Jang et al. *Advances in Difference Equations* (2020) 2020:278
https://doi.org/10.1186/s13662-020-02746-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

p-Adic integral on $\mathbb{Z}_p$ associated with degenerate Bernoulli polynomials of the second kind

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Abstract

In this paper, by means of p-adic Volkenborn integrals we introduce and study two different degenerate versions of Bernoulli polynomials of the second kind, namely partially and fully degenerate Bernoulli polynomials of the second kind, and also their higher-order versions. We derive several explicit expressions of those polynomials and various identities involving them.

MSC: 11B68; 11B73; 11B83; 11S80

Keywords: Degenerate Bernoulli polynomials of the second kind; Bernoulli polynomials of the second; Bernoulli numbers; Degenerate Stirling numbers; p-Adic integral

1 Introduction and preliminaries

In [1, 2], Carlitz studied degenerate versions of Bernoulli and Euler polynomials, namely the degenerate Bernoulli and Euler polynomials, and obtained some interesting arithmetic and combinatorial results. In recent years, various degenerate versions of many special polynomials and numbers regained interest of some mathematicians, and quite a few results have been discovered. These include the degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Cauchy numbers, and so on (see [3, 10, 13, 16, 18, 19] and the references therein). Here we would like to mention that the study of degenerate versions can be done not only for polynomials but also for transcendental functions like gamma functions. For this, we let the reader refer to the paper [14].

The aim of this paper is to study two degenerate versions of Bernoulli polynomials of the second kind, namely the partially and fully degenerate Bernoulli polynomials of the second kind, and their higher-order versions by using p-adic Volkenborn integrals. We derive several explicit expressions for those polynomials and identities involving them and some other special numbers and polynomials. The possible applications of our results are discussed in the last section.

The paper is organized as follows. In this section, we recall what is needed in the rest of the paper, which includes the p-adic Volkenborn integrals, the ordinary and higher-order Bernoulli polynomials, the Bernoulli polynomials of the second kind, the degenerate exponential functions, the Daehee numbers, the Stirling numbers of both kinds, the degenerate Stirling numbers of both kinds, and the degenerate Bernoulli polynomials. In Sect. 2, we define the partially degenerate Bernoulli polynomials of the second kind and their higher-order versions by using p-adic Volkenborn integrals. We derive several explicit expressions for those polynomials. Further, we obtain identities involving those polynomials and some other polynomials including the higher-order Bernoulli polynomials, the Daehee numbers, and the usual and degenerate Stirling numbers of both kinds. In Sect. 3, we define the fully degenerate Bernoulli polynomials of the second kind and their higher-order versions by using p-adic Volkenborn integrals. We deduce several explicit expressions for those polynomials. Moreover, we obtain identities involving those polynomials and some other special numbers and polynomials. Here we observe that, for $x = 0$, both partial degenerate Bernoulli polynomials of the second kind and fully degenerate Bernoulli polynomials of the second kind become the same degenerate Bernoulli numbers of the second kind.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$.

The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let f be a $\mathbb{C}_p$-valued uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral of f on $\mathbb{Z}_p$ is defined by (see [8, 23–25])

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_0\bigl(x+p^N\mathbb{Z}_p\bigr) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x). \tag{1}$$

From (1), we note that (see [8, 9, 23, 25, 26])

$$I_0(f_1) - I_0(f) = f'(0), \tag{2}$$

where $f_1(x) = f(x+1)$, $f'(0) = \frac{d}{dx}f(x)|_{x=0}$.

By (2), we get (see [8, 23, 26])

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n\,d\mu_0(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+y)t}\,d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{3}$$

where $B_n(x)$ are the Bernoulli polynomials and $B_n = B_n(0)$ are the Bernoulli numbers.

For $r \in \mathbb{N}$, we note that (see [8, 24])

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_r+x)^n\,d\mu_0(x_1)\,d\mu_0(x_2)\cdots d\mu_0(x_r)\frac{t^n}{n!}\\
&\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r+x)t}\,d\mu_0(x_1)\,d\mu_0(x_2)\cdots d\mu_0(x_r)\\
&\quad = \left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},
\end{aligned} \tag{4}$$

where $B_n^{(r)}(x)$ are the Bernoulli polynomials of order r, and $B_n^{(r)} = B_n^{(r)}(0)$ are the Bernoulli numbers of order r.

The Bernoulli polynomials of the second kind (also called the Cauchy polynomials) are defined by (see [2, 7, 10, 11, 17, 22])

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{5}$$

More generally, for any $r \in \mathbb{N}$, the Bernoulli polynomials of the second kind of order r are given by

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}. \tag{6}$$

It is well known that (see [8, 9, 21])

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!}. \tag{7}$$

From (5) and (7), we note that

$$b_n = B_n^{(n)}(1) \quad (n \geq 0).$$

The degenerate exponential function is defined by (see [12, 14, 16, 18–20])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{8}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$.

We note that (see [12, 14])

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \tag{9}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$.

As is known, the Daehee numbers are defined by (see [4, 5, 15])

$$\int_{\mathbb{Z}_p} (1+t)^x \, d\mu_0(x) = \frac{1}{t}\log(1+t) = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}. \tag{10}$$

The Stirling numbers of the first kind are defined as (see [3, 6, 10, 15, 25])

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad (n \geq 0), \tag{11}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 1)$.

As an inversion formula of (11), the Stirling numbers of the second kind are defined by (see [18, 21])

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{12}$$

Recently, Kim considered the degenerate Stirling numbers of the second kind given by (see [10])

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0). \tag{13}$$

In light of (11), the degenerate Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0). \tag{14}$$

In [1, 2], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

2 Partially degenerate Bernoulli polynomials of the second kind

In this and next section, we assume that $0 \neq \lambda \in \mathbb{Z}_p$ and $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. Let $\log_\lambda t$ be the compositional inverse of $e_\lambda(t)$ satisfying

$$\log_\lambda\bigl(e_\lambda(t)\bigr) = e_\lambda\bigl(\log_\lambda(t)\bigr) = t.$$

From (8), we note that

$$\log_\lambda(t) = \frac{1}{\lambda}\bigl(t^\lambda - 1\bigr). \tag{16}$$

By (16), we easily see that $\lim_{\lambda\to 0}\log_\lambda(t) = \log(t)$.

From (2) and (16), we can derive the following equation:

$$\frac{t}{\log_\lambda(1+t)} = \frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda x}\,d\mu_0(x). \tag{17}$$

Let us define the partially degenerate Bernoulli polynomials of the second kind as follows:

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{18}$$

Then, from (17), we see that

$$\sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}\,d\mu_0(y). \tag{19}$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}(x) = b_n(x)$ $(n \geq 0)$. For $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers of the second kind.

First, from (18) we note that

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{\log_\lambda(1+t)}(1+t)^x \\ &= \sum_{m=0}^{\infty} b_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_l\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} b_{m,\lambda}(x)_{n-m}\frac{t^n}{n!}. \end{aligned} \tag{20}$$

Thus we get the next result by (20).

Theorem 1 *For $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m} b_{m,\lambda}(x)_{n-m}.$$

By (3), we get

$$\begin{aligned}\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}\,d\mu_0(y) &= \frac{t}{\log(1+t)}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\big(\log(1+t)\big)^m\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)^m d\mu_0(y) \\ &= \sum_{l=0}^{\infty} b_l\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m\left(\frac{x}{\lambda}\right)\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!} \\ &= \sum_{l=0}^{\infty} b_l\frac{t^l}{l!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\lambda^m B_m\left(\frac{x}{\lambda}\right)S_1(k,m)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m\left(\frac{x}{\lambda}\right)S_1(k,m)b_{n-k}\right)\frac{t^n}{n!}. \end{aligned} \tag{21}$$

Therefore, we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m S_1(k,m)b_{n-k}B_m\left(\frac{x}{\lambda}\right).$$

In particular, we have

$$b_{n,\lambda} = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m S_1(k,m)b_{n-k}B_m.$$

From (9), we note that

$$\frac{1}{t}\big(e_\lambda(t)-1\big)e_\lambda^x(t)=\sum_{l=0}^{\infty}\frac{(1)_{l+1,\lambda}}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}. \tag{22}$$

By (14), we get

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{23}$$

Thus, by replacing t by $\log_\lambda(1+t)$ in (22), we get

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}\frac{1}{m!}\big(\log_\lambda(1+t)\big)^m$$
$$=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{24}$$

Therefore, by (18) and (24), we obtain the following theorem.

Theorem 3 *For $n\geq 0$, we have*

$$b_{n,\lambda}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}S_{1,\lambda}(n,m).$$

In particular, we have

$$b_{n,\lambda}=\sum_{m=0}^{n}\frac{(1)_{m+1,\lambda}}{m+1}S_{1,\lambda}(n,m).$$

From (17), we note that

$$\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}\,d\mu_0(y)=\frac{\log(1+t)}{t}\frac{t}{\log_\lambda(1+t)}(1+t)^x=\sum_{l=0}^{\infty}\frac{D_l}{l!}t^l\sum_{m=0}^{\infty}b_{m,\lambda}(x)\frac{t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}(x)D_{n-m}\right)\frac{t^n}{n!}. \tag{25}$$

On the other hand,

$$\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}\,d\mu_0(y)=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}\,d\mu_0(y)t^n. \tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 4 *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{\lambda y + x}{n} d\mu_0(y) = \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} b_{m,\lambda}(x) D_{n-m}.$$

In particular, we have

$$\int_{\mathbb{Z}_p} \binom{\lambda y}{n} d\mu_0(y) = \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} b_{m,\lambda} D_{n-m}.$$

By replacing t by $e_\lambda(t) - 1$ in (18), we get

$$\begin{aligned} \frac{e_\lambda(t) - 1}{t} e_\lambda^x(t) &= \sum_{m=0}^{\infty} b_{m,\lambda}(x) \frac{1}{m!} \left(e_\lambda(t) - 1\right)^m = \sum_{m=0}^{\infty} b_{m,\lambda}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m) b_{m,\lambda}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

On the other hand, by (22), we get

$$\frac{1}{t}\left(e_\lambda(t) - 1\right) e_\lambda^x(t) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{(1)_{l+1,\lambda}}{l+1} (x)_{n-l,\lambda} \right) \frac{t^n}{n!}. \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m) b_{m,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \frac{(1)_{l+1,\lambda}}{l+1} (x)_{n-l,\lambda}.$$

In particular, we have

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m) b_{m,\lambda} = \frac{1}{n+1} (1)_{n+1,\lambda}.$$

By replacing t by $\log_\lambda(1+t)$ in (15), we get

$$\begin{aligned} \frac{\log_\lambda(1+t)}{t}(1+t)^x &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{1}{m!} \left(\log_\lambda(1+t)\right)^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \beta_{m,\lambda}(x) S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{29}$$

We observe that

$$\begin{aligned} \frac{\log_\lambda(1+t)}{t} &= \frac{1}{\lambda t} \sum_{m=1}^{\infty} \lambda^m \frac{1}{m!} \left(\log(1+t)\right)^m = \frac{1}{\lambda t} \sum_{m=1}^{\infty} \lambda^m \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \frac{1}{\lambda t} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \lambda^m S_1(n,m) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sum_{m=1}^{n+1} \lambda^{m-1} S_1(n+1,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{30}$$

From (30), we obtain

$$\begin{aligned}\frac{\log_{\lambda}(1+t)}{t}(1+t)^{x} &= \sum_{m=0}^{\infty}\frac{1}{m+1}\left(\sum_{k=1}^{m+1}\lambda^{k-1}S_{1}(m+1,k)\right)\frac{t^{m}}{m!}\sum_{l=0}^{\infty}(x)_{l}\frac{t^{l}}{l!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}\left(\sum_{k=1}^{m+1}\lambda^{k-1}S_{1}(m+1,k)\right)(x)_{n-m}\frac{t^{n}}{n!}. \end{aligned}\tag{31}$$

Therefore, by (29) and (31), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}\sum_{k=1}^{m+1}\lambda^{k-1}S_{1}(m+1,k)(x)_{n-m} = \sum_{m=0}^{n}\beta_{m,\lambda}(x)S_{1,\lambda}(n,m).$$

In particular, we have

$$\frac{1}{n+1}\sum_{k=1}^{n+1}\lambda^{k-1}S_{1}(n+1,k) = \sum_{m=0}^{n}\beta_{m,\lambda}S_{1,\lambda}(n,m).$$

From (21), we note that

$$\begin{aligned}\frac{t^{k}}{k!} &= \sum_{m=k}^{\infty}S_{1,\lambda}(m,k)\frac{1}{m!}\bigl(e_{\lambda}(t)-1\bigr)^{m}\\ &= \sum_{m=k}^{\infty}S_{1,\lambda}(m,k)\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^{n}}{n!}\\ &= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}S_{1,\lambda}(m,k)S_{2,\lambda}(n,m)\right)\frac{t^{n}}{n!}\quad (k\geq 0). \end{aligned}\tag{32}$$

By comparing the coefficients on both sides of (32), we obtain the following theorem.

Theorem 7 *For $k \geq 0$, we have*

$$\sum_{m=k}^{n}S_{1,\lambda}(m,k)S_{2,\lambda}(n,m) = \begin{cases}1, & \text{if } n=k,\\ 0, & \text{if } n>k.\end{cases}$$

For $r \in \mathbb{N}$, we define the partially degenerate Bernoulli polynomials of the second kind of order r by the following multiple p-adic integrals on $\mathbb{Z}_p$:

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^{r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)+x}\,d\mu_0(x_1)\,d\mu_0(x_2)\cdots d\mu_0(x_r)\\ &\quad= \left(\frac{t}{\log_{\lambda}(1+t)}\right)^{r}(1+t)^{x} = \sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}. \end{aligned}\tag{33}$$

For $x = 0$, $b_{n,\lambda}^{(r)} = b_{n,\lambda}^{(r)}(0)$ are called the degenerate Bernoulli numbers of the second kind of order r.

On the other hand, (33) is also equal to

$$\begin{aligned}
&\left(\frac{t}{\log(1+t)}\right)^r \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+t)^{\lambda(x_1+x_2+\cdots+x_r)+x}\, d\mu_0(x_1)\, d\mu_0(x_2)\cdots d\mu_0(x_r) \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)\frac{1}{m!}\big(\log(1+t)\big)^m \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!} \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)S_1(k,m)\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)S_1(k,m)B_{n-k}^{(n-k-r+1)}(1)\right)\frac{t^n}{n!}.
\end{aligned} \tag{34}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, we have*

$$b_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)S_1(k,m)B_{n-k}^{(n-k-r+1)}(1).$$

In particular, we have

$$b_{n,\lambda}^{(r)} = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}S_1(k,m)B_{n-k}^{(n-k-r+1)}(1).$$

By replacing t by $e_\lambda(t)-1$ in (33), we get

$$\begin{aligned}
\sum_{m=0}^{\infty} b_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m &= \frac{r!}{t^r}\frac{1}{r!}\big(e_\lambda(t)-1\big)^r e_\lambda^x(t) \\
&= \frac{r!}{t^r}\sum_{n=0}^{\infty}S_{2,\lambda}(n+r,r)\frac{t^{n+r}}{(n+r)!}e_\lambda^x(t) \\
&= \sum_{m=0}^{\infty}\frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda}\frac{t^n}{n!}.
\end{aligned} \tag{35}$$

On the other hand,

$$\sum_{m=0}^{\infty} b_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} b_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{36}$$

From (35) and (36), we obtain the following theorem.

Theorem 9 *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda} = \sum_{m=0}^{n} b_{m,\lambda}^{(r)}(x) S_{2,\lambda}(n,m).$$

In particular, we have

$$S_{2,\lambda}(n+r,r) = \binom{n+r}{r} \sum_{m=0}^{n} b_{m,\lambda}^{(r)} S_{2,\lambda}(n,m).$$

From (13), we note that

$$\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{37}$$

Thus, by (35), we get

$$\frac{(e_\lambda(t)-1)^r}{t^r} e_\lambda^x(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda} \frac{t^n}{n!}. \tag{38}$$

By replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned}
&\left(\frac{t}{\log_\lambda(1+t)}\right)^r (1+t)^x \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_{2,\lambda}(k+r,r)(x)_{m-k,\lambda} \frac{1}{m!}\bigl(\log_\lambda(1+t)\bigr)^m \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_{2,\lambda}(k+r,r) S_{1,\lambda}(n,m)(x)_{m-k,\lambda} \right) \frac{t^n}{n!}.
\end{aligned} \tag{39}$$

Therefore, by (33) and (39), we obtain the following theorem.

Theorem 10 *For $n \geq 0$, we have*

$$b_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_{2,\lambda}(k+r,r) S_{1,\lambda}(n,m)(x)_{m-k,\lambda}.$$

In particular, we have

$$b_{n,\lambda}^{(r)} = \sum_{m=0}^{n} \frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}} S_{1,\lambda}(n,m).$$

By (14), we get

$$\frac{1}{k!}\bigl(\log_\lambda(1+t)\bigr)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{40}$$

Thus, by (40), we have

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x = \sum_{m=0}^\infty \frac{S_{1,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^m}{m!}\sum_{l=0}^\infty (x)_l\frac{t^l}{l!}$$
$$= \sum_{n=0}^\infty\sum_{m=0}^n \frac{\binom{n}{m}}{\binom{m+r}{r}}S_{1,\lambda}(m+r,r)(x)_{n-m,\lambda}\frac{t^n}{n!}. \tag{41}$$

By replacing t by $e_\lambda(t)-1$ in (41), we get

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{m=0}^\infty\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)(x)_{m-k,\lambda}\frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m$$
$$= \sum_{n=0}^\infty\left(\sum_{m=0}^n\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)S_{2,\lambda}(n,m)(x)_{m-k,\lambda}\right)\frac{t^n}{n!}. \tag{42}$$

As is well known, the degenerate Bernoulli polynomials of order r are defined by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^\infty \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 2])}. \tag{43}$$

Therefore, by (42) and (43), we obtain the following theorem.

Theorem 11 *For $n\ge 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{m=0}^n\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)S_{2,\lambda}(n,m)(x)_{m-k,\lambda}.$$

In particular, we have

$$\beta_{n,\lambda}^{(r)} = \sum_{m=0}^n \frac{S_{1,\lambda}(m+r,r)}{\binom{m+r}{r}}S_{2,\lambda}(n,m).$$

From (33), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)+x}\,d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$= \left(\frac{\log(1+t)}{t}\right)^r\left(\frac{t}{\log_\lambda(1+t)}\right)^r(1+t)^x = \sum_{l=0}^\infty \frac{S_1(l+r,r)}{\binom{l+r}{r}}\frac{t^l}{l!}\sum_{m=0}^\infty b_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}$$
$$= \sum_{n=0}^\infty\left(\sum_{l=0}^n \frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)b_{n-l,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}. \tag{44}$$

Thus, by (44), we obtain the following theorem.

Theorem 12 *For $n\ge 0$, we have*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\lambda(x_1+\cdots+x_r)+x}{n}\,d\mu_0(x_1)\cdots d\mu_0(x_r) = \frac{1}{n!}\sum_{l=0}^n \frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)b_{n-l,\lambda}^{(r)}(x).$$

In particular, we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\lambda(x_1+\cdots+x_r)}{n}\,d\mu_0(x_1)\cdots d\mu_0(x_r)=\frac{1}{n!}\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)b_{n-l,\lambda}^{(r)}.$$

Observe from (30) with $\lambda = 1$ that $b_{n,1}^{(r)}(x)=(x)_n$, $b_{n,1}^{(r)}=\delta_{n,0}$.

Now, let us take $\lambda = 1$ in Theorem 12. Then we have, for $n \geq 0$,

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_r+x)_n\,d\mu_0(x_1)\cdots d\mu_0(x_r)=\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)(x)_{n-l}, \tag{45}$$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_r)_n\,d\mu_0(x_1)\cdots d\mu_0(x_r)=\frac{S_1(n+r,r)}{\binom{n+r}{r}}. \tag{46}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_n\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\sum_{l=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^l\,d\mu_0(x)\cdots d\mu_r(x)\\&=\sum_{l=0}^{n}S_1(n,l)B_l^{(r)}(x).\end{aligned} \tag{47}$$

Thus, by (45), (46), and (47), for $n \geq 0$, we get

$$\sum_{l=0}^{n}S_1(n,l)B_l^{(r)}(x)=\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)(x)_{n-l}, \tag{48}$$

$$\sum_{l=0}^{n}S_1(n,l)B_l^{(r)}=\frac{S_1(n+r,r)}{\binom{n+r}{r}}. \tag{49}$$

By replacing t by $\log_\lambda(1+t)$ in (43), we get

$$\begin{aligned}\left(\frac{1}{t}\log_\lambda(1+t)\right)^r(1+t)^x&=\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(r)}(x)\frac{1}{m!}\bigl(\log_\lambda(1+t)\bigr)^m\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,\lambda}^{(r)}(x)S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{50}$$

Therefore, by (41) and (50), we obtain the following theorem.

Theorem 13 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{1,\lambda}(m+r,r)(x)_{n-m,\lambda}=\sum_{m=0}^{n}\beta_{m,\lambda}^{(r)}(x)S_{1,\lambda}(n,m).$$

In particular, we have

$$S_{1,\lambda}(n+r,r) = \binom{n+r}{r}\sum_{m=0}^{n}\beta_{m,\lambda}^{(r)}S_{1,\lambda}(n,m).$$

3 Fully degenerate Bernoulli polynomials of the second kind

Let us define the fully degenerate Bernoulli polynomials of the second kind as follows:

$$\frac{t}{\log_\lambda(1+t)}e^{x\log_\lambda(1+t)} = \sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{51}$$

Then, from (17), we see that

$$\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y}\,d\mu_0(y)e^{x\log_\lambda(1+t)}. \tag{52}$$

Note that $\lim_{\lambda\to 0}\mathbf{b}_{n,\lambda}(x) = b_n(x)$ $(n \geq 0)$. We note that $b_{n,\lambda} = \mathbf{b}_{n,\lambda}(0)$ are the degenerate Bernoulli numbers of the second kind.

We note here that

$$e^{x\log_\lambda(1+t)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}S_{1,\lambda}(n,k)x^k\frac{t^n}{n!}. \tag{53}$$

Here, recalling (14), one should compare (53) with the following:

$$\begin{aligned} e^{x\log(1+t)} = (1+t)^x &= \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{54}$$

From (51) and (53), we note that

$$\begin{aligned} \sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}b_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}S_{1,\lambda}(m,k)x^k\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k\frac{t^n}{n!}. \end{aligned} \tag{55}$$

Thus we get the next result by (55).

Theorem 14 *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k.$$

By Theorem 2 and (53), we get

$$\begin{aligned}
&\frac{t}{\log_\lambda(1+t)}e^{x\log_\lambda(1+t)} \\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{m}{k}\lambda^i B_i S_1(k,i)b_{m-k}\right)\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}S_{1,\lambda}(l,j)x^j\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{n-m}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^i B_i b_{m-k}x^j\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^i B_i b_{m-k}x^j\right)\frac{t^n}{n!}.
\end{aligned} \tag{56}$$

Therefore, we obtain the following theorem.

Theorem 15 *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x) = \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^i B_i b_{m-k}\right)x^j.$$

From (9), we note that

$$\begin{aligned}
\frac{1}{t}\bigl(e_\lambda(t)-1\bigr)e^{xt} &= \sum_{l=0}^{\infty}\frac{(1)_{l+1,\lambda}}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{57}$$

Thus, by replacing t by $\log_\lambda(1+t)$ in (57) and making use of (23), we get

$$\begin{aligned}
\frac{t}{\log_\lambda(1+t)}e^{x\log_\lambda(1+t)} &= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{m-l}\frac{1}{m!}\bigl(\log_\lambda(1+t)\bigr)^m \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{m-l}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{m-l}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=l}^{n}\binom{m}{l}\frac{(1)_{m-l+1,\lambda}}{m-l+1}S_{1,\lambda}(n,m)x^l\right)\frac{t^n}{n!}.
\end{aligned} \tag{58}$$

Therefore, by (51) and (58), we obtain the following theorem.

Theorem 16 *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=l}^{n}\binom{m}{l}\frac{(1)_{m-l+1,\lambda}}{m-l+1}S_{1,\lambda}(n,m)x^l.$$

From (17), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{\lambda y}\,d\mu_0(y)e^{x\log_\lambda(1+t)}&=\frac{\log(1+t)}{t}\frac{t}{\log_\lambda(1+t)}e^{x\log_\lambda(1+t)}\\&=\sum_{l=0}^{\infty}\frac{D_l}{l!}t^l\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}(x)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathbf{b}_{m,\lambda}(x)D_{n-m}\right)\frac{t^n}{n!}.\end{aligned}\tag{59}$$

On the other hand, from (53) we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{\lambda y}\,d\mu_0(y)e^{x\log_\lambda(1+t)}&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(\lambda y)_m\,d\mu_0(y)\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{l}S_{1,\lambda}(l,k)x^k\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}S_{1,\lambda}(l,k)\int_{\mathbb{Z}_p}(\lambda y)_{n-l}\,d\mu_0(y)x^k\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}(l,k)\int_{\mathbb{Z}_p}(\lambda y)_{n-l}\,d\mu_0(y)x^k\frac{t^n}{n!}.\end{aligned}\tag{60}$$

Therefore, by (59) and (60), we obtain the following theorem.

Theorem 17 *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\mathbf{b}_{m,\lambda}(x)D_{n-m}=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}(l,k)\int_{\mathbb{Z}_p}(\lambda y)_{n-l}\,d\mu_0(y)x^k.$$

By replacing t by $e_\lambda(t)-1$ in (51), we get

$$\frac{e_\lambda(t)-1}{t}e^{xt}=\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}(x)\right)\frac{t^n}{n!}.\tag{61}$$

Therefore, by (57) and (61), we obtain the following theorem.

Theorem 18 *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{n-l}.$$

For $r\in\mathbb{N}$, we define the fully degenerate Bernoulli polynomials of the second kind of order r by the following multiple p-adic integrals on $\mathbb{Z}_p$:

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)}\,d\mu_0(x_1)\,d\mu_0(x_2)\cdots d\mu_0(x_r)e^{x\log_\lambda(1+t)}\\&\quad=\left(\frac{t}{\log_\lambda(1+t)}\right)^re^{x\log_\lambda(1+t)}=\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{62}$$

Note here that $b_{n,\lambda}^{(r)} = \mathbf{b}_{n,\lambda}^{(r)}(0)$ are the degenerate Bernoulli numbers of the second of order r.

On the other hand, we have

$$\begin{aligned}
&\left(\frac{t}{\log(1+t)}\right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda(x_1+x_2+\cdots+x_r)}\, d\mu_0(x_1)\, d\mu_0(x_2)\cdots d\mu_0(x_r) \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!} \sum_{m=0}^{\infty} \lambda^m B_m^{(r)} \frac{1}{m!}\bigl(\log(1+t)\bigr)^m \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!} \sum_{m=0}^{\infty} \lambda^m B_m^{(r)} \sum_{k=m}^{\infty} S_1(k,m)\frac{t^k}{k!} \\
&= \sum_{l=0}^{\infty} B_l^{(l-r+1)}(1)\frac{t^l}{l!} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \lambda^m B_m^{(r)} S_1(k,m)\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\sum_{m=0}^{k} \binom{n}{k} \lambda^m B_m^{(r)} S_1(k,m) B_{n-k}^{(n-k-r+1)}(1)\right)\frac{t^n}{n!}.
\end{aligned} \tag{63}$$

Therefore, by (53), (62), and (63), we obtain the following theorem.

Theorem 19 *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}^{(r)}(x) = \sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{m}\sum_{i=0}^{k} \binom{n}{m}\binom{m}{k} S_1(k,i) S_{1,\lambda}(n-m,j)\lambda^i B_i^{(r)} B_{m-k}^{(m-k-r+1)}(1) x^j.$$

By replacing t by $e_\lambda(t)-1$ in (62), we get

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathbf{b}_{m,\lambda}^{(r)}(x)\frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m &= \frac{r!}{t^r}\frac{1}{r!}\bigl(e_\lambda(t)-1\bigr)^r e^{xt} \\
&= \sum_{m=0}^{\infty} \frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^m}{m!} \sum_{l=0}^{\infty} x^l \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r,r) x^{n-m}\frac{t^n}{n!}.
\end{aligned} \tag{64}$$

On the other hand,

$$\sum_{m=0}^{\infty} \mathbf{b}_{m,\lambda}^{(r)}(x)\frac{1}{m!}\bigl(e_\lambda(t)-1\bigr)^m = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \mathbf{b}_{m,\lambda}^{(r)}(x) S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{65}$$

From (64) and (65), we obtain the following theorem.

Theorem 20 *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{m+r}{r}} S_{2,\lambda}(m+r,r) x^{n-m} = \sum_{m=0}^{n} S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}^{(r)}(x).$$

From (37), we recall here that

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad(k\geq 0). \tag{66}$$

Thus, by (66), we get

$$\frac{(e_\lambda(t)-1)^r}{t^r}e^{xt}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)x^{n-m}\frac{t^n}{n!}. \tag{67}$$

By replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned}\left(\frac{t}{\log_\lambda(1+t)}\right)^r e^{x\log_\lambda(1+t)}&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)x^{m-k}\frac{1}{m!}\left(\log_\lambda(1+t)\right)^m\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)S_{1,\lambda}(n,m)x^{m-k}\frac{t^n}{n!}.\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=k}^{n}\frac{\binom{m}{k}}{\binom{m-k+r}{r}}S_{2,\lambda}(m-k+r,r)S_{1,\lambda}(n,m)x^k\right)\frac{t^n}{n!}.\end{aligned} \tag{68}$$

Therefore, by (62) and (68), we obtain the following theorem.

Theorem 21 *For $n\geq 0$, we have*

$$\mathbf{b}_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\sum_{m=k}^{n}\frac{\binom{m}{k}}{\binom{m-k+r}{r}}S_{2,\lambda}(m-k+r,r)S_{1,\lambda}(n,m)x^k.$$

From (62), we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)}\,d\mu_0(x_1)\cdots d\mu_0(x_r)e^{x\log_\lambda(1+t)}\\&=\left(\frac{\log(1+t)}{t}\right)^r\left(\frac{t}{\log_\lambda(1+t)}\right)^r e^{x\log_\lambda(1+t)}\\&=\sum_{l=0}^{\infty}\frac{S_1(l+r,r)}{\binom{l+r}{r}}\frac{t^l}{l!}\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)\mathbf{b}_{n-l,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{69}$$

On the other hand, (69) is also equal to

$$\begin{aligned}&\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\lambda(x_1+\cdots+x_r)\right)_m d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{l}S_{1,\lambda}(l,k)x^k\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}S_{1,\lambda}(n-m,k)x^k\\&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\lambda(x_1+\cdots+x_r)\right)_m d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}.\end{aligned} \tag{70}$$

Thus, by (69) and (70), we obtain the following theorem.

Theorem 22 *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r)\mathbf{b}_{n-l,\lambda}^{(r)}(x) \\ = \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m} S_{1,\lambda}(n-m,k)x^k \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\big(\lambda(x_1+\cdots+x_r)\big)_m\,d\mu_0(x_1)\cdots d\mu_0(x_r).$$

Observe from (62) with $\lambda = 1$ that $\mathbf{b}_{n,1}^{(r)}(x) = x^n$, $b_{n,1}^{(r)} = \mathbf{b}_{n,1}^{(r)}(0) = \delta_{n,0}$.

Now, let us take $\lambda = 1$ in Theorem 22. Then we have, for $n \geq 0$,

$$\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r)x^{n-l} \\ = \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m} S_{1,1}(n-m,k)x^k \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)_m\,d\mu_0(x_1)\cdots d\mu_0(x_r). \tag{71}$$

In addition, we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)_m\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ = \sum_{l=0}^{m} S_1(m,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)^l\,d\mu_0(x)\cdots d\mu_r(x) \\ = \sum_{l=0}^{m} S_1(m,l)B_l^{(r)}. \tag{72}$$

Thus, by (71) and (72), for $n \geq 0$, we get the following theorem.

Theorem 23

$$\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r)x^{n-l} = \sum_{m=0}^{n}\sum_{k=0}^{n-m}\sum_{l=0}^{m}\binom{n}{m} S_{1,1}(n-m,k)S_1(m,l)B_l^{(r)}x^k.$$

4 Conclusion

In this paper, we defined the partially and fully degenerate Bernoulli polynomials of the second kind and their higher-order versions by means of Volkenborn p-adic integrals. We derived several explicit expressions of those polynomials and identities involving them and some other special numbers and polynomials.

Next, we would like to mention three possible applications of our results. The first one is their possible application to probability theory. Indeed, in [18] we demonstrated that both the degenerate Stirling polynomials of the second and the r-truncated degenerate Stirling polynomials of the second kind appear in certain expressions of the probability distributions of appropriate random variables. The second one is their possible application to differential equations from which some useful identities follow. For example, in [7] an infinite family of nonlinear differential equations, having the generating function of the degenerate Bernoulli numbers of the second kind as a solution, were derived. As a result, it was possible to derive an identity involving the ordinary and higher-order degenerate Bernoulli numbers of the second kind and generalized harmonic numbers (see also [4]). The third one is their possible application to identities of symmetry. For instance, in [13] we obtained many symmetric identities in three variables related to degenerate Euler polynomials and alternating generalized falling factorial sums. Each of these possible applications of the special polynomials considered in this paper requires considerable amount of work and hence needs to appear in the form of separate papers.

Finally, as one of our future projects, we will continue to study various degenerate versions of special polynomials and numbers and investigate their possible applications to physics, science, and engineering as well as to mathematics.

Acknowledgements
We thank the reviewers for their valuable comments and suggestions. In addition, the authors would like to thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare no conflict of interest.

Authors' contributions
DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; LCJ and HL checked the results of the paper and typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

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Received: 4 April 2020 Accepted: 2 June 2020 Published online: 09 June 2020

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