\|u - v\|, \\
&\int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau) - v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|q\||\nu|(\upsilon(e) - \upsilon(d))}{p\Gamma(\alpha+2)}\|u - v\|, \\
&\int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau)) - f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&\quad \le \frac{T^{\alpha+1}|\mathcal{E}|\|h\|\mathcal{K}|\nu|(\upsilon(e) - \upsilon(d))}{p\Gamma(\alpha+2)}\|u - v\| \\
&\int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|q(\tau)||u(\tau) - v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \le \frac{T^{\alpha+1}\|q\|}{p\Gamma(\alpha+2)}\|u - v\|,
\end{aligned}$$

$$\int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))-f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \le \frac{T^{\alpha+1}\mathcal{K}\|h\|}{p\Gamma(\alpha+2)}\|u-v\|,$$

and $|\tilde{p}(t)||\tilde{f}(u(t))-\tilde{f}(v(t))| \le \widetilde{\mathcal{K}}\|\tilde{p}\|\|u-v\|$. Thus,

$$\begin{aligned}
&\bigl|Hu(t)-Hv(t)\bigr| \\
&\quad\le \int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)-v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\qquad+ \int_a^c \int_0^\theta \int_0^s \frac{|\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))-f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\varpi(\theta) \\
&\qquad+ \int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)-v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&\qquad+ \int_d^e \int_0^\theta \int_0^s \frac{|\nu||\mathcal{E}|(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))-f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds\,d\upsilon(\theta) \\
&\qquad+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|q(\tau)||u(\tau)-v(\tau)|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds \\
&\qquad+ \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))-f(v(\tau))|}{|p(s)|\Gamma(\alpha)}\,d\tau\,ds + \bigl|\tilde{p}(t)\bigr|\bigl|\tilde{f}(u(t))-\tilde{f}(v(t))\bigr|.
\end{aligned}$$

Hence, $|Hu(t)-Hv(t)| \le \mathcal{A}_1^*\|u-v\|$. This implies that $\|\Theta u-\Theta v\| \le \psi(\|u-v\|)$ for all $u,v \in B_r$ where $\psi(t) = ((2\mathcal{A}_1^* r+\mathcal{A}_2^*)\|\phi\|+g_0\mathcal{A}_1^*)t$. By a similar method to that in the proof of Theorem (3), we can conclude that Θ is an α-ψ-contraction, α-admissible map, $\alpha(u_n,u) \ge 1$ for all n whenever $\alpha(u_{n-1},u_n) \ge 1$ for all $n \ge 1$, $u_n \to u \in C(I,\mathbb{R})$ and $\alpha(u_0,\Theta u_0) \ge 1$ with $u_0 \in B_r \subset X$. Now, by using Lemma 1, Θ has a fixed point in $C(I,\mathbb{R})$ which is a solution for problem (2). □

Example 2 Consider the fractional hybrid Sturm–Liouville problem

$$\begin{cases}
{}^cD^{\frac{999}{1000}}\bigl(e^{-\sqrt[3]{t}}\bigl(\frac{u(t)}{\frac{t}{101}|u(t)|+\frac{1}{1+t}e^{-\pi t}} - \frac{\sin t}{50}(\frac{1}{40}u(t)+3)\bigr)'\bigr) + \frac{e^{-t}}{300(1+t^2)}u(t) \\
\quad = \frac{1}{60}e^{\frac{t^2}{1+t^2}}\cot^{-1}(u(t)+\sqrt{3}), \\
\bigl(\frac{u(t)}{\frac{t}{101}|u(t)|+\frac{1}{1+t}e^{-\pi t}} - \frac{\sin t}{50}(\frac{1}{40}u(t)+3)\bigr)'_{t=0} = 0, \\
\int_0^{\frac{1}{3}}\bigl(\frac{u(\theta)}{\frac{\theta}{101}|u(\theta)|+\frac{1}{1+\theta}e^{-\pi\theta}} - \frac{\sin\theta}{50}(\frac{1}{40}u(\theta)+3)\bigr)\,d(3\theta+1) \\
\quad = \frac{1}{200}\int_{\frac{1}{2}}^1\bigl(\frac{u(\theta)}{\frac{\theta}{101}|u(\theta)|+\frac{1}{1+\theta}e^{-\pi\theta}} - \frac{\sin\theta}{50}(\frac{1}{40}u(\theta)+3)\bigr)\,d(4\theta+2).
\end{cases} \tag{31}$$

Put $\alpha = \frac{999}{1000}$, $T=1$, $r=1$, $\varpi(\theta)=3\theta+1$, $\upsilon(\theta)=4\theta+2$, $p(t)=e^{-\sqrt[3]{t}}$, $\tilde{p}(t)=\frac{\sin t}{50}$, $q(t)=\frac{e^{-t}}{300(1+t^2)}$, $h(t)=e^{\frac{t^2}{1+t^2}}$, $f(u(t))=\cot^{-1}(u(t)+\sqrt{3})$, $\tilde{f}(u(t))=\frac{1}{40}u(t)+3$, and $g(t,u(t))=\frac{t}{101}|u(t)|+\frac{1}{1+t}e^{-\pi t}$. Then $|\frac{\partial f(u)}{\partial u}| \le 1 = \mathcal{K}$, $f_0=\frac{\pi}{6}$, $|\frac{\partial\tilde{f}(u)}{\partial u}| \le \frac{1}{40} = \tilde{\mathcal{K}}$, $\tilde{f}_0 = 3$, $p=1$, $\|\tilde{p}\| = \frac{1}{50}$, $\|q\| = \frac{1}{300}$, $\|h\| = \frac{\sqrt{e}}{60}$, $\varpi(0)=1$, $\varpi(\frac{1}{3})=2$, $\upsilon(\frac{1}{2})=4$, $\upsilon(1)=6$. Also, $\|\phi\| = \frac{1}{100}$, $g_0=1$ and $|g(t,u(t))-g(t,v(t))| \le \frac{t}{100}|u(t)-v(t)|$. Moreover, $\varpi(\frac{1}{3})-\varpi(0) = 1 \ne \frac{1}{100} = \nu(\upsilon(1)-\upsilon(\frac{1}{2}))$, $\frac{\varpi(\frac{1}{3})-\varpi(0)+|\nu|(\upsilon(1)-\upsilon(\frac{1}{2}))}{|\varpi(\frac{1}{3})-\varpi(0)-\nu(\upsilon(1)-\upsilon(\frac{1}{2}))|}+1 = \frac{200}{99}$, and so $\mathcal{A}_1^* = \frac{1+5\sqrt{e}}{300\Gamma(\frac{2999}{1000})} \times \frac{200}{99} + \frac{1}{40}\times\frac{1}{50} = 0.0316519809$, $\mathcal{A}_2^* = \frac{\pi\sqrt{e}}{360\Gamma(\frac{2999}{1000})}\times\frac{200}{99} + \frac{3}{50} = 0.0745465529$, $(\|\phi\|r+g_0)(\mathcal{A}_1^* r+\mathcal{A}_2^*) \approx 0.1072605191 \le 1 = r$, and $(2\mathcal{A}_1^* r+\mathcal{A}_2^*)\|\phi\| + g_0\mathcal{A}_1^* \approx 0.033030486 < 1$. Now, by using Theorem 7, problem (31) has a solution.

Corollary 3 *Assume that hypotheses* (L_1)–(L_3) *hold and there exists* $r > 0$ *such that* $(\|\phi\|r + g_0)(\mathcal{B}_1^* r + \mathcal{B}_2^*) \leq r$ *and* $(2\mathcal{B}_1^* r + \mathcal{B}_2^*)\|\phi\| + g_0\mathcal{B}_1^* < 1$, *where*

$$\mathcal{B}_1^* = \frac{T^{\alpha+1}(\|q\| + \mathcal{K}\|h\|)}{p\Gamma(\alpha+2)}\left(\frac{\varpi(c) - \varpi(a) + |\nu|(\upsilon(e) - \upsilon(d))}{|\varpi(c) - \varpi(a) - \nu(\upsilon(e) - \upsilon(d))|} + 1\right),$$
$$\mathcal{B}_2^* = \frac{T^{\alpha+1}\|h\|f_0}{p\Gamma(\alpha+2)}\left(\frac{\varpi(c) - \varpi(a) + |\nu|(\upsilon(e) - \upsilon(d))}{|\varpi(c) - \varpi(a) - \nu(\upsilon(e) - \upsilon(d))|} + 1\right),$$

$f_0 = |f(0)|$, *and* $g_0 = \sup_{t\in I} g(t,0)$. *Then* $u \in C(I,\mathbb{R})$ *is a solution for the hybrid Sturm–Liouville problem*

$$\begin{cases} {}^cD^\alpha(p(t)(\frac{u(t)}{g(t,u(t))})') + q(t)u(t) = h(t)f(u(t)), \\ (\frac{u(t)}{g(t,u(t))})'_{t=0} = 0, \\ \int_a^c \frac{u(\theta)\,d\varpi(\theta)}{g(\theta,u(\theta))} = \nu \int_d^e \frac{u(\theta)\,d\upsilon(\theta)}{g(\theta,u(\theta))}, \end{cases} \tag{32}$$

where $\varpi(c) - \varpi(a) \neq \nu(\upsilon(e) - \upsilon(d))$, $\varpi(\theta)$ *and* $\upsilon(\theta)$ *are increasing functions, the integrals are in the Riemann–Stieltjes sense, and* $0 \leq a < c \leq d < e \leq T$. *Moreover, if* $(g(t,u(t)))' \in C(I,\mathbb{R})$, *then* $u \in C^1(I,\mathbb{R})$.

Proof In fact, problem (32) is a special case of problem (2) with $\tilde{p}(t) = 0$ for all $t \in I$ and $\tilde{f}(x) = 0$ for all $x \in \mathbb{R}$. Now, by applying Theorem 7, we can conclude that problem 32 has a solution. □

Corollary 4 *Assume that hypotheses* (M_1)–(M_2) *hold and there exists* $r > 0$ *such that* $\frac{\mathcal{C}_2^*}{1-\mathcal{C}_1^*} \leq r$, *where* $\mathcal{C}_1^* = \frac{T^{\alpha+1}(\|q\|+\mathcal{K}\|h\|)}{p\Gamma(\alpha+2)}(\frac{\varpi(c)-\varpi(a)+|\nu|(\upsilon(e)-\upsilon(d))}{|\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))|} + 1) + \widetilde{\mathcal{K}}\|\tilde{p}\| < 1$, $\mathcal{C}_2^* = \frac{T^{\alpha+1}\|h\|f_0}{p\Gamma(\alpha+2)} \times (\frac{\varpi(c)-\varpi(a)+|\nu|(\upsilon(e)-\upsilon(d))}{|\varpi(c)-\varpi(a)-\nu(\upsilon(e)-\upsilon(d))|} + 1) + \tilde{f}_0\|\tilde{p}\|, f_0 = |f(0)|$, *and* $\tilde{f}_0 = |\tilde{f}(0)|$. *Then* $u \in C^1(I,\mathbb{R})$ *is a solution for the hybrid Sturm–Liouville problem*

$$\begin{cases} {}^cD^\alpha((p(t)u'(t) - \tilde{p}(t)\tilde{f}(u(t))) + q(t)u(t) = h(t)f(u(t)), \\ (u(t) - \tilde{p}(t)\tilde{f}(u(t)))'_{t=0} = 0, \\ \int_a^c(u(\theta) - \tilde{p}(\theta)\tilde{f}(u(\theta)))\,d\varpi(\theta) = \nu\int_d^e(u(\theta) - \tilde{p}(\theta)\tilde{f}(u(\theta)))\,d\upsilon(\theta), \end{cases}$$

where $\varpi(c) - \varpi(a) \neq \nu(\upsilon(e) - \upsilon(d))$, $\varpi(\theta)$ *and* $\upsilon(\theta)$ *are increasing functions, the integrals are in the Riemann–Stieltjes sense, and* $0 \leq a < c \leq d < e \leq T$.

5 Conclusion

More natural phenomena and processes in the world are modeled by different types of fractional differential equations. This diversity factor in studying complicate fractional integro-differential equations increases our ability for exact modeling of more phenomena. This path will lead us in future for making modern software which will help us to allow for more cost-free testing and less material consumption. In this work, by using the technique of α-admissible α-ψ-contractions, we study a fractional hybrid version of the Sturm–Liouville equation. In fact, we investigate the existence of solutions for the fractional hybrid Sturm–Liouville equation by using the multi-point boundary value conditions. Also, we review the existence of solutions for a fractional hybrid version of the problem under the integral boundary value conditions. We provide two examples to illustrate our main results.

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