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A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model

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Abstract

In this paper, we study the rubella disease model with the Caputo–Fabrizio fractional derivative. The mathematical solution of the liver model is presented by a three-step Adams–Bashforth scheme. The existence and uniqueness of the solution are discussed by employing fixed point theory. Finally some numerical simulations are showed to underpin the effectiveness of the used derivative.

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1 Introduction

Rubella was first described in the mid-eighteenth century. Friedrich Hoffmann made the first clinical description of rubella in 1740, which was confirmed by de Bergen in 1752 and Orlow in 1758 [1]. Rubella, also known as German measles or three-day measles, is an infection caused by the rubella virus and has symptoms that are similar to those of flu. However, the primary symptom of rubella virus infection is the appearance of a rash (exanthem) on the face which spreads to the trunk and limbs and usually fades after three days [2]. It usually spreads through the air via coughs of people who are infected. People are infectious during the week before and after the appearance of the rash [3]. This disease is often mild with half of people not realizing that they are infected [4]. Rubella is a common infection in many areas of the world, and each year about 100,000 cases of congenital rubella syndrome occur [5].

The mathematical model of measles and rubella has been studied by a number of mathematicians (see, for example, [6–10]). It has been demonstrated by many scientists and mathematicians that fractional extensions of mathematical models of integer order represent the natural fact in a very systematic way such as in the approach of Caputo [11], Podlubny [12], Baleanu et al. [13], Haq et al. [14], Atangana et al. [15], Erturk et al. [16], Kilbas et al. [17], Zafar et al. [18–22]. In a very recent attempt, Caputo and Fabrizio [11]

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propounded a novel fractional derivative having exponential kernel; in addition, Losada and Nieto [23] analyzed the properties of a newly presented fractional derivative. The classical fractional derivatives, especially the Caputo and Riemann derivatives, have their own limitation because their kernel is singular. Since the kernel is employed to describe the memory effect of the physical system, it is obvious that due to this weakness, both derivatives cannot precisely describe the full effect of the memory. Recently, many works related to the fractional equations and applications have been published (see, for example, [24–39]).

Therefore, we use the novel Caputo–Fabrizio (CF) fractional derivative to study the rubella disease model and explain this problem in a better and more efficient manner. We recall some fundamental notions. The Caputo fractional derivative of order α for a continuous function f is defined by

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds \quad (n = [\alpha] + 1).$$

Our second notion is a fractional derivative without singular kernel introduced by Caputo and Fabrizio [11, 34]. Let $b > 0$, $u \in H^1(a,b)$, and $\alpha \in (0,1)$. The Caputo–Fabrizio derivative of order α for a function f is defined by

$$^{\mathrm{CF}}D^{\alpha}f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right)f'(s)\, ds,$$

where $t > 0$ and $M(\alpha)$ is a normalization constant depending on α such that $M(0) = M(1) = 1$. It is well known that the Laplace transform plays an important role in the study of ordinary differential equations [23]. Let $\alpha \in (0,1)$ and $n \geq 1$. The Laplace transform of $^{\mathrm{CF}}D$ is defined by

$$L\big[^{\mathrm{CF}}D^{(\alpha+1)}u(t)\big](s) = \frac{1}{1-\alpha}L\big[u^{(\alpha+n)}(t)\big]\exp\left(\frac{-\alpha}{\alpha-1}t\right)$$
$$= \frac{s^{n+1}L[u(t)] - s^n u(0) - s^{n-1}u'(0) - \cdots - u^{(n)}(0)}{s+\alpha(1-s)}$$

and $L[^{\mathrm{CF}}D^{(\alpha)}u(t)](s) = \frac{sL[u(t)]-u(0)}{s+\alpha(1-s)}$ (for $n = 0$),

$$L\big[^{\mathrm{CF}}D^{(\alpha+1)}u(t)\big](s) = \frac{s^2L[u(t)] - su(0) - u'(0)}{s+\alpha(1-s)}$$

(for $n = 1$). The Riemann–Liouville fractional integral of order $0 < \alpha < 1$ is defined by [12]

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds$$

whenever the integral exists. Also, the fractional integral of Caputo–Fabrizio is defined by [23]

$$^{\mathrm{CF}}I^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s)\, ds.$$

Losada and Nieto gave an explicit formula for $M(\alpha)$ as $M(\alpha) = \frac{2}{2-\alpha}$ (for $0 < \alpha < 1$). Thus, the fractional Caputo–Fabrizio derivative of order $0 < \alpha < 1$ for a function u is given by $^{\mathrm{CF}}D^\alpha u(t) = \frac{1}{1-\alpha} \int_0^t \exp(\frac{-\alpha}{1-\alpha}(t-s))u'(s)\,ds$. For $n \geq 1$ and $\alpha \in (0,1)$, the fractional derivatives $^{\mathrm{CF}}D^{\alpha+n}$ of order $n + \alpha$ are defined by $^{\mathrm{CF}}D^{\alpha+n}u(t) := {}^{\mathrm{CF}}D^\alpha(D^n u(t))$ [26].

2 Mathematical model for the spread of rubella disease

In this section, we investigate the rubella disease model discussed by Koca [8]. He considered the model of rubella disease by employing the Atangana–Baleanu fractional derivative as follows:

$$
\begin{cases}
{}^{\mathrm{ABC}}_0 D_t^\alpha S(t) = B(a) - [\lambda(a,t) + P(a) + \mu(a)]S(t), \\
{}^{\mathrm{ABC}}_0 D_t^\alpha E(t) = \lambda(a,t)S(t) - (\sigma + \mu(a))E(t), \\
{}^{\mathrm{ABC}}_0 D_t^\alpha I(t) = \sigma E(t) - (\beta + \mu(a))I(t), \\
{}^{\mathrm{ABC}}_0 D_t^\alpha R(t) = \beta I(t) - \mu(a)R(t), \\
{}^{\mathrm{ABC}}_0 D_t^\alpha V(t) = D(a)S(t) - \mu(a)V(t),
\end{cases}
\tag{1}
$$

where $S(t)$, $E(t)$, $I(t)$, $R(t)$, $V(t)$ are susceptible, latent, infectious, recovered, and vaccinated parameters respectively. $P(a)$ is a parameter for which immunized by vaccination and $\lambda(a,t)$ is the force of infection of age a at time t and σ is the latent rate and β is the infection rate [40]. In this section, we moderate the system by substituting the time-derivative by the newly introduced Caputo–Fabrizio derivative [11] for $\alpha \in (0,1)$ given by

$$
\begin{cases}
{}^{\mathrm{CF}}_0 D_t^\alpha S(t) = B(a) - [\lambda(a,t) + P(a) + \mu(a)]S(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha E(t) = \lambda(a,t)S(t) - (\sigma + \mu(a))E(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha I(t) = \sigma E(t) - (\beta + \mu(a))I(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha R(t) = \beta I(t) - \mu(a)R(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha V(t) = D(a)S(t) - \mu(a)V(t)
\end{cases}
\tag{2}
$$

with initial conditions

$$
S(0) = S_0, \qquad E(0) = E_0, \qquad I(0) = I_0, \qquad R(0) = R_0, \qquad V(0) = V_0.
$$

In system (2), the right-hand sides of the equations have dimension $(time)^{-1}$. When we change the order of the equations to α, the dimension of the left-hand side would be $(time)^{(-\alpha)}$. In order to have the dimensions match, we should change the dimensions of the parameters σ, β, and the system we obtain eventually is

$$
\begin{cases}
{}^{\mathrm{CF}}_0 D_t^\alpha S(t) = B(a) - [\lambda(a,t) + P(a) + \mu(a)]S(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha E(t) = \lambda(a,t)S(t) - (\sigma^\alpha + \mu(a))E(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha I(t) = \sigma^\alpha E(t) - (\beta^\alpha + \mu(a))I(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha R(t) = \beta^\alpha I(t) - \mu(a)R(t), \\
{}^{\mathrm{CF}}_0 D_t^\alpha V(t) = D(a)S(t) - \mu(a)V(t).
\end{cases}
\tag{3}
$$

The system state is made up with S, E, I, R, V.

3 Existence and uniqueness of a system of solutions of rubella model

We examine the existence of the system of solutions by applying the fixed point theorem. Employing the fractional integral operator due to Nieto and Losada [23] on equation (3), we obtain

$$
\begin{cases}
S(t) - S(0) = {}_0^{\mathrm{CF}}I_t^\alpha[B(a) - [\lambda(a,t) + P(a) + \mu(a)]S(t)], \\
E(t) - E(0) = {}_0^{\mathrm{CF}}I_t^\alpha[\lambda(a,t)S(t) - (\sigma^\alpha + \mu(a))E(t)], \\
I(t) - I(0) = {}_0^{\mathrm{CF}}I_t^\alpha[\sigma^\alpha E(t) - (\beta^\alpha + \mu(a))I(t)], \\
R(t) - R(0) = {}_0^{\mathrm{CF}}I_t^\alpha[\beta^\alpha I(t) - \mu(a)R(t)], \\
V(t) - V(0) = {}_0^{\mathrm{CF}}I_t^\alpha[D(a)S(t) - \mu(a)V(t)].
\end{cases}
\tag{4}
$$

By using the notation presented by Nieto and Losada [23], we get

$$
\begin{aligned}
S(t) - S(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big\{B(a) - \big[\lambda(a,t) + P(a) + \mu(a)\big]S(t)\Big\} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \Big[B(a) - \big[\lambda(a,y) + P(a) + \mu(a)\big]S(y)\Big]\,dy, \\
E(t) - E(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big\{\lambda(a,t)S(t) - \big(\sigma^\alpha + \mu(a)\big)E(t)\Big\} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \Big[\lambda(a,y)S(y) - \big(\sigma^\alpha + \mu(a)\big)E(y)\Big]\,dy, \\
I(t) - I(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big\{\sigma^\alpha E(t) - \big(\beta^\alpha + \mu(a)\big)I(t)\Big\} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \Big[\sigma^\alpha E(y) - \big(\beta^\alpha + \mu(a)\big)I(y)\Big]\,dy, \\
R(t) - R(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big\{\beta^\alpha I(t) - \mu(a)R(t)\Big\} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \Big[\beta^\alpha I(y) - \mu(a)R(y)\Big]\,dy, \\
V(t) - V(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big\{D(a)S(t) - \mu(a)V(t)\Big\} \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \Big[D(a)S(y) - \mu(a)V(y)\Big]\,dy.
\end{aligned}
$$

For clarity, we write

$$
\begin{cases}
P_1(t,S) = B(a) - [\lambda(a,t) + P(a) + \mu(a)]S(t), \\
P_2(t,E) = \lambda(a,t)S(t) - (\sigma^\alpha + \mu(a))E(t), \\
P_3(t,I) = \sigma^\alpha E(t) - (\beta^\alpha + \mu(a))I(t), \\
P_4(t,R) = \beta^\alpha I(t) - \mu(a)R(t), \\
P_5(t,V) = D(a)S(t) - \mu(a)V(t).
\end{cases}
\tag{5}
$$

Theorem 1 *The kernel P_1 satisfies the Lipschitz condition and contraction if the following inequality holds*:

$$0 < \lambda(a,t) + P(a) + \mu(a) \le 1. \tag{6}$$

Proof Let S and S_1 be two functions, then we assess the following:

$$
\begin{aligned}
\left\| P_1(t,S) - P_1(t,S_1) \right\| &= \left\| -\bigl(\lambda(a,t) + P(a) + \mu(a)\bigr)\bigl(S(t) - S_1(t)\bigr) \right\| \\
&\le \bigl\{ \lambda(a,t) + P(a) + \mu(a) \bigr\} \left\| S(t) - S_1(t) \right\| \\
&\le \gamma_1 \left\| S(t) - S_1(t) \right\|.
\end{aligned}
\tag{7}
$$

Taking $\gamma_1 = \lambda(a,t) + P(a) + \mu(a)$ are bounded functions, we get

$$\left\| P_1(t,S) - P_1(t,S_1) \right\| \le \gamma_1 \left\| S(t) - S_1(t) \right\|. \tag{8}$$

Hence the Lipschitz condition is satisfied for P_1. If additionally

$$0 < \lambda(a,t) + P(a) + \mu(a) \le 1,$$

then it is also a contraction for P_1. $\qquad\square$

Similarly, the kernels P_2, P_2, P_3, P_4, P_5 satisfy the Lipschitz condition given as follows:

$$
\begin{cases}
\|P_2(t,E) - P_2(t,E_1)\| \le \gamma_2 \|E(t) - E_1(t)\|, \\
\|P_3(t,I) - P_3(t,I_1)\| \le \gamma_3 \|I(t) - I_1(t)\|, \\
\|P_4(t,R) - P_4(t,R_1)\| \le \gamma_4 \|R(t) - R_1(t)\|, \\
\|P_5(t,V) - P_5(t,V_1)\| \le \gamma_5 \|V(t) - V_1(t)\|.
\end{cases}
\tag{9}
$$

On consideration of the aforesaid kernels, equation (5) becomes

$$
\begin{aligned}
S(t) &= S(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_1(t,S) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_1(y,S)\,dy, \\
E(t) &= E(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_2(t,E) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_2(y,E)\,dy, \\
I(t) &= I(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_3(t,I) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_3(y,I)\,dy, \\
R(t) &= R(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_4(t,R) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_4(y,R)\,dy, \\
V(t) &= V(0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_5(t,V) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_5(y,V)\,dy.
\end{aligned}
\tag{10}
$$

Now, we present the following recursive formula on consideration of the aforesaid kernels, equation (5) becomes

$$S_n(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} P_1(t,S_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t P_1(y,S_{n-1})\,dy,$$

$$E_n(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}P_2(t, E_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t P_2(y, E_{n-1})\,dy,$$

$$I_n(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}P_3(t, I_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t P_3(y, I_{n-1})\,dy, \tag{11}$$

$$R_n(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}P_4(t, R_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t P_4(y, R_{n-1})\,dy,$$

$$V_n(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}P_5(t, V_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t P_5(y, V_{n-1})\,dy.$$

The initial conditions are given as follows:

$$S_0(t) = S(0), \qquad E_0(t) = E(0), \qquad I_0(t) = I(0),$$

$$R_0(t) = R(0), \qquad V_0(t) = V(0). \tag{12}$$

Now we present the difference between the successive terms in the following manner:

$$\phi_n(t) = S_n(t) - S_{n-1}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\big[P_1(t, S_{n-1}) - P_1(t, S_{n-2})\big]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \big[P_1(y, S_{n-1}) - P_1(y, S_{n-2})\big]\,dy,$$

$$\psi_n(t) = E_n(t) - E_{n-1}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\big[P_2(t, E_{n-1}) - P_2(t, E_{n-2})\big]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \big[P_2(y, E_{n-1}) - P_2(y, E_{n-2})\big]\,dy,$$

$$\xi_n(t) = I_n(t) - I_{n-1}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\big[P_3(t, I_{n-1}) - P_3(t, I_{n-2})\big]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \big[P_3(y, I_{n-1}) - P_3(y, I_{n-2})\big]\,dy, \tag{13}$$

$$\chi_n(t) = R_n(t) - R_{n-1}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\big[P_4(t, R_{n-1}) - P_4(t, R_{n-2})\big]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \big[P_4(y, R_{n-1}) - P_4(y, R_{n-2})\big]\,dy,$$

$$\zeta_n(t) = V_n(t) - V_{n-1}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\big[P_5(t, V_{n-1}) - P_5(t, V_{n-2})\big]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \big[P_5(y, V_{n-1}) - P_5(y, V_{n-2})\big]\,dy.$$

It is worth noticing that

$$S_n(t) = \sum_{i=0}^n \phi_i(t),$$

$$E_n(t) = \sum_{i=0}^n \psi_i(t),$$

$$I_n(t) = \sum_{i=0}^n \xi_i(t), \tag{14}$$

$$R_n(t) = \sum_{i=0}^{n} \chi_i(t),$$

$$V_n(t) = \sum_{i=0}^{n} \zeta_i(t).$$

On the other hand,

$$
\begin{aligned}
\|\phi_n(t)\| &= \|S_n(t) - S_{n-1}(t)\| \\
&= \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big[P_1(t, S_{n-1}) - P_1(t, S_{n-2}) \big] \right. \\
&\quad \left. + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big[P_1(y, S_{n-1}) - P_1(y, S_{n-2}) \big] \, dy \right\|.
\end{aligned}
\tag{15}
$$

By using the triangular inequality, we get

$$
\begin{aligned}
\|S_n(t) - S_{n-1}(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|P_1(t, S_{n-1}) - P_1(t, S_{n-2})\| \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \\
&\quad \times \left\| \int_0^t \big[P_1(y, S_{n-1}) - P_1(y, S_{n-2}) \big] \, dy \right\|.
\end{aligned}
\tag{16}
$$

Since the kernel satisfies the Lipschitz condition, we have

$$
\begin{aligned}
\|S_n(t) - S_{n-1}(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 \|S_{n-1} - S_{n-2}\| \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 \int_0^t \|S_{n-1} - S_{n-2}\| \, dy,
\end{aligned}
\tag{17}
$$

then we get

$$
\|\phi_n(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 \|\phi_{n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 \int_0^t \|\phi_{n-1}(y)\| \, dy.
\tag{18}
$$

Similarly, we get the following results:

$$
\begin{aligned}
\|\psi_n(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_2 \|\psi_{n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_2 \int_0^t \|\psi_{n-1}(y)\| \, dy, \\
\|\xi_n(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_3 \|\xi_{n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_3 \int_0^t \|\xi_{n-1}(y)\| \, dy, \\
\|\chi_n(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_4 \|\chi_{n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_4 \int_0^t \|\chi_{n-1}(y)\| \, dy, \\
\|\zeta_n(t)\| &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_5 \|\zeta_{n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_5 \int_0^t \|\zeta_{n-1}(y)\| \, dy.
\end{aligned}
\tag{19}
$$

By taking the above results, we can present the following theorem.

Theorem 2 *Fractional rubella model* (3) *has a system of solutions under the conditions that we can find t_0 such that*

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\gamma_1 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_1 t_0 \le 1. \tag{20}$$

Proof We have considered that the functions $S(t)$, $E(t)$, $I(t)$, $R(t)$, $V(t)$ are bounded. Additionally, we have proven that the kernels satisfy the Lipschitz condition, hence by taking the results of equations (19) and (20) and by employing the recursive method, we derive the succeeding relation as follows:

$$\begin{aligned}
\|\phi_n(t)\| &\le \|S(0)\| \left[\frac{2(1-\alpha)}{2-\alpha)M(\alpha)}\gamma_1 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_1 t \right]^n, \\
\|\psi_n(t)\| &\le \|E(0)\| \left[\frac{2(1-\alpha)}{2-\alpha)M(\alpha)}\gamma_2 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_2 t \right]^n, \\
\|\xi_n(t)\| &\le \|I(0)\| \left[\frac{2(1-\alpha)}{2-\alpha)M(\alpha)}\gamma_3 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_3 t \right]^n, \\
\|\chi_n(t)\| &\le \|R(0)\| \left[\frac{2(1-\alpha)}{2-\alpha)M(\alpha)}\gamma_4 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_4 t \right]^n, \\
\|\zeta_n(t)\| &\le \|V(0)\| \left[\frac{2(1-\alpha)}{2-\alpha)M(\alpha)}\gamma_5 + \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_5 t \right]^n.
\end{aligned} \tag{21}$$

Therefore, the system of functions (15) exists and is smooth. To show that the above functions are a system of solutions of the system of equation (3), we assume

$$\begin{cases}
S(t) - S(0) = S_n(t) - B_n(t), \\
E(t) - E(0) = E_n(t) - C_n(t), \\
I(t) - I(0) = I_n(t) - D_n(t), \\
R(t) - R(0) = R_n(t) - F_n(t), \\
V(t) - V(0) = V_n(t) - H_n(t).
\end{cases} \tag{22}$$

Therefore, we get

$$\begin{aligned}
\|B_n(t)\| &= \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big(P(t,S) - P(t,S_{n-1}) \big) \right. \\
&\quad \left. + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big(P(y,S) - P(y,S_{n-1}) \big)\, dy \right\| \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \big\| P(t,S) - P(t,S_{n-1}) \big\| \\
&\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \big\| P(y,S) - P(y,S_{n-1}) \big\|\, dy \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 \|S - S_{n-1}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 \|S - S_{n-1}\| t.
\end{aligned} \tag{23}$$

By using this process recursively, it yields

$$\left\| B_n(t) \right\| \leq \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} t \right]^{n+1} \gamma_1^{n+1} k. \tag{24}$$

Now, taking the limit on equation (25) as n tends to infinity, we get

$$\left\| B_n(t) \right\| \to 0. \tag{25}$$

Similarly, we have

$$\left\| C_n(t) \right\| \to 0, \qquad \left\| D_n(t) \right\| \to 0, \qquad \left\| F_n(t) \right\| \to 0, \qquad \left\| H_n(t) \right\| \to 0. \tag{26}$$

This completes the proof. □

To prove the uniqueness of a system of solutions of equation (3), we present the following theorem.

Theorem 3 *The system of equations* (3) *has a unique system of solutions if the following condition holds*:

$$\left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 - \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 t \right) \geq 0. \tag{27}$$

Proof Let there exist another system of solutions of (3)

$$S_1(t), \qquad E_1(t), \qquad I_1(t), \qquad R_1(t), \qquad V_1(t),$$

then

$$S(t) - S_1(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left[P_1(t, S) - P_1(t, S_1) \right]$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left[P_1(y, S) - P_1(y, S_1) \right] dy. \tag{28}$$

Applying the norm on equation (29), we get

$$\left\| S(t) - S_1(t) \right\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left\| P_1(t, S) - P_1(t, S_1) \right\| \tag{29}$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\| P_1(y, S) - P_1(y, S_1) \right\| dy.$$

By employing the Lipschitz conditions of the kernel, we get

$$\left\| S(t) - S_1(t) \right\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 \left\| S(t) - S_1(t) \right\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 t \left\| S(t) - S_1(t) \right\|. \tag{30}$$

It gives

$$\left\| S(t) - S_1(t) \right\| \left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \gamma_1 - \frac{2\alpha}{(2-\alpha)M(\alpha)} \gamma_1 t \right) \leq 0. \tag{31}$$

On the other hand, (28) holds, then

$$\left\| S(t) - S_1(t) \right\| \left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\gamma_1 - \frac{2\alpha}{(2-\alpha)M(\alpha)}\gamma_1 t \geq 0. \right. \tag{32}$$

We conclude from (31) and (32) that

$$\left\| S(t) - S_1(t) \right\| = 0. \tag{33}$$

Then we get $S(t) = S_1(t)$. Similarly, we have

$$E(t) = E_1(t), \qquad I(t) = I_1(t), \qquad R(t) = R_1(t), \qquad V(t) = V_1(t). \tag{34}$$

This completes the proof. □

4 Equilibrium points of the model and asymptotic stability

To determine the equilibrium points of fractional order system (3), we set the right-hand side of the equations to zero

$$^{\mathrm{CF}}D_t^\alpha S(t) = {}^{\mathrm{CF}}D_t^\alpha E(t) = {}^{\mathrm{CF}}D_t^\alpha I(t) = {}^{\mathrm{CF}}D_t^\alpha R(t) = {}^{\mathrm{CF}}D_t^\alpha V(t) = 0. \tag{35}$$

By solving the algebraic equations, we obtain the equilibrium point $E^* = (S^*, E^*, I^*, R^*, V^*)$ such that

$$\begin{aligned}
S^* &= \frac{B(a)}{\lambda(a,t) + P(a) + \mu(a)}, \qquad E^* = \frac{B(a)\lambda(a,t)}{(\lambda(a,t) + P(a) + \mu(a))(\sigma^\alpha(a) + \mu(a))}, \\
I^* &= \frac{\sigma^\alpha(a)B(a)\lambda(a,t)}{(\beta^\alpha\sigma^\alpha(a) + \beta^\alpha\mu(a) + \mu(a)\sigma^\alpha(a) + \mu^2(a))(\lambda(a,t) + P(a) + \mu(a))}, \\
R^* &= \frac{\beta^\alpha\sigma^\alpha(a)B(a)\lambda(a,t)}{(\beta^\alpha\sigma^\alpha(a) + \beta^\alpha\mu(a) + \mu(a)\sigma^\alpha(a) + \mu^2(a))(\lambda(a,t) + P(a) + \mu(a))}, \\
V^* &= \frac{D(a)B(a)}{\mu(a)(\lambda(a,t) + P(a) + \mu(a))}.
\end{aligned} \tag{36}$$

To investigate the stability of equilibrium point, first consider the fractional-order linear system as follows:

$$^{\mathrm{CF}}D_t^\alpha u(t) = Tu(t), \tag{37}$$

where $u(t) \in R^n$, $T \in R^{n \times n}$, $0 < \alpha < 1$.

Definition 4 ([41]) For system (37) with Caputo–Fabrizio fractional derivative, the characteristic equation is given by

$$\det\left(s\left(I - (1-\alpha)T\right) - \alpha T\right) = 0. \tag{38}$$

Theorem 5 ([41]) *If $(I - (1-\alpha)T)$ is invertible, then system* (37) *is asymptotically stable if and only if the roots to the characteristic equation of system* (37) *have negative real parts.*

The Jacobian matrix associated with system (3) is given as follows:

$$
J = \begin{bmatrix}
-(\lambda(a,t) + P(a) + \mu(a)) & 0 & 0 & 0 & 0 \\
\lambda(a,t) & -(\sigma^\alpha(a) + \mu(a)) & 0 & 0 & 0 \\
0 & \sigma^\alpha(a) & -(\beta^\alpha + \mu(a)) & 0 & 0 \\
0 & 0 & \beta^\alpha & -\mu(a) & 0 \\
D(a) & 0 & 0 & 0 & -\mu(a)
\end{bmatrix}.
$$

Thus the characteristic equation of system (3) is

$$
\det\big(s\big(I - (1-\alpha)J\big) - \alpha J\big) = 0. \tag{39}
$$

Theorem 6 *The equilibrium point E^* of model* (3) *is asymptotically stable if and only if real parts of the roots of the characteristic equation* (39) *are negative.*

Proof According to the above Jacobian matrix, we have $J = J(E^*)$. Therefore, it is sufficient that we compute the roots of equation (39). We have

$$
\det\big(s\big(I - (1-\alpha)J\big) - \alpha J\big)
$$
$$
= \big\{s + \big(s(1-\alpha) + \alpha\big)\big(\lambda(a,t) + P(a) + \mu(a)\big)\big\}\big\{s + \big(s(1-\alpha) + \alpha\big)\big(\sigma^\alpha(a) + \mu(a)\big)\big\},
$$
$$
\times \big\{s + \big(s(1-\alpha) + \alpha\big)\big(\beta^\alpha + \mu(a)\big)\big\}\big\{s + \big(s(1-\alpha) + \alpha\big)\mu(a)\big\}^2 = 0.
$$

By solving this algebraic equation, we get

$$
s_1 = \frac{-\alpha(\lambda(a,t) + P(a) + \mu(a))}{1 + (1-\alpha)(\lambda(a,t) + P(a) + \mu(a))},
$$
$$
s_2 = \frac{-\alpha(\sigma^\alpha(a) + \mu(a))}{1 + (1-\alpha)(\sigma^\alpha(a) + \mu(a))},
$$
$$
s_3 = \frac{-\alpha(\beta^\alpha + \mu(a))}{(\beta^\alpha + \mu(a))},
$$
$$
s_4 = \frac{-\alpha\mu(a)}{1 + (1-\alpha)\mu(a)}.
$$

Since $\alpha \in (0,1)$, then s_1, s_2, s_3, s_4 are negative. Hence by using Theorem 5, the equilibrium point E^* of model (3) is asymptotically stable. □

4.1 Numerical method and simulations

In this section, using the Adams–Bashforth scheme, we present a numerical solution for the rubella model (3). Owolabi and Atangana introduced the three-step Adams–Bashforth scheme with the Caputo–Fabrizio fractional derivative [42]. We use this method to find three step Adams–Bashforth scheme for fractional order system (3).

Consider the fractional differential equation with Caputo–Fabrizio derivative

$$
{}^{CF}D_t^\alpha x(t) = f\big(t, x(t)\big), \quad 0 < \alpha < 1. \tag{40}
$$

Applying the Caputo–Fabrizio fractional integral on both sides of equation (40), we have

$$x(t) - x(0) = \frac{(1-\alpha)}{M(\alpha)} f(t, x(t)) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau, x(\tau)) \, d\tau. \tag{41}$$

By discretizing the time interval $[0, t]$ in steps of h, we obtain the sequence $t_0 = 0$, $t_{j+1} = t_j + h$, $j = 0, 1, 2, \ldots, n-1$, $t_j = t$. By replacing $t = t_{j+1}$ and $t = t_j$ in equation (41) and computing the difference of the resulting equations, we obtain

$$x(t_{j+1}) - x(t_j) = \frac{(1-\alpha)}{M(\alpha)} \big[f(t_{j+1}, x(t_{j+1})) - f(t_j, x(t_j)) \big] + \frac{\alpha}{M(\alpha)} \int_{t_j}^{t_{j+1}} f(\tau, x(\tau)) \, d\tau. \tag{42}$$

By approximating the integral $\int_{t_j}^{t_{j+1}} f(\tau, x(\tau)) \, d\tau$ with the approximation of $\int_{t_j}^{t_{j+1}} Q_2(\tau) \, d\tau$, where $Q_2(\tau)$ is the Lagrange interpolating polynomial of degree two passing through the points $(t_{j-2}, f(t_{j-2}, x(t_{j-2})))$, $(t_{j-1}, f(t_{j-1}, x(t_{j-1})))$, and $(t_j, f(t_j, x(t_j)))$. That is,

$$Q_2(\tau) = \sum_{i=0}^{i=2} f(t_{j-i}, x(t_{j-i}))) L_i(\tau),$$

where $L_i(\tau)$ are the Lagrange basis polynomials on the points t_{j-2}, t_{j-1}, t_j. Let $x_j = x(t_j)$, using the change of variable $s = \frac{t_{j+1} - \tau}{h}$, substituting for the Lagrange basis polynomials and integrating, we obtain

$$\int_{t_j}^{t_{j+1}} f(\tau, x(\tau)) \, d\tau = h \int_0^1 f(t_j, x_j) \frac{(s-2)(s-3)}{(1-2)(1-3)}$$
$$+ f(t_{j-1}, x_{j-1}) \frac{(s-1)(s-3)}{(2-1)(2-3)} + f(t_{j-2}, x_{j-2}) \frac{(s-2)(s-1)}{(3-2)(3-1)} \, ds$$
$$= \frac{23h}{12} f(t_j, x_j) - \frac{16h}{12} f(t_{j-1}, x_{j-1}) + \frac{5h}{12} f(t_{j-2}, x_{j-2}). \tag{43}$$

Then

$$x(t_{j+1}) - x(t_j) = \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f(t_j, x_j) - \left(\frac{1-\alpha}{M(\alpha)} \right.$$
$$\left. + \frac{16\alpha h}{12M(\alpha)} \right) f(t_{j-1}, x_{j-1}) + \left(\frac{5\alpha h}{12M(\alpha)} \right) f(t_{j-2}, x_{j-2}). \tag{44}$$

In this method, the error is

$$R_j^\alpha(t) = \frac{\alpha}{M(\alpha)} \int_{t_j}^{t_{j+1}} \frac{3}{8} h^3 f^{(4)}(\eta) \, d\eta = \frac{3\alpha h^3}{8M(\alpha)} f^{(3)}(\lambda_j, x(\lambda_j)), \quad \lambda_j \in (t_j, t_{j+1}). \tag{45}$$

In the following, we obtain the numerical simulations of model (3) using the three-step Adams–Bashforth scheme for Caputo–Fabrizio fractional derivative in equation (44). Consider the vectors $\mathbf{x}(t) = (S(t), E(t), I(t), R(t), V(t))$ and $\mathbf{f}(t, \mathbf{x}(t)) = (f_1(t, \mathbf{x}(t)), f_2(t, \mathbf{x}(t)), f_3(t, \mathbf{x}(t)), f_4(t, \mathbf{x}(t)), f_5(t, \mathbf{x}(t)))$, where $f_i(t, \mathbf{x}(t))$, $i = 1, 2, 3, 4$, are scalar functions that are de-

fined from system (3) as follows:

$$
\begin{cases}
f_1(t, \mathbf{x}(t)) = B(a) - [\lambda(a, t) + P(a) + \mu(a)]S(t), \\
f_2(t, \mathbf{x}(t)) = \lambda(a, t)S(t) - (\sigma^\alpha + \mu(a))E(t), \\
f_3(t, \mathbf{x}(t)) = \sigma^\alpha E(t) - (\beta^\alpha + \mu(a))I(t), \\
f_4(t, \mathbf{x}(t)) = \beta^\alpha I(t) - \mu(a)R(t), \\
f_5(t, \mathbf{x}(t)) = D(a)S(t) - \mu(a)V(t).
\end{cases}
\tag{46}
$$

We write system (3) in the vector form as follows:

$$
{}^{\mathrm{CF}}D_t^\alpha \mathbf{x}(t) = \mathbf{f}\big(t, \mathbf{x}(t)\big), \quad 0 < \alpha < 1.
\tag{47}
$$

Using equation (44), we obtain the solution of system (3) as the following iterative formula:

$$
\begin{aligned}
\mathbf{x}(t_{j+1}) = \mathbf{x}(t_j) &+ \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_j, \mathbf{x}_j) \\
&- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_{j-1}, \mathbf{x}_{j-1}) + \left(\frac{5\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_{j-2}, \mathbf{x}_{j-2}).
\end{aligned}
\tag{48}
$$

Assume $\mathbf{x}_0 = \mathbf{x}(t_0) = [S(t_0), E(t_0), I(t_0), R(t_0), V(t_0)]^T$, $\mathbf{x}_{j-2} = \mathbf{x}(t_{j-2})$, $\mathbf{x}_{j-1} = \mathbf{x}(t_{j-1})$, $\mathbf{x}_j = \mathbf{x}(t_j)$, $\mathbf{x}_{j+1} = \mathbf{x}(t_{j+1})$, then

$$
\begin{aligned}
\mathbf{x}_{j+1} = \mathbf{x}_j &+ \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_j, \mathbf{x}_j) \\
&- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_{j-1}, \mathbf{x}_{j-1}) + \left(\frac{5\alpha h}{12M(\alpha)} \right) \mathbf{f}(t_{j-2}, \mathbf{x}_{j-2}).
\end{aligned}
\tag{49}
$$

Thus, we obtain the iterative formulas

$$
\begin{aligned}
S_{j+1} = S_j &+ \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f_1(S_j, E_j, I_j, R_j, V_j) \\
&- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) f_1(S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}, V_{j-1}) \\
&+ \left(\frac{5\alpha h}{12M(\alpha)} \right) f_1(S_{j-2}, E_{j-2}, I_{j-2}, R_{j-2}, V_{j-2}), \\
E_{j+1} = S_j &+ \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f_2(S_j, E_j, I_j, R_j, V_j) \\
&- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) f_2(S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}, V_{j-1}) \\
&+ \left(\frac{5\alpha h}{12M(\alpha)} \right) f_2(S_{j-2}, E_{j-2}, I_{j-2}, R_{j-2}, V_{j-2}), \\
I_{j+1} = I_j &+ \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f_3(S_j, E_j, I_j, R_j, V_j)
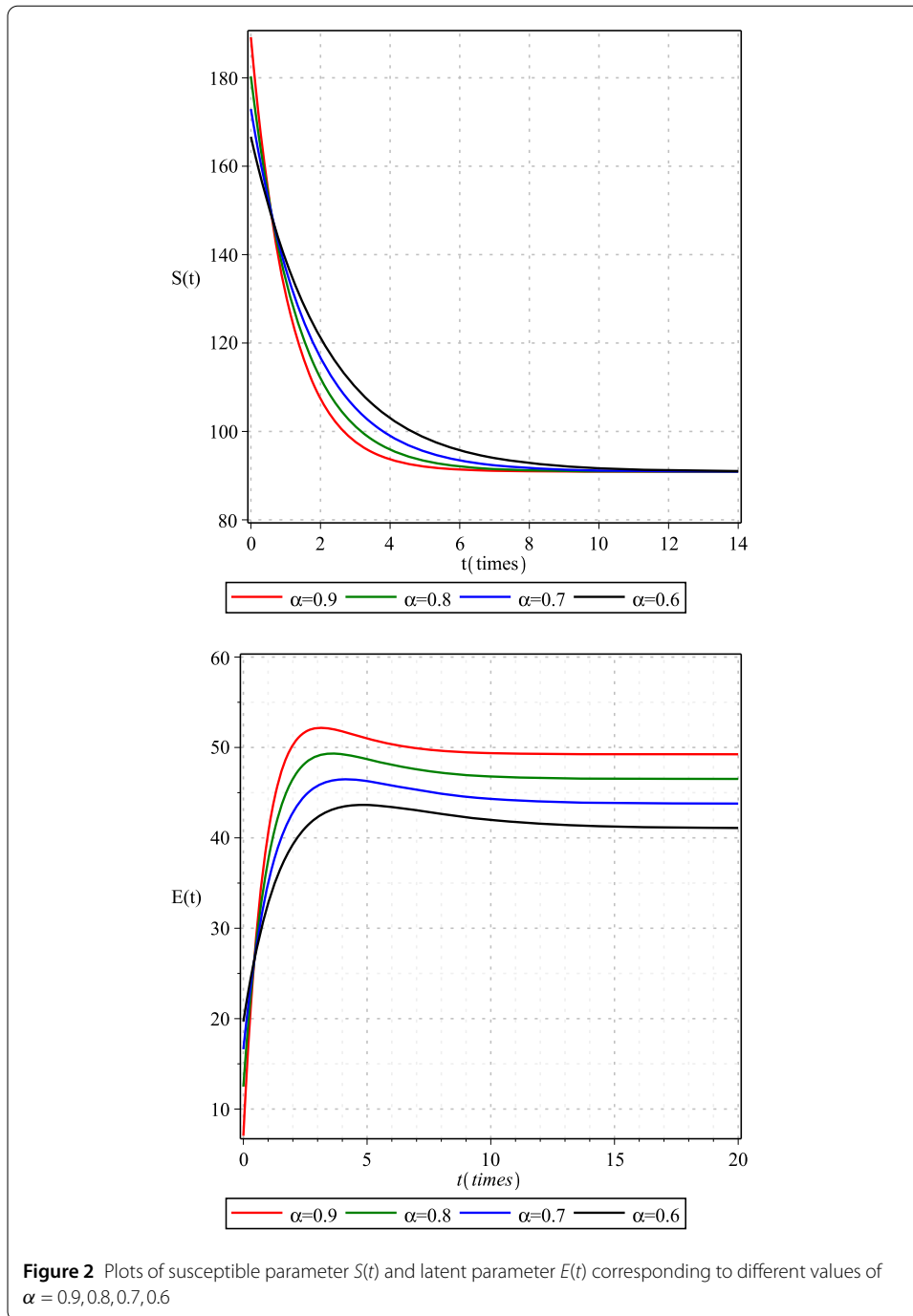\end{aligned}
$$
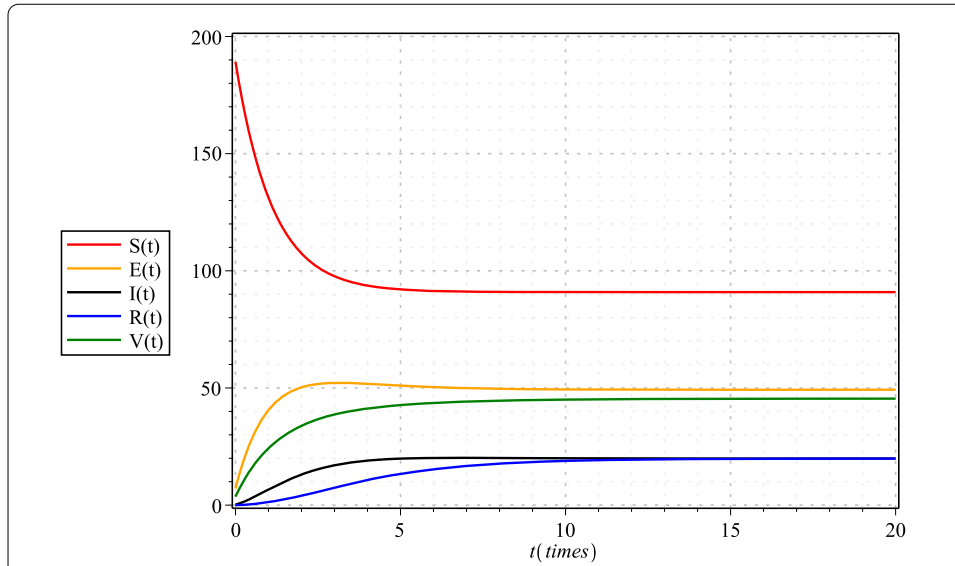
Figure 1 Plots of all variables in model (3) with $\alpha = 0.9$

$$- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) f_3(S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}, V_{j-1})$$

$$+ \left(\frac{5\alpha h}{12M(\alpha)} \right) f_3(S_{j-2}, E_{j-2}, I_{j-2}, R_{j-2}, V_{j-2}),$$

$$R_{j+1} = R_j + \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f_4(S_j, E_j, I_j, R_j, V_j)$$

$$- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) f_4(S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}, V_{j-1})$$

$$+ \left(\frac{5\alpha h}{12M(\alpha)} \right) f_4(S_{j-2}, E_{j-2}, I_{j-2}, R_{j-2}, V_{j-2}),$$

$$V_{j+1} = V_j + \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)} \right) f_5(S_j, E_j, I_j, R_j, V_j)$$

$$- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)} \right) f_5(S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}, V_{j-1})$$

$$+ \left(\frac{5\alpha h}{12M(\alpha)} \right) f_5(S_{j-2}, E_{j-2}, I_{j-2}, R_{j-2}, V_{j-2}).$$

For numerical simulations, we utilize the values of the parameters $B = 100$, $P = 0.3$, $\lambda = 0.4$, $\mu = 0.4$, $\sigma = 0.3$, $\beta = 0.4$, $D = 0.2$, and the initial conditions are given by $S_0 = 300$, $E_0 = 0$, $I_0 = 0$, $R_0 = 0$, $V_0 = 0$.

The equilibrium point is $E^* = (S^*, E^*, I^*, R^*, V^*) = (90.9, 49.24, 19.87, 21.78, 45.454)$. Fig. (1) shows the plots of the solutions of model (3) for $\alpha = 0.9$. As can be seen, the system is stable at equilibrium point. Figures 2–4 show plots for $S(t), E(t), I(t), R(t), V(t)$ in model (3) for the fractional orders $\alpha = 0.9, 0.8, 0.7, 0.6$, respectively. We can observe from these plots that the curves of each variable have the same trend when α is changed. However, their values are slightly different. We can observe from Fig. 2 that the curves of $S(t)$ are decreasing, and they finally converge to the equilibrium point $S^* = 90.9$. Figure 2 shows that all the graphs of $E(t)$ increase with time and tend to the equilibrium point $E^* = 49.24$.

Figure 2 Plots of susceptible parameter $S(t)$ and latent parameter $E(t)$ corresponding to different values of $\alpha = 0.9, 0.8, 0.7, 0.6$

Figures 3 and 4 show that all graphs of $I(t)$, $R(t)$, and $V(t)$ increase with time and then converge to the equilibrium points $I^* = 19.87$, $R^* = 21.78$, and $V^* = 45.454$, respectively.

Tables 1–5 present the comparative study between the standard derivative, Caputo derivatives, and the Caputo–Fabrizio derivative. It can easily be observed from Tables 1–5 that the Caputo–Fabrizio fractional derivative shows the new nature compared to the standard derivative and Caputo fractional derivative. The graphical representation shows that the model depends notably on the fractional order. Figures 2–4 and Tables 1–5 show the clear difference at different values of α.
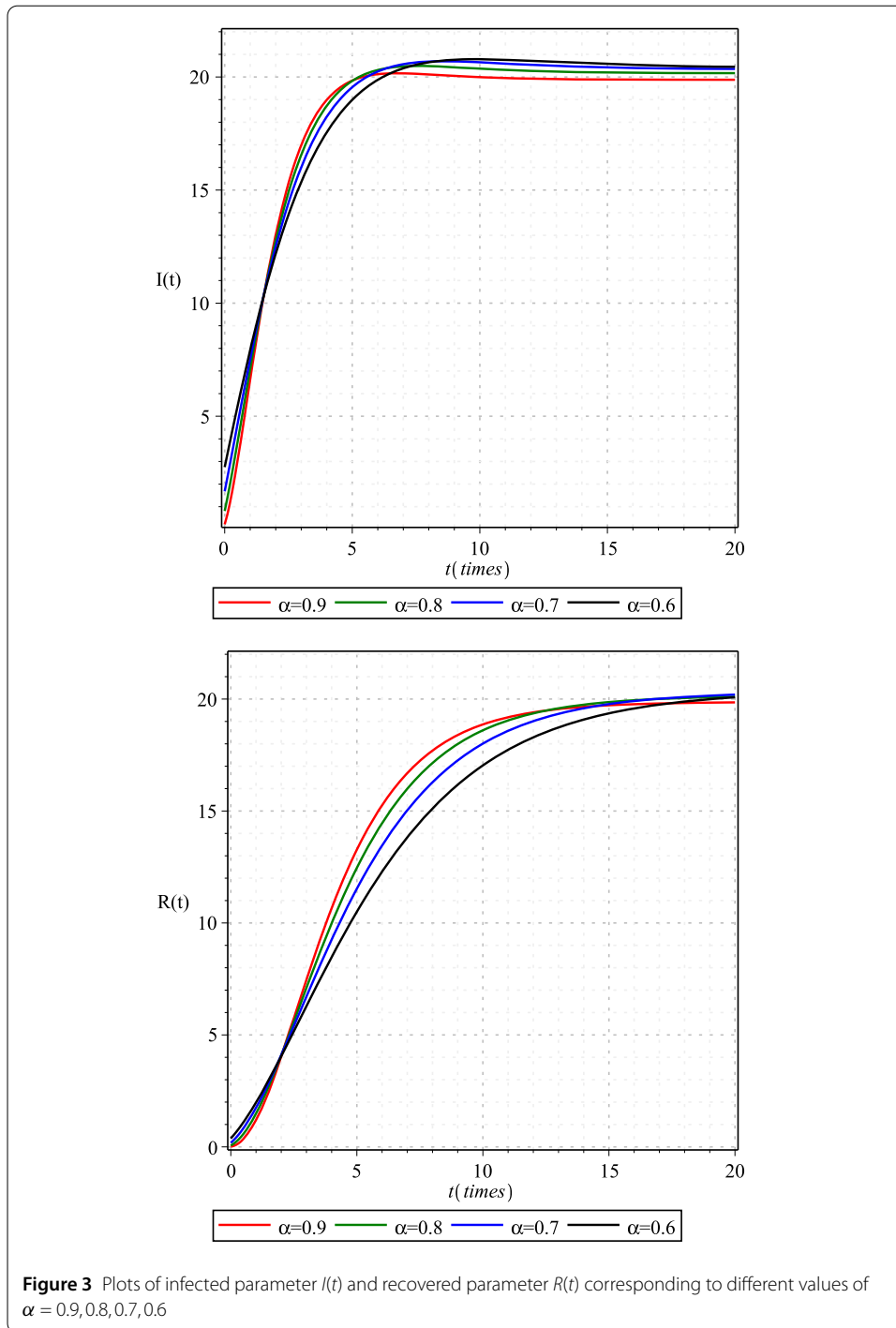
Figure 3 Plots of infected parameter *I*(*t*) and recovered parameter *R*(*t*) corresponding to different values of $\alpha = 0.9, 0.8, 0.7, 0.6$

5 Conclusion

In this paper, we have investigated a Caputo–Fabrizio fractional differential equation model for the spread of rubella disease. Using fixed point theory, we have demonstrated the existence of a unique solution. Also, we have determined the equilibrium point of the model and investigated its stability. We have used a three-step fractional Adams–Bashforth scheme to obtain numerical results of the fractional system of rubella model. Eventually, we have presented the numerical simulations for different values of the

Figure 4 Plots of vaccinated parameter $V(t)$ corresponding to different values of $\alpha = 0.9, 0.8, 0.7, 0.6$
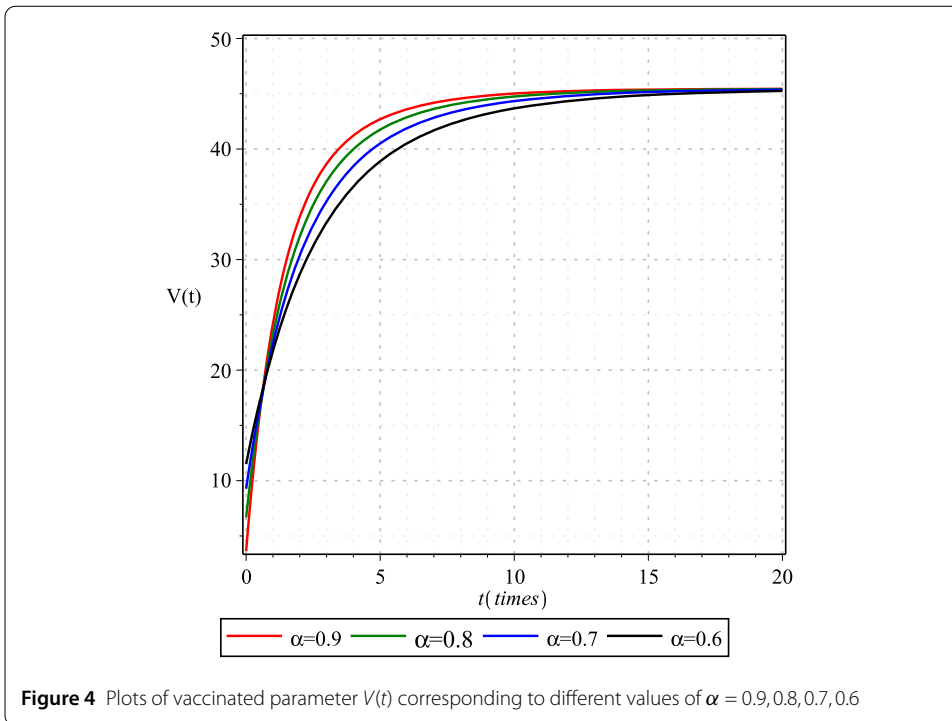
Table 1 Comparison between the standard derivative, Caputo fractional derivatives, and Caputo–Fabrizio fractional derivative for $S(t)$

	t					
	0	1	2	3	4	5
D^{α} ($\alpha = 1$)	50	76.4116	86.4888	89.1413	90.1514	90.6258
$^{c}D^{\alpha}$ ($\alpha = 0.95$)	50	77.8345	85.689	88.4888	91.1344	92.678
$^{cf}D^{\alpha}$ ($\alpha = 0.95$)	52.1327	77.5116	86.47658	89.9403	90.9814	91.0058

Table 2 Comparison between the standard derivative, Caputo fractional derivatives, and Caputo–Fabrizio fractional derivative for $E(t)$

	t					
	0	1	2	3	4	5
D^{α} ($\alpha = 1$)	9.5612	12.0014	9.1120	6.021	3.998	2.8791
$^{c}D^{\alpha}$ ($\alpha = 0.95$)	9.8721	13.9864	10.2341	7.234	4.5432	3.5402
$^{cf}D^{\alpha}$ ($\alpha = 0.95$)	10.5704	13.0185	9.9593	6.7502	4.4770	3.0694

Table 3 Comparison between the standard derivative, Caputo fractional derivatives, and Caputo–Fabrizio fractional derivative for $I(t)$

	t					
	0	1	2	3	4	5
D^{α} ($\alpha = 1$)	8.6753	7.84	6.3251	3.987	3.001	2.765
$^{c}D^{\alpha}$ ($\alpha = 0.95$)	8.9674	7.983	5.372	3.865	2.0216	3.2157
$^{cf}D^{\alpha}$ ($\alpha = 0.95$)	9.6789	7.6784	5.934	3.213	2.8764	2.1451

fractional-order $\alpha = 0.9, 0.8, 0.7, 0.6$ and have compared the numerical results of the standard derivative with two fractional derivatives of rubella model for $\alpha = 0.95$.

Table 4 Comparison between the standard derivative, Caputo fractional derivatives, and Caputo–Fabrizio fractional derivative for $R(t)$

	t					
	0	1	2	3	4	5
D^α ($\alpha = 1$)	2.15	18.6573	28.1124	44.8761	69.9672	107.8351
$^c D^\alpha$ ($\alpha = 0.95$)	2.1511	18.9631	29.6782	47.9246	70.5361	110.1133
$^{cf} D^\alpha$ ($\alpha = 0.95$)	2.1731	19.8653	28.9632	45.3197	68.5167	108.3261

Table 5 Comparison between the standard derivative, Caputo fractional derivatives, and Caputo–Fabrizio fractional derivative for $V(t)$

	t					
	0	1	2	3	4	5
D^α ($\alpha = 1$)	19.1246	36.8261	56.9181	85.2111	125.6753	186.7658
$^c D^\alpha$ ($\alpha = 0.95$)	19.5831	37.9461	59.7263	88.0051	129.8584	190.6483
$^{cf} D^\alpha$ ($\alpha = 0.95$)	20.8921	37.6961	58.1155	86.5283	127.6918	187.9709

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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