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Nonlocal boundary value problems for Hilfer-type pantograph fractional differential equations and inclusions

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Abstract

In this paper, we study boundary value problems, involving the Hilfer fractional derivative, for pantograph fractional differential equations and inclusions supplemented by nonlocal integral boundary conditions. Existence and uniqueness results are obtained by using well-known fixed point theorems for single and multi-valued functions. Examples illustrating our results are also presented.

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1 Introduction

In recent years, the theory of fractional differential equations has played a very important role in a new branch of applied mathematics, which has been utilized for mathematical models in engineering, physics, chemistry, signal analysis, etc. For details and applications we refer the reader to the classical reference texts such as [1–7]. Fractional differential equations are considered valuable tools to model many real world problems. Boundary value problems of differential equations represent an important class of applied analysis. Most of the researchers have given attention to study fractional differential equations by taking Caputo or Riemann–Liouville derivatives. Engineers and scientists have developed some new models that involve fractional differential equations for which the Riemann–Liouville derivative is not considered appropriate. Therefore certain modifications were introduced to avoid the difficulties and some new types of fractional order derivative operators were introduced in the literature by authors like Caputo, Hadamard, and Erdely-Kober. A generalization of derivatives of both Riemann–Liouville and Caputo was given by Hilfer in [8], known as *the Hilfer fractional derivative* of order α and a type $\beta \in [0,1]$, which can be reduced to the Riemann–Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. Such derivative interpolates between the Riemann–Liouville and Caputo derivative. Fractional differential equations involving the Hilfer derivative have many applications; see [9–12] and the references cited therein.

An important class of differential equations containing proportional delays are called *pantograph equations*. This important class was named after Ockendon and Tayler [13]. Numbers of applications have been studied by many researchers of these equations in applied sciences including biology, physics, economics, and electrodynamics. For more details as regards the aforesaid equations, we refer to [14, 15].

Initial value problems involving Hilfer fractional derivatives were studied by several authors; see for example [16–18] and the references therein. Nonlocal boundary value problems for the Hilfer fractional derivative were studied in [19]. Initial value problems for pantograph equations with the Hilfer fractional derivative were studied in [15, 20].

To the best of our knowledge, there is no work on boundary value problems for pantograph equations with the Hilfer fractional derivative in the literature. This paper comes to fill this gap, by introducing a new class of boundary value problems of pantograph equations with Hilfer-type fractional differential equations and nonlocal integral boundary conditions, of the form

$$ {}^{H}D^{\alpha,\beta}x(t) = f\big(t, x(t), x(\lambda t)\big), \quad t \in [a,b], \tag{1.1} $$

$$ x(a) = 0, \qquad Ax(b) + BI^{\delta}x(\eta) = c, \quad \eta \in (a,b), \tag{1.2} $$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, $1 < \alpha < 2$ and parameter β, $0 \le \beta \le 1$, $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, I^{δ} is the Riemann–Liouville fractional integral of order $\delta > 0$, $a \ge 0$, $A, B, c \in \mathbb{R}$ and $0 < \lambda < 1$.

Existence and uniqueness results are proved by using well-known fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while the nonlinear alternative of Leray–Schauder type [21] and Krasnoselskii's fixed point theorem [22] are applied to obtain the existence results for the problem (1.1)–(1.2).

After that we study the multi-valued version of the problem (1.1)–(1.2) by considering the inclusion problem

$$ {}^{H}D^{\alpha,\beta}x(t) \in F\big(t, x(t), x(\lambda t)\big), \quad t \in [a,b], \tag{1.3} $$

$$ x(a) = 0, \qquad Ax(b) + BI^{\delta}x(\eta) = c, \quad \eta \in (a,b), \tag{1.4} $$

where $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

For the problem (1.3)–(1.4) we prove existence results for both cases, convex valued (upper semicontinuous case) and non-convex valued (Lipschitz case) multifunctions. For the case when the multi-valued F has convex values we use the Bohnenblust–Karlin fixed point theorem, Martelli's fixed point theorem and the nonlinear alternative for Kakutani maps. For the lower semicontinuous case the existence result is based in nonlinear alternative of Leray–Schauder type together with a selection theorem for lower semicontinuous maps with decomposable values. Finally in the case of possible non-convex valued multi-valued map we apply a fixed point theorem for contractive multi-valued maps due to Covitz and Nadler.

The outline of the paper is as follows: We present our main work for single-valued case in Sect. 3, and for multi-valued case in Sect. 4, while Sect. 2 contains some preliminary concepts related to our problem. Examples are constructed to illustrate the main results.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and multi-valued analysis and present preliminary results needed in our proofs later [2, 5].

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $u : [a, \infty) \to \mathbb{R}$, is defined by

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds,$$

provided the right-hand side exists on (a, ∞).

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function u, is defined by

$${}^{\mathrm{RL}}D^{\alpha} u(t) := D^n I^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a, ∞).

Definition 2.3 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function u, is defined by

$${}^{C}D^{\alpha} u(t) := I^{n-\alpha} D^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, \quad n-1 < \alpha < n,$$

provided the right-hand side is point-wise defined on (a, ∞).

In [8] (see also [9]) another new definition of the fractional derivative was suggested. *The generalized Riemann–Liouville fractional derivative* is defined as follows.

Definition 2.4 The generalized Riemann–Liouville fractional derivative or the Hilfer fractional derivative of order α and parameter β of a function u is defined by

$${}^{H}D^{\alpha,\beta} u(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} u(t),$$

where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, $t > a$, $D = \frac{d}{dt}$.

Remark 2.5 When $\beta = 0$ the Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative

$${}^{H}D^{\alpha,0} u(t) = D^n I^{n-\alpha} u(t),$$

while when $\beta = 1$ the Hilfer fractional derivative corresponds to the Caputo fractional derivative

$${}^{H}D^{\alpha,1} u(t) = I^{n-\alpha} D^n u(t).$$

In the following lemma we present the compositional property of Riemann–Liouville fractional integral operator with the Hilfer fractional derivative operator.

Lemma 2.6 ([9]) *Let* $f \in L(a,b)$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $0 \leq \beta \leq 1$, $I^{(n-\alpha)(1-\beta)}f \in AC^k[a,b]$. *Then*

$$\left(I^{\alpha\,H}D^{\alpha,\beta}f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t\to a^+} \frac{d^k}{dt^k}\left(I^{(1-\beta)(n-\alpha)}f\right)(t).$$

The following lemma deals with a linear variant of the boundary value problem (1.1)–(1.2).

Lemma 2.7 *Let* $a \geq 0$, $1 < \alpha < 2$, $\gamma = \alpha + 2\beta - \alpha\beta$, $h \in C([a,b],\mathbb{R})$ *and*

$$\Lambda = \frac{A(b-a)^{\gamma-1}}{\Gamma(\gamma)} + \frac{B(\eta-a)^{\gamma+\delta-1}}{\Gamma(\gamma+\delta)} \neq 0. \tag{2.1}$$

Then the function x *is a solution of the boundary value problem*

$$^{H}D^{\alpha,\beta}x(t) = h(t), \quad t \in [a,b], \tag{2.2}$$

$$x(a) = 0, \qquad Ax(b) + BI^{\delta}x(\eta) = c, \quad \eta \in (a,b), \tag{2.3}$$

if and only if

$$x(t) = I^{\alpha}h(t) + \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[c - AI^{\alpha}h(b) - BI^{\alpha+\delta}h(\eta)\right]. \tag{2.4}$$

Proof Assume that x is a solution of the nonlocal (2.2)–(2.3). Operating the fractional integral I^{α} on both sides of equation (2.2) and using Lemma 2.6, we obtain

$$\begin{aligned} x(t) &= c_0\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} + I^{\alpha}h(t) \\ &= c_0\frac{(t-a)^{\gamma-2}}{\Gamma(\gamma-1)} + c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + I^{\alpha}h(t), \end{aligned}$$

since $(1-\beta)(2-\alpha) = 2-\gamma$, where c_0 and c_1 are some real constants.

From the first boundary condition $x(a) = 0$ we can obtain $c_0 = 0$, since $\lim_{t\to a}(t-a)^{\gamma-2} = \infty$. Then we get

$$x(t) = c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + I^{\alpha}h(t). \tag{2.5}$$

From $Ax(b) + BI^{\delta}x(\eta) = c$ we found

$$c_1 = \frac{1}{\Lambda}\left[c - AI^{\alpha}h(b) - BI^{\alpha+\delta}h(\eta)\right].$$

Substituting the values of c_1 in (2.5), we obtain the solution (2.4). The converse follows by direct computation. This completes the proof. □

3 Main results for the single-valued problem (1.1)–(1.2)

In view of Lemma 2.7, we define an operator $\mathcal{A}:\mathcal{C}\to\mathcal{C}$ by

$$
\begin{aligned}
(\mathcal{A}x)(t) = {} & \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(c - AI^{\alpha}f\bigl(s,x(s),x(\lambda s)\bigr)(b) \\
& - BI^{\alpha+\delta}f\bigl(s,x(s),x(\lambda s)\bigr)(\eta)\bigr) + I^{\alpha}f\bigl(s,x(s),x(\lambda s)\bigr)(t).
\end{aligned} \tag{3.1}
$$

It should be noticed that problem (1.1)–(1.2) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

In the following, for the sake of convenience, we set

$$
\Omega = \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + |B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.2}
$$

We prove existence as well as existence and uniqueness results, for the boundary value problem (1.1)–(1.2) by using well-known fixed point theorems.

Our existence and uniqueness result is based on Banach's fixed point theorem.

Theorem 3.1 *Assume that*:

(H_1) *there exists a constant $L > 0$ such that*

$$
\bigl|f(t,x_1,x_2) - f(t,y_1,y_2)\bigr| \le L\bigl(|x_1-y_1| + |x_2-y_2|\bigr)
$$

for each $t \in [a,b]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$.

If

$$
2L\Omega < 1, \tag{3.3}
$$

where Ω is defined by (3.2), then the boundary value problem (1.1)–(1.2) has a unique solution on $[a,b]$.

Proof We transform the boundary value problem (1.1)–(1.2) into a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (1.1)–(1.2). Applying the Banach contraction mapping principle, we shall show that $\mathcal{A}$ has a unique fixed point.

We let $\sup_{t\in[a,b]}|f(t,0,0)| = M < \infty$, and choose

$$
r \ge \frac{M\Omega + |c|(b-a)^{\gamma-1}/|\Lambda|\Gamma(\gamma)}{1-2L\Omega}. \tag{3.4}
$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For any $x \in B_r$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{A}x)(t)\bigr| \\
&\quad \le \sup_{t\in[a,b]}\left\{\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\bigl(|c| + |B|I^{\alpha+\delta}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(\eta) + |A|I^{\alpha}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(b)\bigr)\right. \\
&\qquad \left. + I^{\alpha}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(t)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \big(|c| + |B| I^{\alpha+\delta} \big(\big| f\big(s,x(s),x(\lambda s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(\eta) \right. \\
&\quad + |A| I^{\alpha} \big(\big| f\big(s,x(s),x(\lambda s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(b) \big) \\
&\quad \left. + I^{\alpha} \big(\big| f\big(s,x(s),x(\lambda s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(b) \right\} \\
&\leq \big(2L\|x\| + M\big) \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left[|A| \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + |B| \frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right\} \\
&\quad + |c| \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \\
&\leq (2Lr + M)\Omega + |c| \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \leq r,
\end{aligned}
$$

which implies that $\mathcal{A}B_r \subset B_r$.

Next, we let $x, y \in \mathcal{C}$. Then, for $t \in [a,b]$, we have

$$
\begin{aligned}
&\big| (\mathcal{A}x)(t) - (\mathcal{A}y)(t) \big| \\
&\leq \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \big(|B| I^{\alpha+\delta} \big| f\big(s,x(s),x(\lambda s)\big) - f\big(s,y(s),y(\lambda s)\big) \big|(\eta) \right. \\
&\quad + |A| I^{\alpha} \big| f\big(s,x(s),x(\lambda s)\big) - f\big(s,y(s),y(\lambda s)\big) \big|(b) \big) \\
&\quad \left. + I^{\alpha} \big| f\big(s,x(s),x(\lambda s)\big) - f\big(s,y(s),y(\lambda s)\big) \big|(b) \right\} \\
&\leq 2L \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \left[|A| \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + |B| \frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right\} \|x-y\| \\
&= 2L\Omega \|x-y\|,
\end{aligned}
$$

which implies that $\|\mathcal{A}x - \mathcal{A}y\| \leq 2L\Omega \|x-y\|$. As $2L\Omega < 1$, $\mathcal{A}$ is a contraction. Therefore, we deduce by the Banach contraction mapping principle that $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (1.1)–(1.2). The proof is completed. □

Example 3.2 Consider the nonlocal boundary value problem for the Hilfer-type pantograph fractional differential equation of the form

$$
\begin{cases}
{}^{H}D^{\frac{3}{2},\frac{2}{5}} x(t) = \frac{e^{\frac{1}{2}-t}}{17+2t} \left(\frac{x^2(t)+|x(t)|}{1+|x(t)|} + 2\sin x(\tfrac{1}{2}t) \right) + t^2 + 1, & t \in [\tfrac{1}{2}, \tfrac{5}{2}], \\
x(\tfrac{1}{2}) = 0, \qquad \tfrac{2}{3} x(\tfrac{5}{2}) + \tfrac{3}{4} I^{\frac{1}{2}} x(\tfrac{3}{2}) = \tfrac{4}{5}.
\end{cases}
\tag{3.5}
$$

Here $\alpha = 3/2$, $\beta = 2/5$, $\lambda = 1/2$, $a = 1/2$, $b = 5/2$, $A = 2/3$, $B = 3/4$, $\delta = 1/2$, $\eta = 3/2$, $c = 4/5$. The setting yields $\gamma = 17/10$, $\Lambda = 1.872599119$ and $\Omega = 4.129461300$. Now, we put

$$
f(t,x_1,x_2) = \frac{e^{\frac{1}{2}-t}}{17+2t} \left(\frac{x_1^2 + |x_1|}{1+|x_1|} + 2\sin x_2 \right) + t^2 + 1,
$$

which satisfies (H_1) as

$$
\big| f(t,x_1,x_2) - f(t,y_1,y_2) \big| \leq \frac{2e^{\frac{1}{2}-t}}{17+2t} \big(|x_1 - y_1| + |x_2 - y_2| \big) \leq \frac{1}{9} \big(|x_1 - y_1| + |x_2 - y_2| \big).
$$

Setting $L = 1/9$, we obtain $2L\Omega \approx 0.9176580667 < 1$ which shows that inequality (3.3) is true. Then, by the conclusion of Theorem 3.1, we deduce that the boundary value problem (3.5) has a unique solution on $[1/2, 5/2]$.

Next we present two existence results. The first is based on the well-known Krasnoselskii fixed point theorem ([22]).

Theorem 3.3 *Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that*:

(H_2) $|f(t,x,y)| \leq \varphi(t)$, $\forall (t,x,y) \in [a,b] \times \mathbb{R} \times \mathbb{R}$, *and* $\varphi \in C([a,b], \mathbb{R}^+)$.

Then the boundary value problem (1.1)–(1.2) *has at least one solution on* $[a,b]$, *provided*

$$L\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] < 1. \tag{3.6}$$

Proof Setting $\sup_{t\in[a,b]} \varphi(t) = \|\varphi\|$ and choosing

$$\rho \geq \|\varphi\|\Omega + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \tag{3.7}$$

(where Ω is defined by (3.2)), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$. We define the operators $\mathcal{A}_1$, $\mathcal{A}_2$ on B_ρ by

$$\mathcal{A}_1 x(t) = I^\alpha f\big(s, x(s), x(\lambda s)\big)(t), \quad t \in [a,b],$$

and

$$\mathcal{A}_2 x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - BI^{\alpha+\delta} f\big(s,x(s),x(\lambda s)\big)(\eta) - AI^\alpha f\big(s,x(s),x(\lambda s)\big)(b)\big), \quad t \in [a,b].$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}
&\big|(\mathcal{A}_1 x)(t) + (\mathcal{A}_2 y)(t)\big| \\
&\quad\leq \sup_{t\in[a,b]}\left\{\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\big(|c| + |B|I^{\alpha+\delta}\big|f\big(s,x(s),x(\lambda s)\big)\big|(\eta) + |A|I^\alpha\big|f\big(s,x(s),x(\lambda s)\big)\big|(b)\big)\right. \\
&\qquad\left. + I^\alpha\big|f\big(s,x(s),x(\lambda s)\big)\big|(t)\right\} \\
&\quad\leq \|\varphi\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right) + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \\
&\quad= \|\varphi\|\Omega + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \leq \rho.
\end{aligned}$$

This shows that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_\rho$. It is easy to see, using (3.6), that $\mathcal{A}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Also, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1 x\| \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{A}_1$.

We define $\sup_{(t,x)\in[a,b]\times B_\rho\times B_\rho} |f(t,x,y)| = \bar{f} < \infty$, and consequently, for any $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, we have

$$\begin{aligned}
\left|(\mathcal{A}_1x)(t_2) - (\mathcal{A}_1x)(t_1)\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]f\big(s,x(s),x(\lambda s)\big)\,ds \right.\\
&\quad \left. + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\big(s,x(s),x(\lambda s)\big)\,ds\right| \\
&\leq \frac{\bar{f}}{\Gamma(\alpha+1)}\left[2(t_2-t_1)^\alpha + \left|(t_2-a)^\alpha - (t_1-a)^\alpha\right|\right],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{A}_1$ is equicontinuous. So $\mathcal{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá–Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Thus all the assumptions of Krasnoselskii's fixed point theorem ([22]) are satisfied. So the conclusion of Krasnoselskii's fixed point theorem implies that the boundary value problem (1.1)–(1.2) has at least one solution on $[a,b]$. □

Example 3.4 Consider the nonlocal boundary value problem for the Hilfer-type pantograph fractional differential equation of the form

$$\begin{cases} {}^HD^{\frac{5}{3},\frac{1}{2}}x(t) = \frac{5\tan^{-1}|x(t)|}{9+3t} + \frac{2\sin|x(t/4)|}{4+3t} + e^{-2t}, & t \in [\frac{1}{3}, \frac{5}{3}], \\ x(\frac{1}{3}) = 0, \qquad \frac{3}{5}x(\frac{5}{3}) + \frac{1}{4}I^{\frac{3}{2}}x(\frac{2}{3}) = \frac{3}{4}. \end{cases} \tag{3.8}$$

Here $\alpha = 5/3$, $\beta = 1/2$, $\lambda = 1/4$, $a = 1/3$, $b = 5/3$, $A = 3/5$, $B = 1/4$, $\delta = 3/2$, $\eta = 2/3$ and $c = 3/4$. Then we compute that $\gamma = 11/6$, $\Lambda \approx 0.8175877260$, $\Omega \approx 2.139687890$ and

$$\Omega_1 := \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{|B|(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \frac{|A|(b-a)^\alpha}{\Gamma(\alpha+1)}\right] \approx 1.066150326.$$

From (3.8), we can find that

$$\begin{aligned}
\left|f(t,x_1,x_2) - f(t,y_1,y_2)\right| &\leq \frac{5}{9+3t}|x_1-y_1| + \frac{2}{4+3t}|x_2-y_2| \\
&\leq \frac{9}{10}\big(|x_1-y_1| + |x_2-y_2|\big).
\end{aligned}$$

This means that the condition (H_1) is satisfied with $L = 9/10$. We get $L\Omega_1 \approx 0.9595352934 < 1$ and

$$\left|f(t,x,y)\right| \leq \frac{5\pi}{2(9+3t)} + \frac{2}{4+3t} + e^{-2t},$$

which satisfy conditions (3.6) and (H_2), respectively. Applying Theorem 3.3, the boundary value problem (3.8) has at least one solution on $[1/3, 5/3]$.

Remark 3.5 In Theorem 3.1, the existence and uniqueness of solutions for the given problem is established by means of the Banach contraction mapping principle. In Theorem 3.3 the existence of solutions is established via the Krasnoselskii fixed point theorem. The proof of this result is based on the idea of splitting the operator $\mathcal{A}$ into the sum of two operators $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1$ is contractive and $\mathcal{A}_2$ is compact. One can notice that the entire operator $\mathcal{A}$ is not required to be contractive. On the other hand, Theorem 3.1 deals with the existence of a unique solution of the given problem via the Banach contraction mapping principle, in which the entire operator $\mathcal{A}$ is shown to be contractive. Thus, the linkage between contractive conditions imposed in Theorems 3.3 and 3.1 provides a precise estimate to pass onto a unique solution from the existence of a solution for the problem at hand. In Example 3.8 above we note that Theorem 3.1 is not applicable because $2L\Omega \approx 3.851438202 > 1$.

The Leray–Schauder's nonlinear alternative ([21]) is used for our next existence result.

Theorem 3.6 *Assume that*:

(H_3) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([a,b],\mathbb{R}^+)$ *such that*

$$\bigl|f(t,u,v)\bigr| \le p(t)\psi\bigl(|u|+|v|\bigr) \quad \textit{for each } (t,u,v) \in [a,b]\times\mathbb{R}\times\mathbb{R};$$

(H_4) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(2M)\|p\|\Omega + |c|(b-a)^{\gamma-1}/|\Lambda|\Gamma(\gamma)} > 1,$$

where Ω *is defined by* (3.2).

Then the boundary value problem (1.1)–(1.2) *has at least one solution on* $[a,b]$.

Proof Let the operator $\mathcal{A}$ be defined by (3.1). Firstly, we shall show that $\mathcal{A}$ maps bounded sets (balls) into bounded set in $\mathcal{C}$. For a number $r > 0$, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [a,b]$ we have

$$\begin{aligned}
&\bigl|(\mathcal{A}x)(t)\bigr| \\
&\quad\le \sup_{t\in[a,b]}\biggl\{\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\bigl(|c| + |B|I^{\alpha+\delta}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(\eta) + |A|I^{\alpha}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(b)\bigr) \\
&\qquad + I^{\alpha}\bigl|f\bigl(s,x(s),x(\lambda s)\bigr)\bigr|(t)\biggr\} \\
&\quad\le \psi\bigl(2\|x\|\bigr)\|p\|\biggl(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\biggl[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\biggr] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\biggr) \\
&\qquad + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)},
\end{aligned}$$

and, consequently,

$$\|\mathcal{A}x\| \le \psi(2r)\|p\|\Omega + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}.$$

Next we will show that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $\tau_1, \tau_2 \in [a,b]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&\big|(\mathcal{A}x)(\tau_2) - (\mathcal{A}x)(\tau_1)\big| \\
&\quad\leq \frac{(\tau_2 - a)^{\gamma-1} - (\tau_1 - a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \big(|B| I^{\alpha+\delta} \big|f\big(s, x(s), x(\lambda s)\big)\big|(\eta) \\
&\qquad + |A| I^{\alpha} \big|f\big(s, x(s), x(\lambda s), x(\lambda s)\big)\big|(b)\big) \\
&\qquad + \frac{1}{\Gamma(\alpha)} \bigg| \int_a^{\tau_1} \big[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\big] f\big(s, x(s), x(\lambda s)\big)\, ds \\
&\qquad + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f\big(s, x(s), x(\lambda s)\big)\, ds \bigg| \\
&\quad\leq \frac{(\tau_2 - a)^{\gamma-1} - (\tau_1 - a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \|p\| \psi(2r) \bigg[|B| \frac{(\eta - a)^{\alpha+\delta}}{\Gamma(\alpha + \delta + 1)} + |A| \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \bigg] \\
&\qquad + \frac{\|p\| \psi(2r)}{\Gamma(\alpha+1)} \big[2(t_2 - t_1)^{\alpha} + \big|(t_2 - a)^{\alpha} - (t_1 - a)^{\alpha}\big| \big].
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá–Ascoli theorem, the operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative ([21]) once we have proved the boundedness of the set of all solutions to equations $x = \nu \mathcal{A} x$ for $\nu \in (0,1)$.

Let x be a solution. Then, for $t \in [a,b]$, and following computations similar to the first step, we have

$$\big|x(t)\big| \leq \psi\big(2\|x\|\big) \|p\| \Omega + |c| \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)},$$

which leads to

$$\frac{\|x\|}{\psi(2\|x\|)\|p\|\Omega + |c|(b-a)^{\gamma-1}/|\Lambda|\Gamma(\gamma)} \leq 1.$$

In view of (H_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \big\{ x \in C\big([a,b], \mathbb{R}\big) : \|x\| < M \big\}.$$

We see that the operator $\mathcal{A} : \bar{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu \mathcal{A} x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type ([21]), we deduce that $\mathcal{A}$ has a fixed point $x \in \bar{U}$ which is a solution of the boundary value problem (1.1)–(1.2). This completes the proof. □

Example 3.7 Consider the nonlocal boundary value problem for the Hilfer-type pantograph fractional differential equation of the form

$$\begin{cases} {}^H D^{\frac{7}{4},\frac{4}{5}} x(t) = \frac{e^{\frac{1}{4}-t}}{7+4t}\big(\big(\frac{|x^9(t)|}{x^8(t)+1} + \frac{x^8(t/5)}{|x^7(t/5)|+1}\big)^2 + 1\big), \quad t \in [\frac{1}{4}, \frac{3}{4}], \\ x(\frac{1}{4}) = 0, \qquad \frac{1}{3} x(\frac{3}{4}) + \frac{3}{7} I^{\frac{5}{2}} x(\frac{1}{2}) = \frac{2}{5}. \end{cases} \tag{3.9}$$

Here $\alpha = 7/4$, $\beta = 4/5$, $\lambda = 1/5$, $a = 1/4$, $b = 3/4$, $A = 1/3$, $B = 3/7$, $\delta = 5/2$, $\eta = 1/2$, $c = 2/5$. Now, we find constants $\gamma = 39/20$, $\Lambda \approx 0.1764175022$ and $\Omega \approx 0.3694499725$. Also, the nonlinear function can be expressed as

$$\big|f(t,x,y)\big| \leq \frac{e^{\frac{1}{4}-t}}{7+4t}\big(\big(|x|+|y|\big)^2+1\big).$$

Setting $p(t) = (e^{(1/4)-t})/(7+4t)$ and $\psi(u) = u^2+1$, we have $\|p\| = 1/8$ and $\psi(|x|+|y|) = (|x|+|y|)^2+1$. Thus we can compute that there exists a constant $M \in (1.937052574, 3.476400259)$ satisfying inequality in (H_4). Therefore, all conditions in Theorem 3.6 are fulfilled. Thus the boundary value problem (3.9) has at least one solution on $[1/4, 3/4]$.

4 Existence results for the multi-valued problem (1.3)–(1.4)

By $\mathcal{C} = C([a,b],\mathbb{R})$ we denote the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|x\| := \sup\big\{\big|x(t)\big| : t \in [a,b]\big\}.$$

Also by $L^1([a,b],\mathbb{R})$ we denote the space of functions $x : [a,b] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_a^b |x(t)|\,dt$.

For a normed space $(X,\|\cdot\|)$, we define $\mathcal{P}_q(X) = \{Y \in \mathcal{P}(X) : Y \text{ has the property } q\}$. Thus, for example, $\mathcal{P}_{\mathrm{cl},b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{\mathrm{cp},c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

For each $y \in C([a,b],\mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \big\{v \in L^1\big([a,b],\mathbb{R}\big) : v(t) \in F\big(t,x(t),x(\lambda t)\big) \text{ on } [a,b]\big\}.$$

The following lemma will be used in the sequel.

Lemma 4.1 ([23]) *Let X be a separable Banach space. Let $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp},c}(X)$ be an L^1-Carathéodory multi-valued map and let Θ be a linear continuous mapping from $L^1([a,b],X)$ to $C([a,b],X)$. Then the operator*

$$\Theta \circ S_F : C\big([a,b],X\big) \to \mathcal{P}_{\mathrm{cp},c}\big(C\big([a,b],X\big)\big), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([a,b],X) \times C([a,b],X)$.

Before stating and proving our main existence results for problem (1.3)–(1.4), we will give the definition of its solution.

Definition 4.2 A function $x \in AC^2([a,b],\mathbb{R})$ is said to be a solution of the problem (1.3)–(1.4) if $x(a) = 0$, $Ax(b) + BI^\delta x(\eta) = c$, and there exists a function $v \in L^1([a,b],\mathbb{R})$ with $v \in F(t,x,y)$ a.e. on $[a,b]$ such that

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\big) + I^\alpha v(s)(t).$$

4.1 The upper semicontinuous case

Consider first the case when F has convex values. Our first result is based on Bohnenblust–Karlin fixed point theorem.

Lemma 4.3 ((Bohnenblust–Karlin) [24]) *Let X be a Banach space, D a nonempty subset of X, witch is bounded, closed and convex. Suppose $G : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is upper semi-continuous with closed, convex values, and $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

Theorem 4.4 *Assume that*:

(A_1) $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp},c}(\mathbb{R})$ *is L^1-Carathéodory; i.e.*

(i) $t \longmapsto F(t,x,y)$ *is measurable for each $x, y \in \mathbb{R}$;*

(ii) $(x,y) \longmapsto F(t,x,y)$ *is upper semicontinuous for almost all $t \in [a,b]$;*

(iii) *for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a,b],\mathbb{R}^+)$ such that*

$$\bigl\|F(t,x,y)\bigr\| = \sup\bigl\{|v| : v \in F(t,x,y)\bigr\} \le \varphi_\rho(t)$$

for all $x, y \in \mathbb{R}$ with $\|x\|, \|y\| \le \rho$ and for a.e. $t \in [a,b]$;

(A_2)

$$\liminf_{\rho\to\infty} \frac{1}{\rho}\int_a^b \phi_\rho(t)\,dt = \mu. \tag{4.1}$$

Then the boundary problem (1.3)–(1.4) *has at least one solution on* $[a,b]$ *provided that*

$$\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left(|A|\frac{b^{\alpha-1}}{\Gamma(\alpha)} + |B|\frac{\eta^{\alpha+\delta-1}}{\Gamma(\alpha+\delta)}\right) + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\right\}\mu < 1. \tag{4.2}$$

Proof In order to transform the problem (1.3)–(1.4) into a fixed point problem, we consider the multi-valued map: $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ defined by

$$N(x) = \left\{\begin{array}{l} h \in C([a,b],\mathbb{R}) : \\ \quad h(t) = \left\{\begin{array}{l} \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}(c - AI^\alpha v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)) \\ \quad + I^\alpha v(s)(t), \quad v \in S_{F,x}. \end{array}\right. \end{array}\right.$$

It is clear that fixed points of N are solutions of problem (1.3)–(1.4). In turn, we need to show that the operator N satisfies all condition of Lemma 4.3. The proof is constructed in several steps.

Step 1. $N(x)$ is convex for each $x \in C([a,b],\mathbb{R})$.

Indeed, if h_1, h_2 belongs to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that, for each $t \in [a,b]$, we have

$$h_i(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(c - AI^\alpha v_i(s)(b) - BI^{\alpha+\delta}v_i(s)(\eta)\bigr) + I^\alpha v_i(s)(t), \quad i = 1,2.$$

Let $0 \le \theta \le 1$. Then, for each $t \in [a,b]$, we have

$$\begin{aligned}
\big[\theta h_1 + (1-\theta)h_2\big](t) = {}& \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}\big[\theta v_1(s) + (1-\theta)v_2(s)\big](b) \\
& - BI^{\alpha+\delta}\big[\theta v_1(s) + (1-\theta)v_2(s)\big](\eta)\big) \\
& + I^{\alpha}\big[\theta v_1(s) + (1-\theta)v_2(s)\big](t).
\end{aligned}$$

Since F has convex values, that is, $S_{F,x}$ is convex, we have

$$\theta h_1 + (1-\theta)h_2 \in N(x).$$

Step 2. $N(x)$ maps bounded sets (balls) into bounded sets in $C([a,b],\mathbb{R})$.

For a positive number ρ, let $B_\rho = \{x \in C([a,b],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([a,b],\mathbb{R})$. We shall prove that there exists a positive number ρ' such that $N(B_{\rho'}) \subseteq B_{\rho'}$. If not, for each positive number ρ, there exists a function $x_\rho(\cdot) \in B_\rho$, $\|N(x_\rho)\| > \rho$ for some $t \in [a,b]$ and

$$h_\rho(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}v_r(s)(b) - BI^{\alpha+\delta}v_r(s)(\eta)\big) + I^{\alpha}v_r(s)(t)$$

for some $v_\rho \in S_{F,x_\rho}$. However, on the other hand, we have

$$\begin{aligned}
\rho &< \big\|N(x_\rho)\big\| \\
&\le \frac{(b-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(|c| + |A|I^{\alpha}\big|v(s)\big|(b) + |B|I^{\alpha+\delta}\big|v(s)\big|(\eta)\big) + I^{\alpha}\big|v(s)\big|(t) \\
&\le \left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left(|A|\frac{b^{\alpha-1}}{\Gamma(\alpha)} + |B|\frac{\eta^{\alpha+\delta-1}}{\Gamma(\alpha+\delta)}\right) + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\right\}\int_a^b \phi_\rho(s)\,ds + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}.
\end{aligned}$$

Dividing both sides by ρ and taking the lower limit as $\rho \to \infty$, we get

$$1 \le \mu\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left(|A|\frac{b^{\alpha-1}}{\Gamma(\alpha)} + |B|\frac{\eta^{\alpha+\delta-1}}{\Gamma(\alpha+\delta)}\right) + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\right\},$$

which contradicts (4.2). Hence there exists a positive number ρ such that $N(B_\rho) \subseteq B_\rho$.

Step 3. $N(x)$ maps bounded sets into equicontinuous sets of $C([a,b],\mathbb{R})$.

Let x be any element in B_ρ and $h \in N(x)$, then there exists a function $v \in S_{F,x}$ such that, for each $t \in [a,b]$, we have

$$h(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\big) + I^{\alpha}v(s)(t).$$

Let $\tau_1, \tau_2 \in [a,b]$, $\tau_1 < \tau_2$. Thus

$$\begin{aligned}
&\big|h(t_2) - h(t_1)\big| \\
&\quad\le \frac{(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\big(|B|I^{\alpha+\delta}\big|v(s)\big|(\eta) + |A|I^{\alpha}\big|v(s)\big|(b)\big) \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left|\int_a^{\tau_1}\big[(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}\big]v(s)\,ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}v(s)\,ds\right|.
\end{aligned}$$

The right-hand side of the above inequality clearly tends to zero independently of $x \in B_\rho$ as $\tau_1 \to \tau_2$. As a consequence of Steps 1–3 together with the Arzelá–Ascoli theorem, we conclude that $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ is completely continuous.

Step 4. $N(x)$ is closed for each $x \in C([a,b],\mathbb{R})$.

Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([a,b],\mathbb{R})$. Then $u \in C([a,b],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$u_n(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v_n(s)(b) - BI^{\alpha+\delta}v_n(s)(\eta)\big) + I^\alpha v_n(s)(t).$$

As F has compact values, we pass onto a subsequence (if necessary) to find that v_n converges to v in $L^1([a,b],\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [a,b]$, we have

$$u_n(t) \to v(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\big) + I^\alpha v(s)(t).$$

Hence, $u \in N(x)$.

Next we show that the operator N is upper semicontinuous. In order to do so, it is enough to establish that N has a closed graph ([25, Proposition 1.2]).

Step 5. N has a closed graph.

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$h_n(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v_n(s)(b) - BI^{\alpha+\delta}v_n(s)(\eta)\big) + I^\alpha v_n(s)(t).$$

We must show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [a,b]$,

$$h_*(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v_*(s)(b) - BI^{\alpha+\delta}v_*(s)(\eta)\big) + I^\alpha v_*(s)(t).$$

Consider the continuous linear operator $\Theta : L^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$v \to \Theta(v)(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\big) + I^\alpha v(s)(t).$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$ and, thus, it follows from Lemma 4.1 that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_*$, Lemma 4.1 implies that

$$h_*(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^\alpha v_*(s)(b) - BI^{\alpha+\delta}v_*(s)(\eta)\big) + I^\alpha v_*(s)(t)$$

for some $v_* \in S_{F,x_*}$

Hence, we conclude that N is a compact multi-valued map, u.s.c. with convex closed values. As a consequence of Lemma 4.3, we deduce that N has a fixed point which is a solution of the boundary problem (1.3)–(1.4). This completes the proof. □

Next, we give an existence result based upon the following form of fixed point theorem which is applicable to completely continuous operators [26].

Lemma 4.5 *Let X a Banach space, and $T : X \to \mathcal{P}_{b,\mathrm{cl},c}(X)$ be a completely continuous multi-valued map. If the set $\mathcal{E} = \{x \in X : \nu x \in T(x), \nu > 1\}$ is bounded, then T has a fixed point.*

Theorem 4.6 *Assume that the following hypotheses hold*:

(A_3) *$F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{b,\mathrm{cl},c}(\mathbb{R})$ is a L^1-Carathéodory multi-valued map*;

(A_4) *there exists a function $h \in C([a,b],\mathbb{R})$ such that*

$$\bigl\|F(t,x,y)\bigr\| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and each } x,y \in \mathbb{R}.$$

Then the problem (1.3)–(1.4) *has at least one solution on* $[a,b]$.

Proof Consider N defined in the proof of Theorem 4.4. As in Theorem 4.4, we can show that N is convex and completely continuous. It remains to show that the set

$$\mathcal{E} = \bigl\{x \in C\bigl([a,b],\mathbb{R}\bigr) : \nu x \in N(x), \nu > 1\bigr\}$$

is bounded. Let $x \in \mathcal{E}$, then $\nu x \in N(x)$ for some $\nu > 1$ and there exists a function $v \in S_{F,x}$ such that

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(c - AI^{\alpha}v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\bigr) + I^{\alpha}v(s)(t).$$

For each $t \in [a,b]$, we have

$$\begin{aligned}\bigl|x(t)\bigr| &\le \frac{(b-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(|c| + |A|I^{\alpha}\bigl|v(s)\bigr|(b) + |B|I^{\alpha+\delta}\bigl|v(s)\bigr|(\eta)\bigr) + I^{\alpha}\bigl|v(s)\bigr|(t) \\ &\le \|h\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\quad + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}.\end{aligned}$$

Taking the supremum over $t \in [a,b]$, we get

$$\begin{aligned}\|x\| &\le \|h\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\quad + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} < \infty.\end{aligned}$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Lemma 4.5 we deduce that N has at least one fixed point which implies that the problem (1.3)–(1.4) has a solution on $[a,b]$. □

Our final existence result in this subsection is based on the Leray–Schauder nonlinear alternative for Kakutani maps ([21]).

Theorem 4.7 *Assume that* (H_4) *and* (A_1) *hold. In addition we assume that*:

(A_5) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([a,b],\mathbb{R}^+)$ *such that* $\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x,y)\} \le p(t)\psi(|x|)$ *for each* $(t,x,y) \in [a,b] \times \mathbb{R} \times \mathbb{R}$.

Then the boundary value problem (1.3)–(1.4) *has at least one solution on* $[a,b]$.

Proof Consider the operator N defined in the proof of Theorem 4.4. Let $x \in \nu N(x)$ for some $\nu \in (0,1)$. We show there exists an open set $U \subseteq C([a,b],\mathbb{R})$ with $x \notin N(x)$ for any $\nu \in (0,1)$ and all $x \in \partial U$. Let $\nu \in (0,1)$ and $x \in \nu N(x)$. Then there exists $v \in L^1([a,b],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [a,b]$, we have

$$x(t) = \nu \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(c - AI^{\alpha}v(s)(b) - BI^{\alpha+\delta}v(s)(\eta)\bigr) + \nu I^{\alpha}v(s)(t).$$

In view of (A_5), we have, for each $t \in [a,b]$,

$$\begin{aligned}\bigl|x(t)\bigr| &\le \frac{(b-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\bigl(|c| + |A|I^{\alpha}\bigl|v(s)\bigr|(b) + |B|I^{\alpha+\delta}\bigl|v(s)\bigr|(\eta)\bigr) + I^{\alpha}\bigl|v(s)\bigr|(t) \\ &\le \|p\|\psi\bigl(\|x\|\bigr)\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\quad + |c|\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}.\end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\psi(2\|x\|)\|p\|\Omega + |c|(b-a)^{\gamma-1}/|\Lambda|\Gamma(\gamma)} \le 1.$$

In view of (H_4), there exists M such that $\|x\| \ne M$. Let us set

$$U = \bigl\{x \in C\bigl([a,b],\mathbb{R}\bigr) : \|x\| < M\bigr\}.$$

Proceeding as in the proof of Theorem 4.4, we claim that the operator $N : \overline{U} \to \mathcal{P}(C([a,b],\mathbb{R}))$ is a compact, upper semicontinuous multi-valued map with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \nu N(x)$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type ([21]), we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the boundary value problem (1.3)–(1.4). This completes the proof. □

4.2 The lower semicontinuous case

Here we study the case when F is not necessarily convex valued, by applying the nonlinear alternative of Leray–Schauder type together with the selection theorem of Bressan and Colombo [27] for lower semicontinuous maps with decomposable values.

Theorem 4.8 *Assume that* (H_4), (A_5) *and the following condition hold*:

(A_6) $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multi-valued map such that* $(t,x,y) \longmapsto F(t,x,y)$ *is* $\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B}$ *measurable and* $x \longmapsto F(t,x,y)$ *is lower semicontinuous for each* $t \in [a,b]$.

Then the boundary value problem (1.3)–(1.4) *has at least one solution on* $[a,b]$.

Proof It follows from (A_5) and (A_6) that F is of l.s.c. type [28]. Then, by the selection theorem of Bressan and Colombo [27], there exists a continuous function $v : AC^1([a,b],\mathbb{R}) \to L^1([a,b],\mathbb{R})$ such that $v(x) \in \mathcal{F}(x)$ for all $v \in C([a,b],\mathbb{R})$, where $\mathcal{F} : C([a,b] \times \mathbb{R}) \to \mathcal{P}(L^1([a,b],\mathbb{R}))$ is the Nemytskii operator associated with F, defined as

$$\mathcal{F}(v) = \{w \in L^1([a,b],\mathbb{R}) : w(t) \in F(t, v(t), v(\lambda t)) \text{ for a.e. } t \in [a,b]\}.$$

Consider the problem

$$^{H}D^{\alpha,\beta}x(t) = f(x(t)), \quad t \in [a,b], \tag{4.3}$$

$$x(a) = 0, \qquad Ax(b) + BI^{\delta}x(\eta) = c, \quad \eta \in (a,b). \tag{4.4}$$

Observe that x is a solution to the boundary value problem (1.3)–(1.4) if $x \in AC^2([a,b],\mathbb{R})$ is a solution of the problem (4.3)–(4.4). In order to transform the problem (4.3)–(4.4) into a fixed point problem, we define an operator $\overline{N}$ as

$$\overline{N}(x) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}f(x(s))(b) - BI^{\alpha+\delta}f(x(s))(\eta)\big) + I^{\alpha}f(x(s))(t).$$

It can easily be shown that $\overline{N}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.7. So we omit it. This completes the proof. □

4.3 The Lipschitz case

In this subsection, we prove the existence of solutions for the boundary value problem (1.3)–(1.4) with a non-convex valued right-hand side by applying a fixed point theorem for multi-valued maps due to Covitz and Nadler [29].

Theorem 4.9 *Assume that the following conditions hold*:

- (B_1) $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x, y) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x, y \in \mathbb{R}$.
- (B_2) $H_d(F(t,x,y), F(t,\bar{x}),\bar{y}) \le m(t)(|x-\bar{x}| + |y-\bar{y}|)$ *for almost all* $t \in [a,b]$ *and* $x,\bar{x},y,\bar{y} \in \mathbb{R}$ *with* $m \in C([a,b],\mathbb{R}^+)$ *and* $d(0, F(t,0,0)) \le m(t)$ *for almost all* $t \in [a,b]$.

Then the boundary value problem (1.3)–(1.4) *has at least one solution on* $[a,b]$ *if*

$$2\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\|m\| < 1.$$

Proof We transform the boundary value problem (1.3)–(1.4) into a fixed point problem by considering the operator $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ defined at the beginning of the proof of Theorem 4.4. We show that the operator N satisfies the assumptions of Covitz and Nadler Theorem [29] in two steps.

Step I. N is nonempty and closed for every $v \in S_{F,x}$.

Note that the set-valued map $F(\cdot, x(\cdot))$ is measurable by the measurable selection theorem (e.g., [30, Theorem III.6]) and it admits a measurable selection $v : [a,b] \to \mathbb{R}$. Moreover, by the assumption (B_1), we have

$$|v(t)| \le m(t) + m(t)|x(t)|,$$

i.e. $v \in L^1([a,b],\mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,x} \neq \emptyset$. Moreover $N(x) \in \mathcal{P}_{cl}(C([a,b],\mathbb{R}))$ for each $x \in C([a,b],\mathbb{R})$, as proved in Step 4 of Theorem 4.4.

Step II. Next we show that there exists $\theta < 1$ such that

$$H_d\big(N(x),N(\bar{x})\big) \leq \theta\big(\|x-\bar{x}\| + \|y-\bar{y}\|\big) \quad \text{for each } x,\bar{x}y,\bar{y} \in AC^2\big([a,b],\mathbb{R}\big).$$

Let $x,\bar{x},y,\bar{y} \in AC^2([a,b],\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t,x(t),y(t))$ such that, for each $t \in [a,b]$,

$$h_1(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}v_1(s)(b) - BI^{\alpha+\delta}v_1(s)(\eta)\big) + I^{\alpha}v_1(s)(t).$$

By (B_2), we have

$$H_d\big(F(t,x,y),F(t,\bar{x},\bar{y})\big) \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big).$$

So, there exists $w(t) \in F(t,\bar{x}(t),\bar{y}(t))$ such that

$$\big|v_1(t) - w\big| \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big), \quad t \in [a,b].$$

Define $U : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \big\{w \in \mathbb{R} : \big|v_1(t) - w\big| \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big)\big\}.$$

Since the multi-valued operator $U(t) \cap F(t,\bar{x}(t),\bar{y}(t))$ is measurable (Proposition III.4 [30]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t,\bar{x}(t),\bar{y}(t))$ and for each $t \in [a,b]$, we have $|v_1(t) - v_2(t)| \leq m(t)(|x(t)-\bar{x}(t)| + |y(t)-\bar{y}(t)|)$.

For each $t \in [a,b]$, let us define

$$h_2(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(c - AI^{\alpha}v_2(s)(b) - BI^{\alpha+\delta}v_2(s)(\eta)\big) + I^{\alpha}v_2(s)(t).$$

Thus,

$$\begin{aligned}
&\big|h_1(t) - h_2(t)\big| \\
&\quad\leq \frac{(b-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\big(|A|I^{\alpha}\big|v_2(s) - v_1(s)\big|(b) + |B|I^{\alpha+\delta}\big|v_2(s) - v_1(s)\big|(\eta)\big) \\
&\qquad + I^{\alpha}\big|v_2(s) - v_1(s)\big|(t) \\
&\quad\leq 2\bigg(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\bigg[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg)\|m\|\|x-\overline{x}\|.
\end{aligned}$$

Hence

$$\|h_1 - h_2\| \leq 2\bigg(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\bigg[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + |A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg)\|m\|\|x-\overline{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d\big(N(x),N(\bar{x})\big)$$
$$\leq 2\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[|B|\frac{(\eta-a)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}+|A|\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\|m\|\|x-\overline{x}\|.$$

Since N is a contraction, it follows by Covitz and Nadler's lemma ([29]) that N has a fixed point x which is a solution of (1.3)–(1.4). This completes the proof. □

Example 4.10 Consider the nonlocal boundary value problem for the Hilfer-type pantograph fractional differential inclusion of the form

$$\begin{cases} {}^{H}D^{\frac{8}{5},\frac{2}{3}}x(t)\in F(t,x(t),x(t/2)), & t\in[\frac{3}{5},\frac{9}{5}],\\ x(\frac{3}{5})=0, & \frac{1}{2}x(\frac{9}{5})+\frac{3}{5}I^{\frac{7}{2}}x(\frac{7}{5})=\frac{2}{3}. \end{cases} \tag{4.5}$$

Here $\alpha=8/5$, $\beta=2/3$, $\lambda=1/2$, $a=3/5$, $b=9/5$, $A=1/2$, $B=3/5$, $\delta=7/2$, $\eta=7/5$, $c=2/3$. Next, by direct computations, we have $\gamma=28/15$, $\Lambda\approx 0.6212581095$, $\Omega\approx 1.867435349$ and $\Omega_1\approx 0.9310211429$.

(i) Let $F(t,x(t),x(t/2))$ be defined by

$$F(t,x,y)=\left[\left(\frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}\right)e^{-t},\left(\frac{x^2}{1+|x|}+\frac{y^2}{1+|y|}+1\right)e^{-t}\right]. \tag{4.6}$$

It is easy to see that the condition (A_1) in Theorem 4.4 is satisfied. Indeed, we obtain from (4.6) that $\|F(t,x,y)\|\leq(2\rho+1)e^{-t}$ for all $x,y\in\mathbb{R}$ with $\|x\|,\|y\|\leq\rho$ and for *a.e.* $t\in[3/5,9/5]$. Next, we can find that

$$\mu=\liminf_{\rho\to\infty}\frac{1}{\rho}\int_{\frac{3}{5}}^{\frac{9}{5}}(2\rho+1)e^{-t}\,dt\approx 0.7670254956,$$

which leads to

$$\mu\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left(|A|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+|B|\frac{\eta^{\alpha+\delta-1}}{\Gamma(\alpha+\delta)}\right)+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\right\}\approx 0.7141169533<1.$$

Therefore, by Theorem 4.4, the boundary value problem (4.5) with (4.6) has at least one solution on $[3/5,9/5]$.

(ii) Let $F(t,x(t),x(t/2))$ be defined by

$$F(t,x,y)=\left[\frac{1+\sin|x|+\sin|y|}{(15+t)^2},\left(\frac{t+1}{41}\right)\left(1+\frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}\right)\right]. \tag{4.7}$$

It is clear that F, defined in (4.7), is measurable for all $x,y\in\mathbb{R}$. Next, we can compute that

$$H_d\big(F(t,x,y),F(t,\bar{x},\bar{y})\big)\leq\left(\frac{t+1}{41}\right)\big(|x-\bar{x}|+|y-\bar{y}|\big),\quad x,y,\bar{x},\bar{y}\in\mathbb{R},t\in\left[\frac{3}{5},\frac{9}{5}\right].$$

By setting $m(t)=(t+1)/41$, we get $\|m\|=14/205$. Also, we have $d(0,F(t,0,0))\leq m(t)$, $t\in[3/5,9/5]$. Hence we get $2\Omega\|m\|\approx 0.2550643403<1$. Thus, applying the conclusion of

Theorem 4.9, we deduce that the boundary value problem (4.5) with (4.7) has at least one solution on $[3/5, 9/5]$.

5 Conclusion

In this paper we initiated the study of a new class of boundary value problems, involving the Hilfer fractional derivative, for pantograph fractional differential equations and inclusions supplemented by nonlocal integral boundary conditions. Existence and uniqueness results are proved in the single-valued case. Banach's fixed point theorem is used to obtain the uniqueness result, while the nonlinear alternative of Leray–Schauder type and Krasnoselskii's fixed point theorem are applied to obtain the existence results. For the multi-valued problem we prove existence results for both convex valued and non-convex valued multifunctions. For the case when the multi-valued F has convex values we use the Bohnenblust–Karlin fixed point theorem, Martelli's fixed point theorem and the nonlinear alternative for Kakutani maps. For the lower semicontinuous case the existence result is based on the nonlinear alternative of Leray–Schauder type together with a selection theorem for lower semicontinuous maps with decomposable values. Finally in the case of a possible non-convex valued multi-valued map we use a fixed point theorem for contractive multi-valued maps due to Covitz and Nadler. Examples illustrating the obtained results are also presented.

The results presented in this paper are more general and correspond to several new results corresponding to specific values of the parameters involved in the problem (1.1)–(1.2. For instance, the nonlocal boundary condition given by (1.2) with $A = 0$, $c = 0$, $B \neq 0$ can be conceived of as a conserved boundary condition as the sum of the values of the continuous unknown function over the given interval of arbitrary length is zero. In other words, our problem becomes an "average type" boundary value problem for Hilfer-type fractional differential equations or inclusions. With $B = 1$, $A = -1$, $c = 0$ our boundary condition (1.2) becomes $x(b) = BI^{\delta}x(\eta)$, etc. In our future work we plan to investigate the existence of solutions for boundary value problems for other kinds of fractional differential equations and boundary conditions.

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References

1. Diethelm, K.: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics. Springer, New York (2010)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of the Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204 (2006)
3. Lakshmikantham, V., Leela, S., Devi, J.V.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
4. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
5. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
6. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Gordon & Breach, Yverdon (1993)
7. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
8. Hilfer, R. (ed.): Applications of Fractional Calculus in Physics World Scientific, Singapore (2000)
9. Hilfer, R., Luchko, Y., Tomovski, Z.: Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives. Fract. Calc. Appl. Anal. **12**, 299–318 (2009)
10. Soong, T.T.: Random Differential Equations in Science and Engineering. Academic Press, New York (1973)
11. Almeida, R., Malinowska, A.B., Monteiro, M.T.T.: Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. Math. Methods Appl. Sci. **41**, 336–352 (2018)
12. Hilfer, R.: Experimental evidence for fractional time evolution in glass forming materials. J. Chem. Phys. **284**, 399–408 (2002)
13. Ockendon, J.R., Taylor, A.B.: The dynamics of a current collection system for an electric locomotive. Proc. R. Soc. Lond. Ser. A **322**, 447–468 (1971)
14. Balachandran, K., Kiruthika, S., Trujillo, J.J.: Existence of solutions of nonlinear fractional pantograph equations. Acta Math. Sci. **33**, 712–720 (2013)
15. Vivek, D., Kanagarajan, K., Harikrishnan, S.: Existence and uniqueness results for pantograph equations with generalized fractional derivative. J. Nonlinear Anal. Appl. **2**, 105–112 (2017)
16. Furati, K.M., Kassim, N.D., Tatar, N.E.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. **64**, 1616–1626 (2012)
17. Gu, H., Trujillo, J.J.: Existence of mild solution for evolution equation with Hilfer fractional derivative. Appl. Math. Comput. **257**, 344–354 (2015)
18. Wang, J., Zhang, Y.: Nonlocal initial value problems for differential equations with Hilfer fractional derivative. Appl. Math. Comput. **266**, 850–859 (2015)
19. Asawasamrit, S., Kijjathanakorn, A., Ntouyas, S.K., Tariboon, J.: Nonlocal boundary value problems for Hilfer fractional differential equations. Bull. Korean Math. Soc. **55**, 1639–1657 (2018)
20. Vivek, D., Elsayed, E.M., Kanagarajan, K.: Existence, uniqueness and stability of Hilfer type neutral pantograph differential equations with nonlocal conditions. Int. J. Sci. Innov. Math. Res. **6**, 42–53 (2018)
21. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
22. Krasnoselskii, M.A.: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123–127 (1955)
23. Lasota, A., Opial, Z.: An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781–786 (1965)
24. Bohnenblust, H.F., Karlin, S.: On a theorem of Ville. In: Contributions to the Theory of Games, vol. I, pp. 155–160. Princeton University Press, Princeton (1950)
25. Deimling, K.: Multivalued Differential Equations. de Gruyter, Berlin (1992)
26. Martelli, M.: A Rothe's theorem for non compact acyclic-valued maps. Boll. Unione Mat. Ital. (4) **11**(3, suppl.), 70–76 (1975)
27. Bressan, A., Colombo, G.: Extensions and selections of maps with decomposable values. Stud. Math. **90**, 69–86 (1988)
28. Frigon, M.: Théorèmes d'existence de solutions d'inclusions différentielles. In: Granas, A., Frigon, M. (eds.) Topological Methods in Differential Equations and Inclusions. NATO ASI Series C, vol. 472, pp. 51–87. Kluwer Academic, Dordrecht (1995)
29. Covitz, H., Nadler, S.B. Jr.: Multivalued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5–11 (1970)
30. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)