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Monotonicity properties for a ratio of finite many gamma functions

Feng Qi[1,2,3] and Dongkyu Lim[4*]

Dedicated to people facing and fighting COVID-19

*Correspondence: dklim@anu.ac.kr
[4]Department of Mathematics Education, Andong National University, Andong, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In the paper, the authors consider a ratio of finite many gamma functions and find its monotonicity properties such as complete monotonicity, the Bernstein function property, and logarithmically complete monotonicity.

1 Preliminaries

Let $f(x)$ be an infinite differentiable function on an infinite interval $(0,\infty)$.

(1) If $(-1)^k f^{(k)}(x) \ge 0$ for all $k \ge 0$ and $x \in (0,\infty)$, then we call $f(x)$ a completely monotonic function on $(0,\infty)$. See the review papers [22, 31, 36] and [35, Chapter IV].
(2) If $(-1)^k[\ln f(x)]^{(k)} \ge 0$ for all $k \ge 1$ and $x \in (0,\infty)$, or say, if the logarithmic derivative $[\ln f(x)]' = \frac{f'(x)}{f(x)}$ is a completely monotonic function on $(0,\infty)$, then we call $f(x)$ a logarithmically completely monotonic function on $(0,\infty)$. See the papers [3, 4, 7, 24] and [33, Chap. 5].
(3) If $f'(x)$ is a completely monotonic function on $(0,\infty)$, then we call $f(x)$ a Bernstein function on $(0,\infty)$. See the paper [28] and the monograph [33].

The classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

See [1, Chap. 6], [15, Chap. 5], the paper [18], and [34, Chap. 3]. In the literature, the logarithmic derivative

$$\psi(z) = \bigl[\ln \Gamma(x)\bigr]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

and its first derivative $\psi'(z)$ are respectively called the digamma and trigamma functions. See the papers [5, 6, 10, 25, 26] and closely related references therein.

2 Motivations

This paper is motivated by a series of papers [2, 11, 12, 16, 19, 21, 27, 29, 32]. For detailed review and survey, please read the papers [19, 27, 29, 32] and closely related references therein.

In the paper [2], motivated by [11, 12], the function

$$x \in (0,\infty) \mapsto \frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((m-k)x+1)} p^{kx}(1-p)^{(m-k)x} \tag{2.1}$$

was considered, where $p \in (0,1)$ and k, m are nonnegative integers with $0 \le k \le m$.

In [16, Theorem 2.1] and [32], the function

$$x \in (0,\infty) \mapsto \frac{\Gamma(1+x\sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \tag{2.2}$$

was independently studied, where $m \ge 2$, $\lambda_i > 0$ for $1 \le i \le m$, $p_i \in (0,1)$ for $1 \le i \le m$, and $\sum_{i=1}^m p_i = 1$. The q-analogue

$$x \in (0,\infty) \mapsto \frac{\Gamma_q(1+x\sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma_q(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \tag{2.3}$$

of the function in (2.2) was investigated in [19], where $q \in (0,1)$, $m \ge 2$, $\lambda_i > 0$ for $1 \le i \le m$, $p_i \in (0,1)$ for $1 \le i \le m$ with $\sum_{i=1}^m p_i = 1$, and Γ_q is the q-analogue of the gamma function Γ.

The functions

$$x \in (0,\infty) \mapsto \frac{\prod_{i=1}^m \Gamma(\nu_i x+1)\prod_{j=1}^n \Gamma(\tau_j x+1)}{\prod_{i=1}^m \prod_{j=1}^n \Gamma(\lambda_{ij} x+1)} \tag{2.4}$$

and

$$x \in (0,\infty) \mapsto \frac{\prod_{i=1}^m \Gamma(1+\nu_i x)\prod_{j=1}^n \Gamma(1+\tau_j x)}{[\prod_{i=1}^m \prod_{j=1}^n \Gamma(1+\lambda_{ij} x)]^\rho} \tag{2.5}$$

were respectively considered in [17, Theorem 2.1] and [29, Theorem 4.1], where $\rho \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$.

In [27], the function

$$x \in (0,\infty) \mapsto \frac{\prod_{i=1}^m [\Gamma(1+\nu_i x)]^{\nu_i^\theta} \prod_{j=1}^n [\Gamma(1+\tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1+\lambda_{ij} x)]^{\rho\lambda_{ij}^\theta}} \tag{2.6}$$

was discussed, where $\rho, \theta \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$.

In this paper, stimulated by the above six functions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6), we consider a new function

$$\mathcal{Q}(x)=\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)=\frac{[\Gamma(1+x\sum_{i=1}^m a_i)]^{(\sum_{i=1}^m a_i)^\theta}}{\prod_{i=1}^m[\Gamma(1+xa_i)]^{\rho a_i^\theta}}\left(\prod_{i=1}^m p_i^{a_i}\right)^{\varrho x} \tag{2.7}$$

on $(0,\infty)$, where $m\ge 2$, $\rho,\varrho,\theta\in\mathbb{R}$, $a=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$, and $p=(p_1,p_2,\dots,p_m)$ with $p_i\in(0,1)$ for $1\le i\le m$ and $\sum_{i=1}^m p_i=1$.

3 Monotonicity properties

In this section, we now start out to find and prove some monotonicity properties for the function $\mathcal{Q}(x)=\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ defined in (2.7). Our main results in this section can be stated in the following theorem.

Theorem 3.1 *Let $m\ge 2$, $a=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$, and $p=(p_1,p_2,\dots,p_m)$ with $\sum_{i=1}^m p_i=1$ and $p_i\in(0,1)$ for $1\le i\le m$. Then*

(1) *when $\rho\le 1$ and $\theta\ge 0$, the second logarithmic derivative*

$$\left[\ln\mathcal{Q}(x)\right]''=\left(\sum_{i=1}^m a_i\right)^{\theta+2}\psi'\left(1+x\sum_{i=1}^m a_i\right)-\rho\sum_{i=1}^m a_i^{\theta+2}\psi'(1+a_ix)$$

is completely monotonic on $(0,\infty)$;

(2) *when $\rho=1$, $\varrho=0$, and $\theta=0$, the function*

$$\mathcal{Q}_{m,a,p,1,0,0}(x)=\frac{\Gamma(1+x\sum_{i=1}^m a_i)}{\prod_{i=1}^m\Gamma(1+xa_i)}$$

is increasing on $(0,\infty)$ and its logarithmic derivative

$$\left[\ln\mathcal{Q}_{m,a,p,1,0,0}(x)\right]'=\left(\sum_{i=1}^m a_i\right)\psi\left(1+x\sum_{i=1}^m a_i\right)-\sum_{i=1}^m a_i\psi(1+a_ix)$$

is a Bernstein function on $(0,\infty)$;

(3) *when $\rho=1$, $\varrho\ge 1$, and $\theta=0$, the function $\mathcal{Q}_{m,a,p,1,\varrho,0}(x)$ is logarithmically completely monotonic on $(0,\infty)$;*

(4) *when $(\rho,\varrho,\theta)\in S$ and*

$$S=\{\rho\le 1,\varrho\ge 0,\theta\ge 0\}\setminus\{\rho=1,\varrho=0,\theta=0\}\setminus\{\rho=1,\varrho\ge 1,\theta=0\},$$

the function $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ has a unique minimum on $(0,\infty)$.

Proof Direct calculation gives

$$\ln\mathcal{Q}(x)=\left(\sum_{i=1}^m a_i\right)^{\theta}\ln\Gamma\left(1+x\sum_{i=1}^m a_i\right)-\rho\sum_{i=1}^m a_i^{\theta}\ln\Gamma(1+a_ix)+\varrho x\sum_{i=1}^m a_i\ln p_i,$$

$$\left[\ln\mathcal{Q}(x)\right]'=\left(\sum_{i=1}^m a_i\right)^{\theta+1}\psi\left(1+x\sum_{i=1}^m a_i\right)-\rho\sum_{i=1}^m a_i^{\theta+1}\psi(1+a_ix)+\varrho\sum_{i=1}^m a_i\ln p_i,$$

and

$$\left[\ln \mathcal{Q}(x)\right]'' = \left(\sum_{i=1}^{m} a_i\right)^{\theta+2} \psi'\left(1 + x\sum_{i=1}^{m} a_i\right) - \rho \sum_{i=1}^{m} a_i^{\theta+2}\psi'(1 + a_i x).$$

As in [27, 29, 32], from

$$\psi'(z) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-zt}\,\mathrm{d}t, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1], it follows that

$$\psi'(1+\tau z) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-(1+\tau z)t}\,\mathrm{d}t = \frac{1}{\tau}\int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vz}\,\mathrm{d}v,$$

where $\tau > 0$ and $h(t) = \frac{t}{e^t - 1}$ is the generating function of the classical Bernoulli numbers, see [20, 23] and [34, Chap. 1]. Accordingly, we have

$$\left[\ln \mathcal{Q}(x)\right]'' = \int_0^\infty \left[\left(\sum_{i=1}^{m} a_i\right)^{\theta+1} h\left(\frac{v}{\sum_{i=1}^{m} a_i}\right) - \rho \sum_{i=1}^{m} a_i^{\theta+1} h\left(\frac{v}{a_i}\right)\right] e^{-vx}\,\mathrm{d}v. \tag{3.1}$$

In [27, Theorem 4.1], it was discovered that

$$\sum_{i=1}^{m} \frac{v_i^\alpha}{e^{x/v_i} - 1} + \sum_{j=1}^{n} \frac{\tau_j^\alpha}{e^{x/\tau_j} - 1} \ge 2\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\lambda_{ij}^\alpha}{e^{x/\lambda_{ij}} - 1}, \tag{3.2}$$

where $\alpha \ge 0$, $x > 0$, $\lambda_{ij} > 0$ for $1 \le i \le m$ and $1 \le j \le n$, $v_i = \sum_{j=1}^{n} \lambda_{ij}$, and $\tau_j = \sum_{i=1}^{m} \lambda_{ij}$. As remarked in [27, Remark 4.1], setting $n = m$ and $\lambda_{1k} = \lambda_{k1} = \lambda_k > 0$ for $1 \le k \le m$ and letting $\lambda_{ij} \to 0^+$ for $2 \le i, j \le m$ in inequality (3.2) result in

$$\frac{(\sum_{k=1}^{m} \lambda_k)^\alpha}{e^{x/\sum_{k=1}^{m} \lambda_k} - 1} \ge \sum_{k=1}^{m} \frac{\lambda_k^\alpha}{e^{x/\lambda_k} - 1} \tag{3.3}$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \ge 0$. Inequality (3.3) can be equivalently formulated as

$$\left(\sum_{k=1}^{m} \lambda_k\right)^{\alpha+1} h\left(\frac{x}{\sum_{k=1}^{m} \lambda_k}\right) \ge \sum_{k=1}^{m} \lambda_k^{\alpha+1} h\left(\frac{x}{\lambda_k}\right) \tag{3.4}$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \ge 0$.

Combining inequality (3.4) with equation (3.1) yields that, when $\rho \le 1$ and $\theta \ge 0$, the second derivative $[\ln \mathcal{Q}(x)]''$ is completely monotonic on $(0, \infty)$.

The complete monotonicity of $[\ln \mathcal{Q}(x)]''$ implies that the first derivative $[\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0, \infty)$. Therefore, by virtue of the limit

$$\lim_{x\to\infty}\left[\ln x - \psi(x)\right] = 0$$

in [8, Theorem 1] and [9, Sect. 1.4], we have

$$\begin{aligned}
\lim_{x\to 0^+}\bigl[\ln \mathcal{Q}(x)\bigr]' &= \lim_{x\to 0^+}\left[\left(\sum_{i=1}^m a_i\right)^{\theta+1}\psi\left(1+x\sum_{i=1}^m a_i\right) - \rho\sum_{i=1}^m a_i^{\theta+1}\psi(1+a_ix)\right] \\
&\quad + \varrho\sum_{i=1}^m a_i \ln p_i \\
&= \psi(1)\left[\left(\sum_{i=1}^m a_i\right)^{\theta+1} - \rho\sum_{i=1}^m a_i^{\theta+1}\right] + \varrho\sum_{i=1}^m a_i\ln p_i \\
&\begin{cases} = 0, & \theta = 0, \rho = 1, \varrho = 0; \\ < 0, & \theta = 0, \rho = 1, \varrho > 0; \\ < 0, & \theta = 0, \rho < 1, \varrho \ge 0; \\ < 0, & \theta > 0, \rho \le 1, \varrho \ge 0; \end{cases}
\end{aligned}$$

where $\psi(1) = -0.577\ldots$, and

$$\begin{aligned}
\lim_{x\to \infty}\bigl[\ln \mathcal{Q}(x)\bigr]' &= \lim_{x\to \infty}\left[\left(\sum_{i=1}^m a_i\right)^{\theta+1}\psi\left(1+x\sum_{i=1}^m a_i\right) - \rho\sum_{i=1}^m a_i^{\theta+1}\psi(1+a_ix)\right] \\
&\quad + \varrho\sum_{i=1}^m a_i \ln p_i \\
&= \lim_{x\to\infty}\left\{\left(\sum_{i=1}^m a_i\right)^{\theta+1}\left[\psi\left(1+x\sum_{i=1}^m a_i\right) - \ln\left(1+x\sum_{i=1}^m a_i\right)\right]\right. \\
&\quad \left. - \rho\sum_{i=1}^m a_i^{\theta+1}\bigl[\psi(1+a_ix) - \ln(1+a_ix)\bigr]\right\} + \varrho\sum_{i=1}^m a_i\ln p_i \\
&\quad + \lim_{x\to\infty}\left[\left(\sum_{i=1}^m a_i\right)^{\theta+1}\ln\left(1+x\sum_{i=1}^m a_i\right) - \rho\sum_{i=1}^m a_i^{\theta+1}\ln(1+a_ix)\right] \\
&= \varrho\sum_{i=1}^m a_i\ln p_i + \lim_{x\to\infty}\ln\frac{(1+x\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m (1+a_ix)^{\rho a_i^{\theta+1}}} \\
&= \ln\lim_{x\to\infty}\frac{(\frac{1}{x}+\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m(\frac{1}{x}+a_i)^{\rho a_i^{\theta+1}}} \\
&\quad + \ln\lim_{x\to\infty} x^{(\sum_{i=1}^m a_i)^{\theta+1}-\rho\sum_{i=1}^m a_i^{\theta+1}} + \varrho\sum_{i=1}^m a_i\ln p_i \\
&= \varrho\sum_{i=1}^m a_i\ln p_i + \ln\frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{(\prod_{i=1}^m a_i^{a_i^{\theta+1}})^\rho} \\
&\quad + \begin{cases} 0, & \rho = \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ -\infty, & \rho > \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ \infty, & \rho < \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}. \end{cases}
\end{aligned}$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ such that $\sum_{i=1}^m \xi_i = 1$ and $\xi_i \in (0,1)$ for $1 \le i \le m$ and $m \ge 2$. Then the first derivative of the function $f(x) = \sum_{i=1}^m \xi_i^x$ is $f'(x) = \sum_{i=1}^m \xi_i^x \ln \xi_i < 0$, which implies that the function $f(x)$ is strictly decreasing on $(-\infty, \infty)$. Since $f(1) = 1$, it follows that $f(x) \lesseqqgtr 1$ if and only if $x \gtreqqless 1$. This means that

$$\sum_{i=1}^m \xi_i^x \lesseqqgtr 1, \quad x \gtreqqless 1.$$

Replacing $\xi_i = \frac{a_i}{\sum_{i=1}^m a_i}$ and $x = \theta + 1$ in the above inequality yields

$$\sum_{i=1}^m \left(\frac{a_i}{\sum_{i=1}^m a_i} \right)^{\theta+1} \lesseqqgtr 1, \quad \theta \gtreqqless 0.$$

This can be further rewritten as

$$\sum_{i=1}^m a_i^{\theta+1} \lesseqqgtr \left(\sum_{i=1}^m a_i \right)^{\theta+1}, \quad \theta \gtreqqless 0, a_i > 0, m \ge 2. \tag{3.5}$$

Considering inequality (3.5) reveals that

(1) when $\theta = 0$, we have

$$\lim_{x\to\infty} \bigl[\ln \mathcal{Q}(x)\bigr]' = \varrho \sum_{i=1}^m a_i \ln p_i + \begin{cases} \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} + 0, & \rho = 1; \\ \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{(\prod_{i=1}^m a_i^{a_i})^\rho} + \infty, & \rho < 1. \end{cases}$$

(2) when $\theta > 0$ and $\rho \le 1$, we have

$$\lim_{x\to\infty} \bigl[\ln \mathcal{Q}(x)\bigr]' = \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{(\prod_{i=1}^m a_i^{a_i^{\theta+1}})^\rho} + \infty = \infty.$$

Hence, when $\theta = 0$ and $\rho < 1$ or when $\theta > 0$ and $\rho \le 1$, we obtain

$$\lim_{x\to\infty} \bigl[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)\bigr]' = \infty;$$

when $\theta = 0$ and $\rho = 1$, we have

$$\begin{aligned} \lim_{x\to\infty} \bigl[\ln \mathcal{Q}(x)\bigr]' &= \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} \\ &= (\varrho - 1) \sum_{i=1}^m a_i \ln p_i + \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) - \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i}. \end{aligned}$$

Let f be a convex function on an interval $I \subseteq \mathbb{R}$, let $m \ge 2$ and $x_i \in I$ for $1 \le i \le m$, and let $q_i > 0$ for $1 \le i \le m$. Then

$$f\left(\frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i x_i \right) \le \frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i f(x_i). \tag{3.6}$$

This inequality is called Jensen's discrete inequality for convex functions in the literature [13, Sect. 1.4] and [14, Chapter I]. Applying (3.6) to $f(x) = x\ln x$ which is convex on $(0,\infty)$, $x_i = \frac{a_i}{p_i}$, and $q_i = p_i$ leads to

$$\left(\sum_{i=1}^{m} p_i \frac{a_i}{p_i}\right)\ln\left(\sum_{i=1}^{m} p_i \frac{a_i}{p_i}\right) \le \sum_{i=1}^{m} p_i \frac{a_i}{p_i}\ln\frac{a_i}{p_i}.$$

Accordingly,

$$\lim_{x\to\infty}\bigl[\ln \mathcal{Q}(x)\bigr]' \le (\varrho-1)\sum_{i=1}^{m} a_i \ln p_i \le 0, \quad \varrho \ge 1.$$

Consequently, when $\theta = 0$, $\rho = 1$, and $\varrho \ge 1$, the function $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ is logarithmically completely monotonic on $(0,\infty)$.

The limit

$$\lim_{x\to 0^+}\bigl[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)\bigr]' = 0$$

obtained above implies that $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' \ge 0$, $\mathcal{Q}_{m,a,p,1,0,0}(x)$ is increasing, and then $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]'$ is a Bernstein function on $(0,\infty)$.

When $(\rho,\varrho,\theta) \in S$, the limits

$$\lim_{x\to 0^+}\bigl[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)\bigr]' < 0$$

and

$$\lim_{x\to \infty}\bigl[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)\bigr]' = \infty$$

derived above mean that the first derivative $[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]'$ has a unique zero on $(0,\infty)$, that is, the functions $\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ and $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ have a unique minimum on $(0,\infty)$. The proof of Theorem 3.1 is complete. □

4 An open problem

Let $m,n \ge 2$, $\rho,\varrho,\theta \in \mathbb{R}$, let $\lambda = (\lambda_{ij})_{m\times n}$ with $\lambda_{ij} > 0$ for $1 \le i \le m$ and $1 \le j \le n$, let $\nu_i = \sum_{j=1}^{n}\lambda_{ij}$ and $\tau_j = \sum_{i=1}^{m}\lambda_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$, and let $p = (p_{ij})_{m\times n}$ with $\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} = 1$ and $p_{ij} \in (0,1)$ for $1 \le i \le m$ and $1 \le j \le n$. Define

$$Q_{m,n;\lambda;p;\rho;\varrho;\theta}(x) = \frac{\prod_{i=1}^{m}[\Gamma(1+\nu_i x)]^{\nu_i^\theta}\prod_{j=1}^{n}[\Gamma(1+\tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^{m}\prod_{j=1}^{n}[\Gamma(1+\lambda_{ij}x)]^{\rho\lambda_{ij}^\theta}}\left(\prod_{i=1}^{m} p_{ij}^{\lambda_{ij}}\right)^{\varrho x} \tag{4.1}$$

on the infinite interval $(0,\infty)$.

Can one find monotonicity properties for the function $Q_{m,n;\lambda;p;\rho;\varrho;\theta}(x)$ defined in equation (4.1)?

Remark 4.1 This paper is a slightly revised version of the electronic preprint [30].

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Author details
[1]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, China. [2]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China. [3]School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin, China. [4]Department of Mathematics Education, Andong National University, Andong, Republic of Korea.

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