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Generation of new fractional inequalities via n polynomials s-type convexity with applications

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Abstract

The celebrated Hermite–Hadamard and Ostrowski type inequalities have been studied extensively since they have been established. We find novel versions of the Hermite–Hadamard and Ostrowski type inequalities for the n-polynomial s-type convex functions in the frame of fractional calculus. Taking into account the new concept, we derive some generalizations that capture novel results under investigation. We present two different general techniques, for the functions whose first and second derivatives in absolute value at certain powers are n-polynomial s-type convex functions by employing $\mathcal{K}$-fractional integral operators have yielded intriguing results. Applications and motivations of presented results are briefly discussed that generate novel variants related to quadrature rules that will be helpful for in-depth investigation in fractal theory, optimization and machine learning.

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1 Introduction and preliminaries

A few decades ago, the classical calculus has been revolutionized by tremendous innovations. The researchers are concurring with the extraordinary excellence and truthfulness in outcomes by the fractional-order equations. If we observed the historical background of fractional calculus, the notion has been instigated from a letter of L'Hospital to Leibniz for the meaning of dr/dr^n for $n = 1/2$. In response, Leibniz saw this is "An obvious conundrum, from which one day valuable outcomes will be shown". Also, presently one witnesses the utilization of fractional calculus in various areas, for example, chaos, simulation, and modeling. Numerous useful definitions and operators show the beauty of the fractional calculus, for instance, the Riemann, Caputo, Hadamard, Katugampola, Erdelyi–Kober, Atangana–Baleanu, Weyl types and many others with potential applications in mathematics and physics [1–11]. For a feature depiction of the origin of fractional calculus, improvements, and applications, we refer the reader to the notable monographs [12, 13] and the interesting articles [14–25].

Convexity has played a crucial role in the advancement of pure and applied mathematics [26–35]. Due to its robustness, convex functions and convex sets have been generalized and extended in many mathematics branches, in particular, many inequalities can be found in the literature [36–48] via convexity theory. To the best of our knowledge, the Hermite–Hadamard inequality is a well-known, paramount and extensively useful inequality in the applied literature of mathematical inequalities [49–60]. This inequality is of pivotal significance because of other classical inequalities such as the Hardy, Opial, Lynger, Ostrowski, Minkowski, Hölder, Ky–Fan, Beckenbach–Dresher, Levinson, arithmetic–geometric, Young, Olsen and Gagliardo–Nirenberg inequalities, which are closely related to the classical Hermite–Hadamard inequality [61]. It can be stated as follows: the double inequality

$$\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right) \leq \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(r)\,dr \leq \frac{\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)}{2} \tag{1.1}$$

holds if $\mathcal{P}$ is a convex function on the interval $[\eta_1,\eta_2]$.

Recently, the Hermite–Hadamard inequality (1.1) and its generalizations, refinements, extensions and variants have attracted the attention of many researchers. It has been proved that the function $\mathcal{P}: I \to \mathbb{R}$ is convex if and only if inequality (1.1) holds for all $\eta_1,\eta_2 \in I$ with $\eta_1 \neq \eta_2$.

Let $\mathcal{I} \subseteq \mathbb{R}$ and $\mathcal{P}:\mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$) such that $\eta_1,\eta_2 \in \mathcal{I}^\circ$ with $\eta_1 < \eta_2$. Then the well-known Ostrowski inequality [62] states that

$$\left|\mathcal{P}(z) - \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)\,dz\right| \leq \left[\frac{1}{4} + \frac{(z-\frac{\eta_1+\eta_2}{2})^2}{(\eta_2-\eta_1)^2}\right](\eta_2-\eta_1)\mathcal{M} \tag{1.2}$$

for all $z \in [\eta_1,\eta_2]$ if $|\mathcal{P}'(\zeta)| \leq \mathcal{M}$ for all $\zeta \in [\eta_1,\eta_2]$.

Ostrowski type inequalities have significant contributions in the numerical analysis as they provide the error estimates of many quadrature rules. In recent years, they have been extended and generalized in many fields.

The uses of variants in applied sciences are generally studied and now it is a profoundly appealing research-oriented area where the researchers also investigate the existence and uniqueness of the solutions of fractional differential equations. Khan et al. [63] derived the Hermite–Hadamard inequality for s-convex functions. In [64], the authors derived several generalizations for the Ostrowski type inequality involving the generalized $\mathcal{K}$-fractional integrals.

In the article, we propose an novel class of functional variants for convex functions and several other new and effectively applicable generalizations for convexity theory and fractional operators. The novel technique is useful to generate the Mandelbrot and Julia sets for quadratic and cubic polynomials with s-convexity [65–67].

Now, we discuss some connections between the class of convex functions and s-convex functions.

Definition 1.1 Let $s \in [0,1]$. Then the real-valued function $\mathcal{P}:\mathcal{I} \to \mathbb{R}$ is said to be s-type convex on $\mathcal{I}$ if the inequality

$$\mathcal{P}\big(\zeta x + (1-\zeta)y\big) \leq \big[1 - s(1-\zeta)\big]\mathcal{P}(x) + (1-s\zeta)\mathcal{P}(y) \tag{1.3}$$

holds for all $x, y \in \mathcal{I}$ and $\zeta \in [0,1]$.

Remark 1.2 From Definition 1.1 we clearly see that:

(1) If we choose $s = 1$, then we get the classical convex function.

(2) If we choose $s = 0$, then we get the definition of P-function in [68].

(3) If $\mathcal{P}$ is s-type convex on $\mathcal{I}$, then the range of the function $\mathcal{P}$ is $[0,\infty)$.

Indeed, let $x \in \mathcal{I}$. Then by the s-type convexity of $\mathcal{P}$ we have

$$\mathcal{P}\big(\zeta\eta_1 + (1-\zeta)x\big) \leq \big[1 - s(1-\zeta)\big]\mathcal{P}(\eta_1) + (1 - s\zeta)\mathcal{P}(x)$$

for all $\eta_1 \in \mathcal{I}$ and $\zeta \in [0,1]$.

If $\zeta = 1$, then we clearly see that

$$\mathcal{P}(\eta_1) \leq \mathcal{P}(\eta_1) + (1-s)\mathcal{P}(x),$$

which leads to the conclusion that $\mathcal{P}(x) \geq 0$.

Proposition 1.3 *Every nonnegative convex function is also an s-type convex function.*

Proof Proposition 1.3 follows easily from the facts that

$$s(1-\zeta) \leq (1-\zeta), \qquad \zeta \geq s\zeta,$$

for all $\zeta \in [0,1]$ and $s \in [0,1]$. □

Next, we introduce the definition of n-polynomial s-type convex function.

Definition 1.4 Let $s \in [0,1]$ and $n \in \mathbb{N}$. Then the real-valued function $\mathcal{P} : \mathcal{I} \to \mathbb{R}$ is said to be a n-polynomial s-type convex function if the inequality

$$\mathcal{P}\big(\zeta x + (1-\zeta)y\big) \leq \frac{1}{n}\sum_{i=1}^{n}\big[1 - \big(s(1-\zeta)\big)^i\big]\mathcal{P}(x) + \frac{1}{n}\sum_{i=1}^{n}\big[1 - (s\zeta)^i\big]\mathcal{P}(y) \tag{1.4}$$

holds for $x, y \in \mathcal{I}$ and $\zeta \in [0,1]$.

Remark 1.5 From Definition 1.4 we clearly see that the following statements are true:

(1) If we choose $s = 0$, then we get the P-functions in [68].

(2) If we choose $s = 1$, then we get Definition 2 in [69].

(3) If we choose $n = s = 1$, then we get Definition 1.1.

(4) If $\mathcal{P}$ is a n-polynomial s-type convex function, then the range of the function $\mathcal{P}$ is $[0,\infty)$.

Remark 1.6 Every nonnegative n-polynomial convex function is also a n-polynomial s-type convex function due to

$$\frac{1}{n}\sum_{i=1}^{n}\big[1 - \big((1-\zeta)\big)^i\big] \leq \frac{1}{n}\sum_{i=1}^{n}\big[1 - s(1-\zeta)^i\big]$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left(1-\zeta^{i}\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left[1-(s\zeta)^{i}\right]$$

for all $\zeta \in [0,1]$, $n \in \mathbb{N}$ and $s \in [0,1]$.

We now demonstrate some essential ideas associated with the fractional integral which is mainly due to Mubeen et al. [70].

Let $\alpha, \mathcal{K} > 0$, $\eta_1 < \eta_2$ and $\mathcal{P} \in L_1([\eta_1, \eta_2])$. Then the $\mathcal{K}$-fractional integrals of order α are defined by

$$\mathcal{J}_{\eta_1}^{\alpha,\mathcal{K}}\mathcal{P}(r) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\alpha)}\int_{\eta_1}^{r}(r-\chi)^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}(\chi)\,d\chi \quad (r > \chi) \tag{1.5}$$

and

$$\mathcal{J}_{\eta_2}^{\alpha,\mathcal{K}}\mathcal{P}(r) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\alpha)}\int_{\eta_1}^{r}(\chi-r)^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}(\chi)\,d\chi \quad (r < \chi), \tag{1.6}$$

where $\Gamma_{\mathcal{K}}(\alpha)$ is the $\mathcal{K}$-Gamma function [71] defined by

$$\Gamma_{\mathcal{K}}(\alpha) = \int_{0}^{\infty}\zeta^{\alpha-1}e^{-\frac{\zeta^{\mathcal{K}}}{\mathcal{K}}}\,d\zeta.$$

Note that

$$\Gamma_{\mathcal{K}}(\alpha+\mathcal{K}) = \alpha\Gamma_{\mathcal{K}}(\alpha)$$

and the $\mathcal{K}$-fractional integrals reduce to the RL-fractional integrals if $\mathcal{K} = 1$.

Next, we recall the definitions of the Beta function $\mathbb{B}$ and the Gaussian hypergeometric function ${}_2\mathcal{F}_1$:

$$\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_{0}^{1}\zeta^{x-1}(1-\zeta)^{y-1}\,d\zeta$$

and

$${}_2\mathcal{F}_1(a,b;c;z) = \frac{1}{\mathbb{B}(b,c-b)}\int_{0}^{1}\zeta^{b-1}(1-\zeta)^{c-b-1}(1-z\zeta)^{-a}\,d\zeta,$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler gamma function [72, 73].

The principal purpose of this article is to derive several novel integral inequalities including the Hermite–Hadamard and Ostrowski type inequalities by using n-polynomial s-type convexity and $\mathcal{K}$-fractional integral operator. By use of the fractional operators, we obtained new estimates for the functions whose first and second derivatives in absolute value at certain powers are n-polynomial s-type convex functions. Interestingly, the special cases of the presented results are RL-fractional integral inequalities and quadrature rules. Our work's consequences are useful in the generation of fractals using iterative procedures, which is an interesting field of research and has utilities in the improvement of machine learning algorithms.

2 Hermite–Hadamard type inequalities for *n*-polynomial *s*-type convex function

The aim of this section is to find some inequalities of Hermite–Hadamard type for n-polynomial s-type convex functions. In what follows, we denote by $L_1([\eta_1,\eta_2])$ the space of (Lebesgue) integrable functions on the interval $[\eta_1,\eta_2]$.

Theorem 2.1 *Let $s\in[0,1]$, $\alpha\in(0,1]$, $\mathcal{K}>0$, $n\in\mathbb{N}$, $\eta_2>\eta_1$ and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a n-polynomial s-type convex function such that $\mathcal{P}\in L_1([\eta_1,\eta_2])$. Then one has*

$$\begin{aligned}&\left(\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\right)\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\\&\quad\le\frac{\Gamma_{\mathcal{K}}(\mathcal{K}+\alpha)}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\left[\mathcal{J}_{\eta_1^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)+\mathcal{J}_{\eta_2^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)\right]\\&\quad\le\frac{[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]}{n}\left[\sum_{i=1}^{n}\left[\frac{\alpha(2-s^i)+2i\mathcal{K}}{\alpha+i\mathcal{K}}-\frac{\alpha s^i}{\mathcal{K}}\mathbb{B}\left(\frac{\alpha}{\mathcal{K}},i+1\right)\right]\right].\end{aligned}\tag{2.1}$$

Proof Let $z_1,z_2\in\mathcal{I}$. Then it follows from the n-polynomial s-type convexity of $\mathcal{P}$ on $\mathcal{I}$ that

$$\mathcal{P}\left(\frac{z_1+z_2}{2}\right)\le\frac{1}{n}\sum_{i=1}^{n}\left[1-\left(\frac{s}{2}\right)^i\right]\left[\mathcal{P}(z_1)+\mathcal{P}(z_2)\right].\tag{2.2}$$

Let $z_1=\zeta\eta_2+(1-\zeta)\eta_1$ and $z_2=\zeta\eta_1+(1-\zeta)\eta_2$. Then (2.2) leads to

$$\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\le\frac{1}{n}\sum_{i=1}^{n}\left[1-\left(\frac{s}{2}\right)^i\right]\left[\mathcal{P}\big(\zeta\eta_2+(1-\zeta)\eta_1\big)+\mathcal{P}\big(\zeta\eta_1+(1-\zeta)\eta_2\big)\right].\tag{2.3}$$

Multiplying on both sides of (2.3) by $\zeta^{\frac{\alpha}{\mathcal{K}}-1}$ and integrating the obtained inequality with respect to ζ from 0 to 1, we get

$$\begin{aligned}&\frac{\mathcal{K}}{\alpha}\left(\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\right)\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\\&\quad\le\left[\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta\eta_2+(1-\zeta)\eta_1\big)\,d\zeta+\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta\eta_1+(1-\zeta)\eta_2\big)\,d\zeta\right]\\&\quad\le\frac{1}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{u-\eta_1}{\eta_2-\eta_1}\right)^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}(u)\,du+\int_{\eta_1}^{\eta_2}\left(\frac{\eta_2-u}{\eta_2-\eta_1}\right)^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}(u)\,du\right]\\&\quad\le\frac{\mathcal{K}\Gamma_{\mathcal{K}}(\alpha)}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\left[\mathcal{J}_{\eta_1^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)+\mathcal{J}_{\eta_2^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)\right],\end{aligned}$$

that is,

$$\left(\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\right)\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\le\frac{\Gamma_{\mathcal{K}}(\mathcal{K}+\alpha)}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\left[\mathcal{J}_{\eta_1^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)+\mathcal{J}_{\eta_2^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)\right],$$

which gives the proof of the first inequality of (2.1).

Next, we prove the second inequality of (2.1). Let $\zeta \in [0,1]$. Then from the fact that $\mathcal{P}$ is a n-polynomial s-type convex function, we get

$$\mathcal{P}\big(\zeta\eta_2+(1-\zeta)\eta_1\big)\le\frac{1}{n}\sum_{i=1}^{n}\big[1-(s\zeta)^i\big]\mathcal{P}(\eta_1)+\frac{1}{n}\sum_{i=1}^{n}\big[1-\big(s(1-\zeta)\big)^i\big]\mathcal{P}(\eta_2)$$

and

$$\mathcal{P}\big(\zeta\eta_1+(1-\zeta)\eta_2\big)\le\frac{1}{n}\sum_{i=1}^{n}\big[1-s\zeta^i\big]\mathcal{P}(\eta_2)+\frac{1}{n}\sum_{i=1}^{n}\big[1-\big(s(1-\zeta)\big)^i\big]\mathcal{P}(\eta_1).$$

Adding the above inequalities gives

$$\begin{aligned}&\mathcal{P}\big(\zeta\eta_2+(1-\zeta)\eta_1\big)+\mathcal{P}\big(\zeta\eta_1+(1-\zeta)\eta_2\big)\\&\quad\le\big[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)\big]\left[\frac{1}{n}\sum_{i=1}^{n}[1-s\zeta]^i+\frac{1}{n}\sum_{i=1}^{n}\big[1-\big(s(1-\zeta)\big)^i\big]\right].\end{aligned}\tag{2.4}$$

Multiplying on both sides of (2.4) by $\zeta^{\frac{\alpha}{\mathcal{K}}-1}$, integrating the obtained inequality with respect to ζ from 0 to 1 and then making the change of the variable, we have

$$\begin{aligned}&\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta\eta_2+(1-\zeta)\eta_1\big)\,d\zeta+\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta\eta_1+(1-\zeta)\eta_2\big)\,d\zeta\\&\quad\le\big[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)\big]\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\left[\frac{1}{n}\sum_{i=1}^{n}\big[1-(s\zeta)^i\big]+\frac{1}{n}\sum_{i=1}^{n}\big[1-\big(s(1-\zeta)\big)^i\big]\right]d\zeta,\end{aligned}$$

which leads to the conclusion that

$$\begin{aligned}&\frac{\Gamma_{\mathcal{K}}(\mathcal{K}+\alpha)}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\big[\mathcal{J}^{\alpha,\mathcal{K}}_{\eta_1^+}\mathcal{P}(\eta_2)+\mathcal{J}^{\alpha,\mathcal{K}}_{\eta_2^-}\mathcal{P}(\eta_1)\big]\\&\quad\le\frac{[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]}{n}\left[\sum_{i=1}^{n}\left[\frac{\alpha(2-s^i)+2i\mathcal{K}}{\alpha+i\mathcal{K}}-\frac{\alpha s^i}{\mathcal{K}}\mathbb{B}\left(\frac{\alpha}{\mathcal{K}},i+1\right)\right]\right].\end{aligned}$$

The proof is completed. □

Let $s=1$ and $s=1=\mathcal{K}$. Then Theorem 2.1 leads to Corollaries 2.2 and 2.3 immediately.

Corollary 2.2 *Under the assumption of Theorem 2.1, we have a new result for $\mathcal{K}$-fractional integral operator*:

$$\begin{aligned}&\left(\frac{n2^n}{2^n(n-1)+1}\right)\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\\&\quad\le\frac{\Gamma_{\mathcal{K}}(\mathcal{K}+\alpha)}{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\big[\mathcal{J}^{\alpha,\mathcal{K}}_{\eta_1^+}\mathcal{P}(\eta_2)+\mathcal{J}^{\alpha,\mathcal{K}}_{\eta_2^-}\mathcal{P}(\eta_1)\big]\\&\quad\le\frac{[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]}{n}\left[\sum_{i=1}^{n}\left[\frac{(\alpha+2i\mathcal{K})}{(\alpha+i\mathcal{K})}-\frac{\alpha}{\mathcal{K}}\mathbb{B}\left(\frac{\alpha}{\mathcal{K}},i+1\right)\right]\right].\end{aligned}$$

Corollary 2.3 *Under the assumption of Theorem 2.1, we have the following new result for RL-fractional integral operator*:

$$\begin{aligned}
&\left(\frac{n2^n}{2^n(n-1)+1}\right)\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\\
&\quad\leq\frac{\Gamma(\alpha+1)}{(\eta_2-\eta_1)^\alpha}\left[\mathcal{J}_{\eta_1^+}^{\alpha}\mathcal{P}(\eta_2)+\mathcal{J}_{\eta_2^-}^{\alpha}\mathcal{P}(\eta_1)\right]\\
&\quad\leq\frac{[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]}{n}\sum_{i=1}^{n}\left[\frac{\alpha+2i}{(\alpha+i)}-\alpha\mathbb{B}(\alpha,i+1)\right].
\end{aligned}$$

Remark 2.4 If $s=\mathcal{K}=\alpha=1$, then Theorem 2.1 becomes Theorem 4 of [69].

3 Ostrowski type inequalities for first-order differentiable functions

The aim of this section is to find new estimates that refine the Ostrowski type inequality for the function whose first derivative in absolute value at a certain power is a n-polynomial s-convex function. It is remarkable that Farid and Usman [74] adopted some ideas to derive the Ostrowski type inequalities involving $\mathcal{K}$-fractional integrals.

Lemma 3.1 (See [74]) *Let $\alpha\in(0,1]$, $\mathcal{K}>0$, $\eta_2>\eta_1$ and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'\in L_1[\eta_1,\eta_2]$. Then*

$$\begin{aligned}
&\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\\
&\quad=\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_1\big)\,d\zeta\\
&\qquad-\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_2\big)\,d\zeta.
\end{aligned}\tag{3.1}$$

Using Lemma 3.1, we can prove Theorem 3.2.

Theorem 3.2 *Let $\alpha,\mathcal{K}>0$, $s\in[0,1]$, $n\in\mathbb{N}$, $\eta_2>\eta_1$ and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'\in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}'(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}$ with $|\mathcal{P}'(z)|\leq\mathcal{M}$ for all $z\in[\eta_1,\eta_2]$. Then we have*

$$\begin{aligned}
&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\
&\quad\leq\mathcal{M}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right]\\
&\qquad\times\frac{1}{n}\sum_{i=1}^{n}\left[\left[\frac{\alpha(1-si)+\mathcal{K}(i+1-si)}{(\alpha+\mathcal{K})(\alpha+(i+1)\mathcal{K})}\right]+\left[\frac{\mathcal{K}}{\alpha+\mathcal{K}}-s^i\mathbb{B}\left(\frac{\alpha}{\mathcal{K}}+1,i+1\right)\right]\right].
\end{aligned}\tag{3.2}$$

Proof From Lemma 3.1 and the fact that $|\mathcal{P}'|$ is a n-polynomial s-type convex function on $\mathcal{I}$ we clearly see that

$$\begin{aligned}
&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right| \\
&\leq \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_1\big)\right|d\zeta \\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_2\big)\right|d\zeta \\
&\leq \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[\frac{1}{n}\sum_{i=1}^{n}\left[1-(s\zeta)^i\right]\left|\mathcal{P}'(\eta_1)\right|+\frac{1}{n}\sum_{i=1}^{n}\left[1-\big(s(1-\zeta)\big)^i\right]\left|\mathcal{P}'(z)\right|\right]d\zeta \\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[\frac{1}{n}\sum_{i=1}^{n}\left[1-(s\zeta)^i\right]\left|\mathcal{P}'(\eta_2)\right|\right. \\
&\quad\left.+\frac{1}{n}\sum_{i=1}^{n}\left[1-\big(s(1-\zeta)\big)^i\right]\left|\mathcal{P}'(z)\right|\right]d\zeta \\
&\leq \left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right] \\
&\quad\times\frac{\mathcal{M}}{n}\sum_{i=1}^{n}\left[\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-(s\zeta)^i\right]d\zeta+\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-\big(s(1-\zeta)\big)^i\right]d\zeta\right] \\
&\leq \left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right] \\
&\quad\times\frac{\mathcal{M}}{n}\sum_{i=1}^{n}\left[\left[\frac{\alpha(1-si)+\mathcal{K}(i+1-si)}{(\alpha+\mathcal{K})((\alpha+(i+1)\mathcal{K}))}\right]+\left[\frac{\mathcal{K}}{\alpha+\mathcal{K}}-s^i\mathbb{B}\left(\frac{\alpha}{\mathcal{K}+1}.i+1\right)\right]\right],
\end{aligned}$$

where we have used the facts that

$$\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-(s\zeta)^i\right]d\zeta=\frac{\alpha(1-si)+\mathcal{K}(i+1-si)}{(\alpha+\mathcal{K})((\alpha+(i+1)\mathcal{K}))}$$

and

$$\int_0^1 \zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-\big(s(1-\zeta)\big)^i\right]d\zeta=\frac{\mathcal{K}}{\alpha+\mathcal{K}}-s^i\mathbb{B}\left(\frac{\alpha}{\mathcal{K}}+1,i+1\right).$$

This completes the proof. □

Let $\mathcal{K}=1$ and $\mathcal{K}=\alpha=1$. Then Theorem 3.2 leads to Corollaries 3.3 and 3.4.

Corollary 3.3 *Let $\alpha>0$, $s\in[0,1]$, $n\in\mathbb{N}$, $\eta_2>\eta_1$ and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'\in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}'(z)|$ is a n-polynomial s-type convex* *function on $\mathcal{I}$ with $|\mathcal{P}'(z)| \le \mathcal{M}$ for all $z \in [\eta_1, \eta_2]$. Then one has*

$$\begin{aligned}&\left|\frac{(z-\eta_1)^{\alpha}+(\eta_2-z)^{\alpha}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma(\alpha+1)}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha}\mathcal{P}(\eta_2)\right]\right|\\ &\quad\le \mathcal{M}\left[\frac{(z-\eta_1)^{\alpha+1}+(\eta_2-z)^{\alpha+1}}{\eta_2-\eta_1}\right]\\ &\qquad\times\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\frac{\alpha(1-si)+(i+1-si)}{(\alpha+1)(\alpha+(i+1))}\right]+\left[\frac{1}{\alpha+1}-s^i\mathbb{B}(\alpha+1,i+1)\right]\right\}.\end{aligned}$$

Corollary 3.4 *Let $s \in [0,1]$, $n \in \mathbb{N}$, $\eta_2 > \eta_1$ and $\mathcal{P} : \mathcal{I} = [\eta_1, \eta_2] \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}' \in L_1[\eta_1, \eta_2]$ and $|\mathcal{P}'(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}$ with $|\mathcal{P}'(z)| \le \mathcal{M}$ for all $z \in [\eta_1, \eta_2]$. Then*

$$\begin{aligned}&\left|\mathcal{P}(z)-\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)\,dz\right|\\ &\quad\le\frac{\mathcal{M}}{n}\left(\frac{(z-\eta_1)^2+(\eta_2-z)^2}{\eta_2-\eta_1}\right)\sum_{i=1}^{n}\left(\frac{4+9i+3i^2-4si^2-6si}{2(i+1)(i+2)}\right).\end{aligned}$$

Remark 3.5 Let $\mathcal{K} = \alpha = 1$ and $n = s = 1$. Then Corollary 3.4 leads to inequality (1.2).

Theorem 3.6 *Let $\alpha, \mathcal{K} > 0$, $s \in [0,1]$, $n \in \mathbb{N}$, $q > 1$, $\eta_2 > \eta_1$, $\mathcal{I} = [\eta_1, \eta_2]$, and $\mathcal{P} : \mathcal{I} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}' \in L_1[\eta_1, \eta_2]$, $|\mathcal{P}'(z)|^q$ is a n-polynomial s-type convex function on $\mathcal{I}$ and $|\mathcal{P}'(z)| \le \mathcal{M}$ for all $z \in [\eta_1, \eta_2]$. Then we have*

$$\begin{aligned}&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\ &\quad\le\left(\frac{\mathcal{K}}{\alpha+\mathcal{K}}\right)^{1-\frac{1}{q}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right]\\ &\qquad\times\left[\frac{\mathcal{M}^q}{n}\sum_{i=1}^{n}\left(\left[\frac{\alpha(1-si)+\mathcal{K}(i+1-si)}{(\alpha+\mathcal{K})((\alpha+(i+1)\mathcal{K}))}\right]\right.\right.\\ &\qquad\left.\left.+\left[\frac{\mathcal{K}}{\alpha+\mathcal{K}}-s^i\mathbb{B}\left(\frac{\alpha}{\mathcal{K}}+1,i+1\right)\right]\right)\right]^{\frac{1}{q}}.\end{aligned}\tag{3.3}$$

Proof It follows from Lemma 3.1 and $|\mathcal{P}'|^q$ is a n-polynomial s-type convex function together with the power mean inequality that

$$\begin{aligned}&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\ &\quad\le\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\left(\zeta z+(1-\zeta)\eta_1\right)\right|d\zeta\\ &\qquad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\left(\zeta z+(1-\zeta)\eta_2\right)\right|d\zeta\\ &\quad\le\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\,d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\left(\zeta z+(1-\zeta)\eta_1\right)\right|^q d\zeta\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\,d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_2\big)\right|^q d\zeta\right)^{\frac{1}{q}}\\
&\leq\left(\frac{\mathcal{K}}{\alpha+\mathcal{K}}\right)^{1-\frac{1}{q}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-(s\zeta)^i\right]d\zeta\left|\mathcal{P}'(\eta_1)\right|^q\right.\right.\\
&\quad\left.+\frac{1}{n}\sum_{i=1}^{n}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-\big(s(1-\zeta)\big)^i\right]d\zeta\left|\mathcal{P}'(z)\right|^q\right)^{\frac{1}{q}}\\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-(s\zeta)^i\right]d\zeta\left|\mathcal{P}'(\eta_2)\right|^q\right.\\
&\quad\left.\left.+\frac{1}{n}\sum_{i=1}^{n}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left[1-\big(s(1-\zeta)\big)^i\right]d\zeta\left|\mathcal{P}'(z)\right|^q\right)^{\frac{1}{q}}\right]\\
&\leq\left(\frac{\mathcal{K}}{\alpha+\mathcal{K}}\right)^{1-\frac{1}{q}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right]\\
&\quad\times\left[\frac{\mathcal{M}^q}{n}\sum_{i=1}^{n}\left(\left[\frac{\alpha(1-si)+\mathcal{K}(i+1-si)}{(\alpha+\mathcal{K})((\alpha+(i+1)\mathcal{K}))}\right]+\left[\frac{\mathcal{K}}{\alpha+\mathcal{K}}-s^i\mathbb{B}\left(\frac{\alpha}{\mathcal{K}}+1,i+1\right)\right]\right)\right]^{\frac{1}{q}}. \qquad\square
\end{aligned}
$$

Remark 3.7 If we choose $q=1$, then Theorem 3.6 reduces to Theorem 3.2.

Theorem 3.8 *Let $\alpha,\mathcal{K}>0$, $s\in[0,1]$, $n\in\mathbb{N}$, $r,q>1$ with $1/r+1/q=1$, $\eta_2>\eta_1$, $\mathcal{I}=[\eta_1,\eta_2]$, and $\mathcal{P}:\mathcal{I}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'\in L_1[\eta_1,\eta_2]$, $|\mathcal{P}'(z)|^q$ is a n-polynomial s-type convex function on $\mathcal{I}$ and $|\mathcal{P}'(z)|\leq\mathcal{M}$ for all $z\in[\eta_1,\eta_2]$. Then one has*

$$
\begin{aligned}
&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\
&\leq\left(\frac{\mathcal{K}}{r\alpha+\mathcal{K}}\right)^{\frac{1}{r}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right]\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2\mathcal{M}^q(i+1-s^i)}{i+1}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof Making use of Lemma 3.1 and $|\mathcal{P}'|^q$ is a n-polynomial s-type convex function on $\mathcal{I}$ together with the Hölder inequality we get

$$
\begin{aligned}
&\left|\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\mathcal{P}(z)-\frac{\Gamma_{\mathcal{K}}(\alpha+\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\
&\leq\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_1\big)\right|d\zeta\\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}}\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_2\big)\right|d\zeta\\
&\leq\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\int_0^1\zeta^{\frac{r\alpha}{\mathcal{K}}}\,d\zeta\right)^{\frac{1}{r}}\left(\int_0^1\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_1\big)\right|^q d\zeta\right)^{\frac{1}{q}}\\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\left(\int_0^1\zeta^{\frac{r\alpha}{\mathcal{K}}}\,d\zeta\right)^{\frac{1}{r}}\left(\int_0^1\left|\mathcal{P}'\big(\zeta z+(1-\zeta)\eta_2\big)\right|^q d\zeta\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(\frac{\mathcal{K}}{r\alpha+\mathcal{K}}\right)^{\frac{1}{r}}\Bigg[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\big[1-(s\zeta)^i\big]\,d\zeta\,\big|\mathcal{P}'(\eta_1)\big|^q \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\int_0^1\big[1-\big(s(1-\zeta)\big)^i\big]\,d\zeta\,\big|\mathcal{P}'(z)\big|^q\Bigg)^{\frac{1}{q}} \\
&\quad+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\big[1-(s\zeta)^i\big]\,d\zeta\,\big|\mathcal{P}'(\eta_2)\big|^q \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\int_0^1\big[1-\big(s(1-\zeta)\big)^i\big]\,d\zeta\,\big|\mathcal{P}'(z)\big|^q\Bigg)^{\frac{1}{q}}\Bigg] \\
&\le \left(\frac{\mathcal{K}}{r\alpha+\mathcal{K}}\right)^{\frac{1}{r}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\right]\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2\mathcal{M}^q(i+1-s^i)}{i+1}\right)^{\frac{1}{q}}. \qquad \square
\end{aligned}
$$

4 New Ostrowski type inequalities for twice differentiable functions

We first establish a fractional integral identity which is the extension of the result established by Park [75].

Lemma 4.1 *Let $\alpha,\mathcal{K}>0$, $\eta_2>\eta_1$, $\mathcal{I}=[\eta_1,\eta_2]$, and $\mathcal{P}:\mathcal{I}\to\mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}''\in L_1[\eta_1,\eta_2]$. Then the identity*

$$
\begin{aligned}
&(1-\lambda)\left[\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}-(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}'(z)+\left(1+\frac{\alpha}{\mathcal{K}}-\lambda\right)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}(z) \\
&\quad+\lambda\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_1)+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_2)}{\eta_2-\eta_1}\right] \\
&\quad-\frac{\Gamma_{\mathcal{K}}(\alpha+2\mathcal{K})}{\eta_2-\eta_1}\big[\mathcal{J}^{\alpha,\mathcal{K}}_{z^-}\mathcal{P}(\eta_1)+\mathcal{J}^{\alpha,\mathcal{K}}_{z^+}\mathcal{P}(\eta_2)\big] \\
&=\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\int_0^1\zeta\big(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\big)\mathcal{P}''\big(\zeta z+(1-\zeta)\eta_1\big)\,d\zeta \\
&\quad+\frac{(\eta_2-\zeta)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\int_0^1\zeta\big(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\big)\mathcal{P}''\big(\zeta z+(1-\zeta)\eta_2\big)\,d\zeta
\end{aligned}
\tag{4.1}
$$

holds for $z\in[\eta_1,\eta_2]$ and $\lambda\in[0,1]$.

Proof Let $\nu=\zeta z+(1-\zeta)\eta_1$ and $\nu=\zeta z+(1-\zeta)\eta_2$, respectively. Then (1.5) and (1.6) lead to

$$
\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta z+(1-\zeta)\eta_1\big)\,d\zeta=\frac{\mathcal{K}\Gamma_{\mathcal{K}}(\alpha)}{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}\mathcal{J}^{\alpha,\mathcal{K}}_{z^-}\mathcal{P}(\eta_1)
\tag{4.2}
$$

and

$$
\int_0^1\zeta^{\frac{\alpha}{\mathcal{K}}-1}\mathcal{P}\big(\zeta z+(1-\zeta)\eta_2\big)\,d\zeta=\frac{\mathcal{K}\Gamma_{\mathcal{K}}(\alpha)}{(z-\eta_2)^{\frac{\alpha}{\mathcal{K}}}}\mathcal{J}^{\alpha,\mathcal{K}}_{z^+}\mathcal{P}(\eta_2).
\tag{4.3}
$$

Integrating by parts, using (4.2) and (4.3), and changing the variables, for $z \neq \eta_1$ we can write

$$\begin{aligned}
&\int_0^1 \zeta\big(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\big)\mathcal{P}''\big(\zeta z + (1-\zeta)\eta_1\big)\,d\zeta \\
&\quad = (\lambda - 1)\frac{\mathcal{P}'(z)}{z-\eta_1} + \left(1 + \frac{\alpha}{\mathcal{K}} - \lambda\right)\frac{\mathcal{P}(z)}{(z-\eta_1)^2} \\
&\qquad + \lambda\frac{\mathcal{P}(\eta_1)}{(z-\eta_1)^2} - \frac{\varGamma_{\mathcal{K}}(\alpha + 2\mathcal{K})}{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}\mathcal{J}^{\alpha,\mathcal{K}}_{z^-}\mathcal{P}(\eta_1).
\end{aligned} \tag{4.4}$$

Similarly, for $z \neq \eta_2$ we get

$$\begin{aligned}
&\int_0^1 \zeta\big(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\big)\mathcal{P}''\big(\zeta z + (1-\zeta)\eta_2\big)\,d\zeta \\
&\quad = (1 - \lambda)\frac{\mathcal{P}'(z)}{\eta_2 - z} + \left(1 + \frac{\alpha}{\mathcal{K}} - \lambda\right)\frac{\mathcal{P}(z)}{(\eta_2 - z)^2} \\
&\qquad + \lambda\frac{\mathcal{P}(\eta_2)}{(\eta_2 - z)^2} - \frac{\varGamma_{\mathcal{K}}(\alpha + 2\mathcal{K})}{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}\mathcal{J}^{\alpha,\mathcal{K}}_{z^+}\mathcal{P}(\eta_2).
\end{aligned} \tag{4.5}$$

Multiplying both sides of (4.4) and (4.5) by $\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}$ and $\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}$, respectively, then adding the obtained identities, we obtain the desired identity (4.1). □

In order to simplicity the notation, in what follows we denote

$$\begin{aligned}
&\big|\varOmega_{\mathcal{P}}(\lambda, \alpha, \mathcal{K}; \eta_1, \eta_2, z)\big| \\
&\quad = (1 - \lambda)\left[\frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}} - (z - \eta_1)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2 - \eta_1}\right]\mathcal{P}'(z) \\
&\qquad + \left(1 + \frac{\alpha}{\mathcal{K}} - \lambda\right)\left[\frac{(z - \eta_1)^{\frac{\alpha}{\mathcal{K}}} + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2 - \eta_1}\right]\mathcal{P}(z) \\
&\qquad + \lambda\left[\frac{(z - \eta_1)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_1) + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_2)}{\eta_2 - \eta_1}\right] \\
&\qquad - \frac{\varGamma_{\mathcal{K}}(\alpha + 2\mathcal{K})}{\eta_2 - \eta_1}\big[\mathcal{J}^{\alpha,\mathcal{K}}_{z^-}\mathcal{P}(\eta_1) + \mathcal{J}^{\alpha,\mathcal{K}}_{z^+}\mathcal{P}(\eta_2)\big].
\end{aligned} \tag{4.6}$$

Theorem 4.2 *Let $\alpha, \mathcal{K} > 0$, $s \in [0,1]$, $n \in \mathbb{N}$, $\eta_2 > \eta_1$, $\mathcal{I} = [\eta_1, \eta_2]$, and $\mathcal{P} : \mathcal{I} \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'' \in L_1[\eta_1, \eta_2]$ and $|\mathcal{P}''(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}$. Then the inequality*

$$\begin{aligned}
&\big|\varOmega_{\mathcal{P}}(\lambda, \alpha, \mathcal{K}; \eta_1, \eta_2, z)\big| \\
&\quad \le \Bigg[\frac{1}{n}\sum_{i=1}^{n}\mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left[\frac{(z - \eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\big|\mathcal{P}''(\eta_1)\big| + \frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\big|\mathcal{P}''(\eta_2)\big|\right] \\
&\qquad + \frac{(z - \eta_1)^{\frac{\alpha}{\mathcal{K}}+2} + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2 - \eta_1)}\sum_{i=1}^{n}\mathcal{A}_2(\alpha, \mathcal{K}; i, s)\big|\mathcal{P}''(z)\big|\Bigg]
\end{aligned}$$

holds for all $z \in [\eta_1, \eta_2]$, *where*

$$\begin{aligned}\mathcal{A}_1(\alpha, \mathcal{K}; i, s) &= \int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[\left(1 - (s\zeta)^i\right)\right] d\zeta \\ &= \left[\frac{\lambda\alpha - 2\mathcal{K}(1-\lambda)}{2(\alpha + 2\mathcal{K})} - s^i \frac{\lambda\alpha - \mathcal{K}(i+2)(1-\lambda)}{(i+2)(\alpha + \mathcal{K}(i+2))}\right]\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\mathcal{A}_2(\alpha, \mathcal{K}; i, s) &= \int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[1 - \left(s(1-\zeta)\right)^i\right] d\zeta \\ &= \left[\frac{\lambda\alpha - 2\mathcal{K}(1-\lambda)}{2(\alpha + 2\mathcal{K})} - s^i\left[\frac{\lambda}{(i+2)(i+1)} - \frac{\Gamma(\frac{\alpha}{\mathcal{K}} + 2)\Gamma(i+1)}{\Gamma(\frac{\alpha}{\mathcal{K}} + i + 3)}\right]\right].\end{aligned} \tag{4.8}$$

Proof Making use of Lemma 4.1, the property of the modulus and the n-polynomial s-type convexity of $|\mathcal{P}''|$ on $\mathcal{I}$, we have

$$\begin{aligned}&\left|(1-\lambda)\left[\frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}} - (z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2 - \eta_1}\right]\mathcal{P}'(z) + \left(1 + \frac{\alpha}{\mathcal{K}} - \lambda\right)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}} + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2 - \eta_1}\right]\mathcal{P}(z)\right. \\ &\quad + \lambda\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_1) + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_2)}{\eta_2 - \eta_1}\right] \\ &\quad \left. - \frac{\Gamma_{\mathcal{K}}(\alpha + 2\mathcal{K})}{\eta_2 - \eta_1}\left[\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1) + \mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right| \\ &\leq \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left|\mathcal{P}''\left(\zeta z + (1-\zeta)\eta_1\right)\right| d\zeta \\ &\quad + \frac{(\eta_2 - \zeta)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2 - \eta_1}\int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left|\mathcal{P}''\left(\zeta z + (1-\zeta)\eta_2\right)\right| d\zeta \\ &\leq \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[\frac{1}{n}\sum_{i=1}^n \left(1 - (s\zeta)^i\right)\left|\mathcal{P}''(\eta_1)\right|\right. \\ &\quad \left. + \frac{1}{n}\sum_{i=1}^n \left[1 - \left(s(1-\zeta)\right)^i\right]\left|\mathcal{P}''(z)\right|\right] d\zeta \\ &\quad + \frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\int_0^1 \zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[\frac{1}{n}\sum_{i=1}^n \left(1 - (s\zeta)^i\right)\left|\mathcal{P}''(\eta_2)\right|\right. \\ &\quad \left. + \frac{1}{n}\sum_{i=1}^n \left[1 - \left(s(1-\zeta)\right)^i\right]\left|\mathcal{P}''(z)\right|\right] d\zeta \\ &= \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2 - \eta_1)}\sum_{i=1}^n \left[\mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(\eta_1)\right| + \mathcal{A}_2(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(z)\right|\right] \\ &\quad + \frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2 - \eta_1)}\sum_{i=1}^n \left[\mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(\eta_2)\right| + \mathcal{A}_2(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(z)\right|\right] \\ &= \left[\frac{1}{n}\sum_{i=1}^n \mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\left|\mathcal{P}''(\eta_1)\right| + \frac{(\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2 - \eta_1}\left|\mathcal{P}''(\eta_2)\right|\right]\right. \\ &\quad \left. + \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2} + (\eta_2 - z)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2 - \eta_1)}\sum_{i=1}^n \mathcal{A}_2(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(z)\right|\right].\end{aligned}$$

□

Theorem 4.3 *Let $\alpha, \mathcal{K} > 0$, $s \in [0,1]$, $q > 1$, $n \in \mathbb{N}$, $\eta_2 > \eta_1$, $\mathcal{I} = [\eta_1, \eta_2]$, and $\mathcal{P} : \mathcal{I} \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'' \in L_1[\eta_1, \eta_2]$ and $|\mathcal{P}''(z)|^q$ is a n-polynomial s-type convex function on $\mathcal{I}$. Then the inequality*

$$\begin{aligned}
&\left|\Omega_{\mathcal{P}}(\lambda, \alpha, \mathcal{K}; \eta_1, \eta_2, z)\right| \\
&\quad \le \mathcal{A}_3^{1-\frac{1}{q}}(\alpha, \mathcal{K}, \lambda)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2-\eta_1)}\sum_{i=1}^{n}\left(\mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(\eta_1)\right|^q + \mathcal{A}_2(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(z)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2-\eta_1)}\sum_{i=1}^{n}\left(\mathcal{A}_1(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(\eta_2)\right|^q + \mathcal{A}_2(\alpha, \mathcal{K}; i, s)\left|\mathcal{P}''(z)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

holds for all $z \in (\eta_1, \eta_2)$, where

$$\begin{aligned}
&\mathcal{A}_3(\alpha, \mathcal{K}, \lambda) \\
&\quad = \int_0^1 \left(\zeta\left(\lambda - \zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right)^q d\zeta \\
&\quad = \frac{\mathcal{K}\lambda^{\frac{(1+q)\mathcal{K}+\alpha q}{\alpha}}}{\alpha}\left[\Gamma(1+q)\Gamma\left(\frac{\mathcal{K}(1+q)+\alpha}{\alpha}\right){}_2\mathcal{F}_1\left(1, 1+q, 2+q+\frac{\mathcal{K}(q+1)}{\alpha}, 1\right)\right. \\
&\qquad \left. + \mathbb{B}\left(1+q, -\frac{\mathcal{K}(1+q)+\alpha q}{\alpha}\right) - \mathbb{B}\left(\lambda, 1+q, -\frac{(1+q)\mathcal{K}+\alpha q}{\alpha}\right)\right],
\end{aligned} \tag{4.9}$$

and $\mathcal{A}_1(\alpha, \mathcal{K}; i, s)$ and $\mathcal{A}_2(\alpha, \mathcal{K}; i, s)$ are given in (4.7) and (4.8), respectively.

Proof Using Lemma 4.1, the power mean inequality and the n-polynomial s-type convexity, we have

$$\begin{aligned}
&\left|(1-\lambda)\left[\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}} - (z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}'(z) + \left(1+\frac{\alpha}{\mathcal{K}}-\lambda\right)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}} + (\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}(z)\right. \\
&\qquad + \lambda\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_1) + (\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_2)}{\eta_2-\eta_1}\right] \\
&\qquad \left. - \frac{\Gamma_{\mathcal{K}}(\alpha+2\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1) + \mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right| \\
&\quad \le \frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\int_0^1 \left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|\left|\mathcal{P}''\left(\zeta z + (1-\zeta)\eta_1\right)\right| d\zeta \\
&\qquad + \frac{(\eta_2-\zeta)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1 \left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|\left|\mathcal{P}''\left(\zeta z + (1-\zeta)\eta_2\right)\right| d\zeta \\
&\quad \le \left(\int_0^1 \zeta^q\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)^q d\zeta\right)^{1-\frac{1}{q}}\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\right. \\
&\qquad \times \left(\int_0^1 \zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[\frac{1}{n}\sum_{i=1}^{n}\left(1-(s\zeta)^i\right)\left|\mathcal{P}''(\eta_1)\right|^q\right.\right. \\
&\qquad \left.\left. + \frac{1}{n}\sum_{i=1}^{n}\left[1-\left(s(1-\zeta)\right)^i\right]\left|\mathcal{P}''(z)\right|^q\right] d\zeta\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\left(\int_0^1 \zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\left[\frac{1}{n}\sum_{i=1}^{n}\left(1-(s\zeta)^i\right)\left|\mathcal{P}''(\eta_2)\right|^q\right.\right.\\
&\left.\left.+\frac{1}{n}\sum_{i=1}^{n}\left[1-\left(s(1-\zeta)\right)^i\right]\left|\mathcal{P}''(z)\right|^q\right]d\zeta\right)^{\frac{1}{q}}\Bigg]\\
&=\mathcal{A}_3^{1-\frac{1}{q}}(\alpha,\mathcal{K},\lambda)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2-\eta_1)}\sum_{i=1}^{n}\left(\mathcal{A}_1(\alpha,\mathcal{K};i,s)\left|\mathcal{P}''(\eta_1)\right|^q+\mathcal{A}_2(\alpha,\mathcal{K};i,s)\left|\mathcal{P}''(z)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{n(\eta_2-\eta_1)}\sum_{i=1}^{n}\left(\mathcal{A}_1(\alpha,\mathcal{K};i,s)\left|\mathcal{P}''(\eta_2)\right|^q+\mathcal{A}_2(\alpha,\mathcal{K};i,s)\left|\mathcal{P}''(z)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{4.10}
$$

□

Theorem 4.4 *Let $\alpha,\mathcal{K}>0$, $s\in[0,1]$, $q,r>1$ with $1/q+1/r=1$, $n\in\mathbb{N}$, $\eta_2>\eta_1$, $\mathcal{I}=[\eta_1,\eta_2]$, and $\mathcal{P}:\mathcal{I}\to\mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}''\in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}''(z)|^q$ is a n-polynomial s-type convex function on $\mathcal{I}$. Then the inequality*

$$
\begin{aligned}
&\left|\Omega_{\mathcal{P}}(\lambda,\alpha,\mathcal{K};\eta_1,\eta_2,z)\right|\\
&\le\mathcal{A}_3^{\frac{1}{r}}(\alpha,\mathcal{K},\lambda)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{(\eta_2-\eta_1)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i+1-s^i}{i+1}\right)\left[\left|\mathcal{P}''(z)\right|^q+\left|\mathcal{P}''(\eta_1)\right|^q\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{(\eta_2-\eta_1)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i+1-s^i}{i+1}\right)\left[\left|\mathcal{P}''(z)\right|^q+\left|\mathcal{P}''(\eta_2)\right|^q\right]\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

holds for all $z\in(\eta_1,\eta_2)$, where $\mathcal{A}_1(\alpha,\mathcal{K};i,s)$, $\mathcal{A}_2(\alpha,\mathcal{K};i,s)$ and $\mathcal{A}_3(\alpha,\mathcal{K};\lambda)$ are given in (4.7), (4.8) and (4.9), respectively.

Proof Using Lemma 4.1 and the Hölder inequality together with the n-polynomial s-type convexity, we have

$$
\begin{aligned}
&\left|(1-\lambda)\left[\frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}-(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}'(z)+\left(1+\frac{\alpha}{\mathcal{K}}-\lambda\right)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}}{\eta_2-\eta_1}\right]\mathcal{P}(z)\right.\\
&+\lambda\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_1)+(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}}\mathcal{P}(\eta_2)}{\eta_2-\eta_1}\right]\\
&\left.-\frac{\Gamma_{\mathcal{K}}(\alpha+2\mathcal{K})}{\eta_2-\eta_1}\left[\mathcal{J}_{z^+}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_1)+\mathcal{J}_{z^-}^{\alpha,\mathcal{K}}\mathcal{P}(\eta_2)\right]\right|\\
&\le\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\int_0^1\left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|\left|\mathcal{P}''\left(\zeta z+(1-\zeta)\eta_1\right)\right|d\zeta\\
&+\frac{(\eta_2-\zeta)^{\frac{\alpha}{\mathcal{K}}+1}}{\eta_2-\eta_1}\int_0^1\left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|\left|\mathcal{P}''\left(\zeta z+(1-\zeta)\eta_2\right)\right|d\zeta\\
&\le\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\left(\int_0^1\left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|^r d\zeta\right)^{\frac{1}{r}}\left(\int_0^1\left|\mathcal{P}''\left(\zeta z+(1-\zeta)\eta_1\right)\right|^q d\zeta\right)^{\frac{1}{q}}\\
&+\frac{(\eta_2-\zeta)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\left(\int_0^1\left|\zeta\left(\lambda-\zeta^{\frac{\alpha}{\mathcal{K}}}\right)\right|^r d\zeta\right)^{\frac{1}{r}}\left(\int_0^1\left|\mathcal{P}''\left(\zeta z+(1-\zeta)\eta_2\right)\right|d\zeta\right)^{\frac{1}{q}}\\
&\le\mathcal{A}_3^{\frac{1}{r}}(\alpha,\mathcal{K};\lambda)\left[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\left(\int_0^1\left[\frac{1}{n}\sum_{i=1}^{n}\left(1-(s\zeta)^i\right)\left|\mathcal{P}''(\eta_1)\right|^q\right.\right.\right.
\end{aligned}
$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\left[1-\left(s(1-\zeta)\right)^{i}\right]\left|\mathcal{P}''(z)\right|^{q}\Bigg]\,d\zeta\Bigg)^{\frac{1}{q}}$$

$$+ \frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{\eta_2-\eta_1}\Bigg(\int_0^1\Bigg[\frac{1}{n}\sum_{i=1}^{n}\left(1-(s\zeta)^{i}\right)\left|\mathcal{P}''(\eta_2)\right|^{q}$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\left[1-\left(s(1-\zeta)\right)^{i}\right]\left|\mathcal{P}''(z)\right|^{q}\Bigg]\,d\zeta\Bigg)^{\frac{1}{q}}\Bigg]$$

$$= \mathcal{A}_3^{\frac{1}{r}}(\alpha,\mathcal{K},\lambda)\Bigg[\frac{(z-\eta_1)^{\frac{\alpha}{\mathcal{K}}+2}}{(\eta_2-\eta_1)}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i+1-s^{i}}{i+1}\right)\left[\left|\mathcal{P}''(z)\right|^{q}+\left|\mathcal{P}''(\eta_1)\right|^{q}\right]\Bigg)^{\frac{1}{q}}$$

$$+ \frac{(\eta_2-z)^{\frac{\alpha}{\mathcal{K}}+2}}{(\eta_2-\eta_1)}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i+1-s^{i}}{i+1}\right)\left[\left|\mathcal{P}''(z)\right|^{q}+\left|\mathcal{P}''(\eta_2)\right|^{q}\right]\Bigg)^{\frac{1}{q}}\Bigg]. \tag{4.11}$$

□

5 Applications

In this section, we provide some applications to the estimations of $\mathcal{K}$-fractional integrals, and the midpoint, trapezoidal and Simpson type inequalities for twice differentiable n-polynomial s-type convex functions by use of our results.

Let $\lambda = 0$ and $z = \frac{\eta_1+\eta_2}{2}$, $\lambda = 1$ and $z = \frac{\eta_1+\eta_2}{2}$, $\lambda = \frac{1}{2}$ and $z = \frac{\eta_1+\eta_2}{2}$, $\lambda = \frac{1}{3}$ and $z = \frac{\eta_1+\eta_2}{2}$. Then Theorem 4.2 leads to Corollaries 5.1–5.4 immediately.

Corollary 5.1 *Let $\alpha,\mathcal{K} > 0$, $s \in [0,1]$, $n \in \mathbb{N}$, $\eta_2 > \eta_1$, and $\mathcal{P} : \mathcal{I} = [\eta_1,\eta_2] \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'' \in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}''(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}^\circ$. Then we have*

$$\left|\Omega_{\mathcal{P}}\left(0,\alpha,\mathcal{K};\eta_1,\eta_2,\frac{\eta_1+\eta_2}{2}\right)\right|$$

$$\leq \left(\frac{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{2^{\frac{\alpha}{\mathcal{K}}+1}n}\right)\Bigg[\sum_{i=1}^{n}\mathcal{A}_1(\alpha,\mathcal{K};i,s)\left[\left|\mathcal{P}''(\eta_1)\right|+\left|\mathcal{P}''(\eta_2)\right|\right]$$

$$+ \sum_{i=1}^{n}\mathcal{A}_2(\alpha,\mathcal{K};i,s)\left|\mathcal{P}''\left(\frac{\eta_1+\eta_2}{2}\right)\right|\Bigg],$$

where $\mathcal{A}_1(\alpha,\mathcal{K};i,s)$ and $\mathcal{A}_2(\alpha,\mathcal{K};i,s)$ are given in (4.7) and (4.8), respectively.

Corollary 5.2 *Let $\alpha,\mathcal{K} > 0$, $s \in [0,1]$, $n \in \mathbb{N}$, $\eta_2 > \eta_1$, and $\mathcal{P} : \mathcal{I} = [\eta_1,\eta_2] \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}'' \in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}''(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}^\circ$. Then one has*

$$\left|\Omega_{\mathcal{P}}\left(1,\alpha,\mathcal{K};\eta_1,\eta_2,\frac{\eta_1+\eta_2}{2}\right)\right|$$

$$\leq \left(\frac{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{2^{\frac{\alpha}{\mathcal{K}}+1}n}\right)\Bigg[\sum_{i=1}^{n}\Bigg[\frac{\alpha}{2(\alpha+2\mathcal{K})}-s^{i}\left(\frac{1}{(i+1)(i+2)}-\frac{\Gamma(\frac{\alpha}{\mathcal{K}}+2)\Gamma(i+1)}{\Gamma(\frac{\alpha}{\mathcal{K}}+i+3)}\right)\Bigg]$$

$$\times \left|\mathcal{P}''\left(\frac{\eta_1+\eta_2}{2}\right)\right| + \sum_{i=1}^{n}\left(\frac{\alpha}{2(\alpha+2\mathcal{K})} - \frac{s^i\alpha}{(i+2)(\alpha+\mathcal{K}(i+2))}\right)\left(\left|\mathcal{P}''(\eta_1)\right| + \left|\mathcal{P}''(\eta_2)\right|\right)\Bigg].$$

Corollary 5.3 *Let $\alpha,\mathcal{K}>0$, $s\in[0,1]$, $n\in\mathbb{N}$, $\eta_2>\eta_1$, and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}''\in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}''(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}^\circ$. Then*

$$\begin{aligned}
&\left|\Omega_{\mathcal{P}}\left(\frac{1}{2},\alpha,\mathcal{K};\eta_1,\eta_2,\frac{\eta_1+\eta_2}{2}\right)\right| \\
&\quad\le \left(\frac{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{2^{\frac{\alpha}{\mathcal{K}}+1}n}\right)\left[\sum_{i=1}^{n}\left[\frac{\alpha-2\mathcal{K}}{4(\alpha+2\mathcal{K})} - \frac{s^i[\alpha-\mathcal{K}(i+2)]}{2(i+2)(\alpha+\mathcal{K}(i+2))}\right]\left[\left|\mathcal{P}''(\eta_1)\right|+\left|\mathcal{P}''(\eta_2)\right|\right]\right. \\
&\qquad\left. + \sum_{i=1}^{n}\left[\frac{\alpha-2\mathcal{K}}{4(\alpha+2\mathcal{K})} - s^i\left(\frac{1}{2(i+1)(i+2)} - \frac{\Gamma(\frac{\alpha}{\mathcal{K}}+2)\Gamma(i+1)}{\Gamma(\frac{\alpha}{\mathcal{K}}+i+3)}\right)\right]\left|\mathcal{P}''\left(\frac{\eta_1+\eta_2}{2}\right)\right|\right].
\end{aligned}$$

Corollary 5.4 *Let $\alpha,\mathcal{K}>0$, $s\in[0,1]$, $n\in\mathbb{N}$, $\eta_2>\eta_1$, and $\mathcal{P}:\mathcal{I}=[\eta_1,\eta_2]\to\mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $\mathcal{P}''\in L_1[\eta_1,\eta_2]$ and $|\mathcal{P}''(z)|$ is a n-polynomial s-type convex function on $\mathcal{I}^\circ$. Then we get*

$$\begin{aligned}
&\left|\Omega_{\mathcal{P}}\left(\frac{1}{3},\alpha,\mathcal{K};\eta_1,\eta_2,\frac{\eta_1+\eta_2}{2}\right)\right| \\
&\quad\le \left(\frac{(\eta_2-\eta_1)^{\frac{\alpha}{\mathcal{K}}+1}}{2^{\frac{\alpha}{\mathcal{K}}+1}n}\right)\left[\sum_{i=1}^{n}\left[\frac{\alpha-4\mathcal{K}}{6(\alpha+2\mathcal{K})} - \frac{s^i[\alpha-2\mathcal{K}(i+2)]}{3(i+2)(\alpha+\mathcal{K}(i+2))}\right]\left(\left|\mathcal{P}''(\eta_1)\right|+\left|\mathcal{P}''(\eta_2)\right|\right)\right. \\
&\qquad\left. + \sum_{i=1}^{n}\left[\frac{\alpha-4\mathcal{K}}{6(\alpha+2\mathcal{K})} - s^i\left(\frac{1}{3(i+1)(i+2)} - \frac{\Gamma(\frac{\alpha}{\mathcal{K}}+2)\Gamma(i+1)}{\Gamma(\frac{\alpha}{\mathcal{K}}+i+3)}\right)\right]\left|\mathcal{P}''\left(\frac{\eta_1+\eta_2}{2}\right)\right|\right].
\end{aligned}$$

6 Conclusion

In this paper, we have introduced a new class of n-polynomial s-type convex functions, derived several new versions of the Hermite–Hadamard and Ostrowski type inequalities using the class of n-polynomial s-type convex functions, provided two integral identities for the first and second order differentiable functions, and obtained some refinements of the Ostrowski type inequality. We have also discussed some special cases for the obtained results which showed that the results obtained are quite unifying one. The outcomes acquired by the future plan are all the more invigorating as contrasted with results accessible in the literature. Finally, our work's consequences have a potential connection with fractal theory and machine learning [65–67].

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