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On substantial fractional difference operator

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Abstract

In this article, new definitions of fractional substantial sum and difference operators are introduced. Some properties are established and used to generate a fixed point operator for an arbitrary real order substantial system with conditions involving fractional order difference. We inspect solution existence and Ulam–Hyers–Rassias stability. A property missing in the literature for delta Laplace transform, namely delta exponential shift, is established and used for finding delta Laplace transform for the newly introduced substantial fractional sum and difference.

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1 Introduction

Substantial fractional order integral and derivative were introduced by Chen and Deng [1] in recent form. The definition in Riemann–Liouville(RL) sense is as follows:

Assume that a function f is $(m-1)$-times continuously-differentiable on interval (a,∞), and mth order derivatives are integrable on some finite subinterval of $[a,\infty)$, where $m-1<\mu<m$ for a positive integer m. Furthermore, assume σ to be a constant, then $D_s^{\mu}f(x)=D_s^m[I_s^{(m-\mu)}f(x)]$, where

$$D_s^m=\left(\frac{\partial}{\partial x}+\sigma\right)^m,\quad \text{and}\quad I_s^{(m-\mu)}=\int_{y=a}^{y=x}\frac{(x-y)^{m-\mu-1}}{\Gamma(m-\mu)}e^{-\sigma(x-y)}f(y)\,dy.$$

Chen and Deng discussed some useful composition properties of substantial fractional integral and derivative in [1]. The majority of researchers considered fractional substantial and tempered derivative with an extensive variety of applications in physics, for instance, we refer to a few of them [2–7]. It is assumed that substantial calculus and tempered fractional calculus are equivalent concepts. Cao et al. [8] presented the fact that the expression of fractional order tempered integral and derivative is similar to that of fractional order substantial integral and derivative respectively, but they are different in nature. However, tempered derivative becomes a special case of substantial derivative for nonnegative values of parameter σ. These operators arise from unassociated physical phenomenon. Mathematically arbitrary order substantial calculus is defined on time and space, but the tempered calculus is different from couple of time and space. However, arbitrary order tempered integral and derivative are mostly utilized in truncated exponential power law phenomenon.

A variety of results that helped in developing the theory of discrete fractional calculus are given in [9–22]. Atici and Eloe carefully evoked the interest in theory of fractional difference (FΔ). Abdeljawad defined FΔ with different types of kernel having discrete power law [23, 24], with discrete exponential and generalized Mittag-Leffler functions [15, 25], with discrete exponential and Mittag-Leffler functions on generalized $h\mathbb{Z}$ time scale [26], and kernel containing product of both power law and exponential function in [27]. A delta Laplace transform has been developed and studied in [17, 20, 21, 28]. However, an important shifting property is missing in this setting. Only few simple cases have been addressed by implication of the definition (see Theorem 2.10 and Theorem 2.11 in [20]). Our proposed shifting property is a modest attempt to fill that void.

First we define the product $e_{c_1}(x,a)e_{c_2}(y,a)$ for $x,y \in \mathbb{N}_a$ as a solution of the delta partial difference problem

$$\begin{aligned} &c_2\Delta_x u(x,y) - c_1\Delta_y u(x,y) = 0, \quad \text{with} \\ &u(x,a) = e_{c_1}(x,a), \qquad u(a,y) = e_{c_2}(y,a), \end{aligned}$$

where $c_1, c_2 \in \mathcal{R}$ for the set of regressive functions $\mathcal{R}$. Note that the product of two exponential functions in continuous calculus enjoys the exponent law, namely $e^{c_1 x}e^{c_2 y} = e^{c_1 x + c_2 y}$. This is the key motivation behind the product of two delta exponential functions in discrete calculus. Surprisingly, analogous result does not hold in general for the discrete case. However, $e_c(x,0)e_c(y,0) = e_c(x+y,0)$ holds for $x,y \in \mathbb{N}_0$. Discrete analogues are practically important and easier to use in real life problems. A brief discussion on the need to study discrete analogue operator is given by Abdeljawad [27]. A discrete version of substantial derivative is a potential candidate to productively describe many physical phenomena. Moreover, in Theorem 3.5 of [29] Lizama derived a relation between RL fractional difference and derivative by applying Poisson transformation. These combined facts allow us to define substantial FΔ in delta fractional setting in the RL sense. By introducing substantial FΔ analogously in the Caputo sense, one can examine the qualitative properties. In discrete setup, Ulam type stability is discussed in [30–32]. Stability analysis can be further explored in the Ulam–Hyers–Rassias sense from [33–36].

The following Cauchy type substantial FΔ problem is discussed in this paper:

$$\begin{cases} {}^s\Delta_a^\nu u(x) + f(x+\nu-1, u(x+\nu-1)) = 0, & \text{for } x \in \mathbb{N}_a, \\ {}^s\Delta^{\nu-i+1} u(x_0 = a+m-\nu) = u_i, & i = 0,1,\ldots,m-1, \end{cases} \tag{1}$$

where $m-1 < \nu \leq m$ with positive integer m. Here we use a type of initial conditions involving noninteger order differences suggested by Heymans and Podlubny [37]. However, these conditions may be converted to whole order conditions by a technique used by Holm [22] in his doctoral dissertation. Physical entity of initial conditions that involves RL derivative has been challenged by few researchers. However, Heymans and Podlubny discussed some expositions and provided a physical interpretation for the initial conditions [37, 38].

This paper is organized in four sections. In Sect. 2, we give some preliminaries from discrete calculus. In the third section, analogous substantial operators are introduced. Composition and delta exponential shift properties along with relations between RL and substantial FΔ are presented. Existence theory and stability conditions are obtained in the last section of the paper.

2 Preliminaries

For convenience, this section comprises some basic definitions and results for later use in the sequel. The sets considered in this paper are $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$, $\mathbb{N}_a^T := \{a, a+1, a+2, \ldots, T\}$, and $[a,T]_{\mathbb{N}_a} := [a,T] \cap \mathbb{N}_a$ for fixed $a, T \in \mathbb{R}$. For $t \in \mathbb{N}_a$, the jump operator is given by $\sigma(t) = t+1$.

Definition 2.1 ([28]) Suppose that $f : \mathbb{N}_a \to \mathbb{R}$, the delta Laplace transform based on a, is given as

$$\mathcal{L}_a\{f\}(y) = \int_a^\infty e_{\ominus y}\big(\sigma(x), a\big) f(x) \Delta x$$

for complex numbers $y \neq -1$.

The following concepts are discussed in [20]. The set $\mathcal{R} = \{p_i : 1 + p_i(x) \neq 0\}$ for $x \in \mathbb{N}_a$. The circle plus sum of $p_1, p_2 \in \mathcal{R}$ is given by $p_1 \oplus p_2 = p_1 + p_2 + p_1 p_2$ and $\ominus p_1(x) = \frac{-p_1(x)}{1+p_1(x)}$ for $x \in \mathbb{N}_a$. $h_\mu(t,s) = \frac{(t-s)^{\underline{\mu}}}{\Gamma(\mu+1)}$ is the Taylor monomial with numerator as a falling function. The definition of delta exponential function for regressive function can be found in [20]. We only refer to an example.

Example 2.2 ([20]) If $p_1(x) = c$ is a constant such that $c \in \mathcal{R}$ (that is, $c \neq -1$), then a delta exponential function for constant is given by $e_{p_1}(x,s) = e_c(x,s) = [1+c]^{x-s}$ for $x \in \mathbb{N}_a$. In particular, for the initial point of the domain of definition $s = a$, we have

$$e_c(x,a) = [1+c]^{x-a} \quad \text{for } x \in \mathbb{N}_a.$$

Lemma 2.3 ([20]) *Assume $p(x) \in \mathcal{R}$. Then $\Delta_x e_{p(x)}(x,y) = p(x) e_{p(x)}(x,y)$.*

Definition 2.4 ([20]) Suppose $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$. Then $\Delta_a^{-\mu} f(x) := \sum_{\tau=a}^{x-\mu} h_{\mu-1}(x, \sigma(\tau)) f(\tau)$ for $x \in \mathbb{N}_{a+\mu}$.

Definition 2.5 ([17, 39]) Suppose $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$. Then $\Delta_a^{\mu} f(x) = \sum_{\tau=a}^{x+\mu} h_{-\mu-1}(x, \sigma(\tau)) \times f(\tau)$, where $x \in \mathbb{N}_{a+m-\mu}$.

Lemma 2.6 ([20]) *Suppose $\nu \geq 0$ and $\mu > 0$. The power law $\Delta_{a+\nu}^{-\mu} (x-a)^{\underline{\nu}} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \times (x-a)^{\underline{\mu+\nu}}$ for $x \in \mathbb{N}_{a+\mu+\nu}$.*

Lemma 2.7 ([22]) *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $m-1 < \mu < m$, where m and k are positive integers. Then $[\Delta^k(\Delta_a^{-\mu} f)](x) = (\Delta_a^{k-\mu} f)(x)$ for $x \in \mathbb{N}_{a+\mu}$.*

Lemma 2.8 ([20] [Leibniz formula]) *Assume $f : \mathbb{N}_{a+\mu} \times \mathbb{N}_a \to \mathbb{R}$ and $\mu > 0$. Then, for $x \in \mathbb{N}_{a+\mu}$, $\Delta \sum_{\tau=a}^{x-\mu} f(x,\tau) = \sum_{\tau=a}^{x-\mu} \Delta_x f(x,\tau) + f(x+1, x-\mu+1)$.*

Lemma 2.9 ([20]) *Assume two functions $v, w : \mathbb{N}_a \to \mathbb{R}$. Let $b_1, b_2 \in \mathbb{N}_a$ such that $b_1 < b_2$. Then we have the summation by parts formula $\sum_{b_1}^{b_2} v(\sigma(t))\Delta w(t)\Delta t = v(t)w(t)|_{b_1}^{b_2+1} - \sum_{b_1}^{b_2} w(t)\Delta v(t)\Delta t$.*

Lemma 2.10 ([20]) *Assume that $g : \mathbb{N}_\ell^m \to \mathbb{R}$ with indefinite sum of g is equal to G on $\mathbb{N}_\ell^{m+1}$. Subsequently*

$$\sum_{t=\ell}^{m} g(t) = \sum_{t=\ell}^{m} \Delta G(t) = G(m+1) - G(\ell).$$

Lemma 2.11 ([20]) *Assume $c_1, c_1 \in \mathcal{R}$ and $x \in \mathbb{N}_a$. Then $e_{c_1}(x,a)e_{c_2}(x,a) = e_{c_1 \oplus c_2}(x,a)$.*

Lemma 2.12 ([20]) *Assume $f, g : \mathbb{N}_a^b \to \mathbb{R}$. Then, for $x \in \mathbb{N}_a^{b-1}$,*

$$\Delta\big[f(x)g(x)\big] = f\big(\sigma(x)\big)\Delta g(x) + \big[\Delta f(x)\big]g(x).$$

Lemma 2.13 ([20]) *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$ and $\mu > 0$.*

$$\text{Then, for } |y+1| > r, \quad \text{we have, } \mathcal{L}_{a+\mu}\big\{\Delta_a^{-\mu} f\big\}(y) = \frac{(y+1)^\mu}{y^\mu}\tilde{F}_a(y).$$

Lemma 2.14 ([20]) *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and $m-1 < \mu < m$ with positive integer m. Then, for $|y+1| > r$,*

$$\mathcal{L}_{a+m-\mu}\big\{\Delta_a^\mu f\big\}(y) = y^\mu(y+1)^{m-\mu}\tilde{F}_a(y) - \sum_{j=0}^{m-1} y^j \Delta_a^{\mu-1-j} f(a+m-\mu).$$

The definitions of Ulam stability for FΔ equations were introduced in [31]. For positive ϵ and $x \in [a, T]_{\mathbb{N}_a}$,

$$\big|{}^s\Delta_a^\nu v(x) + f\big(\rho(x)+\nu, v\big(\rho(x)+\nu\big)\big)\big| \leq \epsilon, \tag{2}$$

$$\big|{}^s\Delta_a^\nu v(x) + f\big(\rho(x)+\nu, v\big(\rho(x)+\nu\big)\big)\big| \leq \epsilon\psi\big(\rho(x)+\nu\big), \tag{3}$$

$$\|v-u\| \leq \epsilon d_f, \tag{4}$$

$$\|v-u\| \leq \epsilon\psi(x)d_{f,\psi}, \tag{5}$$

where $\psi : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$. For some positive $d_f \in \mathbb{R}$, if v satisfies (2) and (4), then u satisfying (1) is stable in the Ulam–Hyers sense. For generalized Ulam–Hyers stability, we replace ϵd_f with $\phi_f(\epsilon) \in C(R^+, R^+)$ such that $\phi_f(0) = 0$. Further, for some positive $d_{f,\psi} \in \mathbb{R}$, if v satisfies (3) and (5), then u satisfying (1) is stable in the Ulam–Hyers–Rassias sense. For generalized Ulam–Hyers–Rassias stability, we replace $\epsilon\psi(x)$ with $\Phi(x)$ in (3) and (5).

3 Substantial fractional sum and difference

Lizama [29] considered abstract FΔ equations with the kernel of Poisson distribution. To define fractional substantial sum, here we use the same kernel in the discrete setting, specifically by using the delta exponential and Taylor monomial on a discrete time scale similar to [27].

Definition 3.1 Assume $f:\mathbb{N}_a \to \mathbb{R}$, $0<\mu\in\mathbb{R}$, and a constant $-p\in\mathcal{R}$. Subsequently substantial sum of order μ is given as

$$^{s}\Delta_a^{-\mu} f(x) := \sum_{\tau=a}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)e_{-p}(x-\tau,0)f(\tau)$$

for $x\in\mathbb{N}_{a+\mu}$.

Definition 3.2 Assume $f:\mathbb{N}_a \to \mathbb{R}$, $m-1<\mu\leq m$ with positive integer m and a constant $-p\in\mathcal{R}$. Then, for $x\in\mathbb{N}_{a+m-\mu}$, the substantial FΔ of f of order μ is defined by $^{s}\Delta^{\mu}f(x) := {}^{s}\Delta^{m}[{}^{s}\Delta_{a+\mu}^{-(m-\mu)}f(x)]$, where $^{s}\Delta^{m} = (\frac{\Delta_x+p}{1-p})^m$, and Δ_x is delta partial difference with respect to x.

Remark 1 Note that, for $p=0$, substantial sum and difference operators reduce to RL sum (Definition 2.4) and RL difference (Definition 2.5), respectively.

Remark 2 Substantial derivative vanishes if and only if all lower integer order substantial derivative vanished (Remark 2.12 in [1]). Unlike in the continuous setting, integer order substantial difference need not be zero for substantial FΔ to be zero. A simple example in the delta discrete setting, where substantial FΔ vanishes, is $f(x)=e_{-p}(x,0)$; however, by using Definition 3.2, we have

$$^{s}\Delta^{0} e_{-p}(x,0) = \left(\frac{\Delta_x+p}{1-p}\right)^0 e_{-p}(x,0) = e_{-p}(x,0),$$

evaluation at $x=0$ is nonzero.

Lemma 3.3 (Composition of fractional sums) *Assume that $f:\mathbb{N}_a\to\mathbb{R}$ and μ, ν are positive real numbers, then for $x\in\mathbb{N}_{a+\mu+\nu}$,*

$$\bigl[{}^{s}\Delta_{a+\nu}^{-\mu}\bigl({}^{s}\Delta_a^{-\nu}f\bigr)\bigr](x) = \bigl({}^{s}\Delta_a^{-(\mu+\nu)}f\bigr)(x) = \bigl[{}^{s}\Delta_{a+\mu}^{-\nu}\bigl({}^{s}\Delta_a^{-\mu}f\bigr)\bigr](x).$$

Proof For $x\in\mathbb{N}_{a+\mu+\nu}$, consider the left-hand side

$$\begin{aligned}
\bigl[{}^{s}\Delta_{a+\nu}^{-\mu}\bigl({}^{s}\Delta_a^{-\nu}f\bigr)\bigr](x) &= \sum_{\tau=a+\nu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)e_{-p}(x-\tau,0)\bigl({}^{s}\Delta_a^{-\nu}f\bigr)(\tau) \\
&= \sum_{\tau=\nu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)e_{-p}(x-\tau,0)\sum_{w=0}^{\tau-\mu} h_{\nu-1}\bigl(\tau,\sigma(w)\bigr) \\
&\quad\times e_{-p}(\tau-w,0)f(w) \\
&= \sum_{\tau=\nu}^{x-\mu}\sum_{w=0}^{\tau-\mu} e_{-p}(x-w,0)\frac{(x-\sigma(\tau))^{\underline{\mu-1}}}{\Gamma(\mu)}\frac{(\tau-\sigma(w))^{\underline{\nu-1}}}{\Gamma(\nu)}f(w) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)}\sum_{w=0}^{x-(\mu+\nu)} e_{-p}(x-w,0)\sum_{\tau=w+\nu}^{x-\mu}\bigl(x-\sigma(\tau)\bigr)^{\underline{\mu-1}} \\
&\quad\times\bigl(\tau-\sigma(w)\bigr)^{\underline{\nu-1}}f(w).
\end{aligned}$$

Let $\tau - \sigma(w) = y$,

$$\begin{aligned}\left[{}^{s}\Delta_{a+\nu}^{-\mu}\left({}^{s}\Delta_{a}^{-\nu}f\right)\right](x) &= \frac{1}{\Gamma(\mu)\Gamma(\nu)}\sum_{w=0}^{x-(\mu+\nu)} e_{-p}(x-w,0) \\ &\quad\times \sum_{y=\nu-1}^{x-\mu-w-1}(x-y-w-2)^{\underline{\mu-1}}(y)^{\underline{\nu-1}}f(w),\end{aligned}$$

$$\begin{aligned}\left[{}^{s}\Delta_{a+\nu}^{-\mu}\left({}^{s}\Delta_{a}^{-\nu}f\right)\right](x) &= \frac{1}{\Gamma(\nu)}\sum_{w=0}^{x-(\mu+\nu)} e_{-p}(x-w,0) \\ &\quad\times \left[\frac{1}{\Gamma(\mu)}\sum_{y=\nu-1}^{x-\mu-w-1}\left(x-w-1-\sigma(y)\right)^{\underline{\mu-1}}(y)^{\underline{\nu-1}}\right]f(w).\end{aligned}$$

By using Definition 2.4, we get

$$\left[{}^{s}\Delta_{a+\nu}^{-\mu}\left({}^{s}\Delta_{a}^{-\nu}f\right)\right](x) = \frac{1}{\Gamma(\nu)}\sum_{w=0}^{x-(\mu+\nu)} e_{-p}(x-w,0)\left[\Delta_{\nu-1}^{-\mu}x^{\underline{\nu-1}}\right]_{x\to x-w-1}f(w).$$

By Lemma 2.6 we have $\Delta_{\nu-1}^{-\mu}x^{\underline{\nu}} = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}x^{\underline{\mu+\nu-1}}$, which yields the following:

$$\begin{aligned}\left[{}^{s}\Delta_{a+\nu}^{-\mu}\left({}^{s}\Delta_{a}^{-\nu}f\right)\right](x) &= \sum_{w=0}^{x-(\mu+\nu)} e_{-p}(x-w,0)\left[\frac{1}{\Gamma(\mu+\nu)}(x-w-1)^{\underline{\mu+\nu-1}}\right]f(w) \\ &= \sum_{w=0}^{x-(\mu+\nu)} h_{\mu+\nu-1}\left(x,\sigma(w)\right)e_{-p}(x-w,0)f(w) \\ &= \left({}^{s}\Delta_{a}^{-(\mu+\nu)}f\right)(x)\end{aligned}$$

for $x \in \mathbb{N}_{a+\mu+\nu}$. We may interchange μ and ν to get

$$\left[{}^{s}\Delta_{a+\mu}^{-\nu}\left({}^{s}\Delta_{a}^{-\mu}f\right)\right](x) = \left({}^{s}\Delta_{a}^{-(\mu+\nu)}f\right)(x).$$ □

Lemma 3.4 (Left inverse property) *Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\mu > 0$ such that, for positive integer m, $m-1 < \mu < m$. Then, for $x \in \mathbb{N}_{a+\mu}$,*

$$\left[{}^{s}\Delta^{\mu}\left({}^{s}\Delta_{a}^{-\mu}f\right)\right](x) = f(x).$$

Proof First, by induction, we prove the identity for integer m

$${}^{s}\Delta^{m}\left\{{}^{s}\Delta_{a}^{-m}f(x)\right\} = f(x). \tag{6}$$

Consider the base case for $m = 1$

$${}^{s}\Delta\left\{{}^{s}\Delta_{a}^{-1}f(x)\right\} = \left(\frac{\Delta_x + p}{1-p}\right)\left[\sum_{\tau=a}^{x-1} h_0\left(x,\sigma(\tau)\right)e_{-p}(x-\tau,0)f(\tau)\right].$$

Since $h_0(x,\sigma(\tau)) = 1$, therefore

$$ {}^{s}\Delta\left\{{}^{s}\Delta_a^{-1}f(x)\right\} = \frac{\Delta_x}{1-p}\left[\sum_{\tau=a}^{x-1} e_{-p}(x-\tau,0)f(\tau)\right] + \frac{p}{1-p}\left[\sum_{\tau=a}^{x-1} e_{-p}(x-\tau,0)f(\tau)\right]. $$

Now, applying the Leibniz formula (Lemma 2.8) on the first bracket, we obtain

$$ \begin{aligned} {}^{s}\Delta\left\{{}^{s}\Delta_a^{-1}f(x)\right\} &= \frac{1}{1-p}\left[\sum_{\tau=a}^{x-1} \Delta_x e_{-p}(x-\tau,0)f(\tau) + e_{-p}(x+1-x,0)f(x)\right] \\ &\quad + \frac{p}{1-p}\left[\sum_{\tau=a}^{x-1} e_{-p}(x-\tau,0)f(\tau)\right] \\ &= \frac{1}{1-p}\left[\sum_{\tau=a}^{x-1} -pe_{-p}(x-\tau,0)f(\tau) + (1-p)^{1-0}f(x)\right] \\ &\quad + \frac{p}{1-p}\left[\sum_{\tau=a}^{x-1} e_{-p}(x-\tau,0)f(\tau)\right] = f(x). \end{aligned} $$

Assume that the statement in Equation (6) is true for m. For induction step, consider

$$ \begin{aligned} {}^{s}\Delta^{m+1}{}^{s}\Delta_a^{-(m+1)}f(x) &= {}^{s}\Delta^{m+1}\left\{{}^{s}\Delta_{a+m}^{-1}\,{}^{s}\Delta_a^{-m}\right\}f(x) \\ &= {}^{s}\Delta^{m}\left\{{}^{s}\Delta\,{}^{s}\Delta_{a+m}^{-1}\right\}{}^{s}\Delta_a^{-m}f(x) \\ &= {}^{s}\Delta^{m}{}^{s}\Delta_a^{-m}f(x) = f(x). \end{aligned} $$

For positive integer m and $m-1 < \mu \le m$, we have

$$ {}^{s}\Delta^{\mu}\left[{}^{s}\Delta_a^{-\mu}f(x)\right] = {}^{s}\Delta^{m}\left\{{}^{s}\Delta_{a+\mu}^{-(m-\mu)}\right\}\left[{}^{s}\Delta_a^{-\mu}f(x)\right]. $$

Finally, using Lemma 3.3, we arrive at

$$ {}^{s}\Delta^{\mu}\left[{}^{s}\Delta_a^{-\mu}f(x)\right] = {}^{s}\Delta^{m}\left\{{}^{s}\Delta_a^{-m}f(x)\right\} = f(x). \qquad \square $$

Lemma 3.5 (Composition of sum with difference) *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$, and $k \in \mathbb{N}_0$. Then, for $x \in \mathbb{N}_{a+\mu}$,*

$$ \begin{aligned} \left[{}^{s}\Delta_a^{-\mu}\left({}^{s}\Delta^k f\right)\right](x) &= \sum_{j=0}^{k}\binom{k}{j}(-p)^{k-j}{}^{s}\Delta^{j-\mu}f(x) - e_{-p}(x-a+1,0) \\ &\quad \times \sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}h_{\mu-j+1}(x,a)\,{}^{s}\Delta^{k-i+1}f(a). \end{aligned} \tag{7} $$

Further let $\nu > 0$ be such that $m-1 < \nu \leq m$ for positive integer m. Then, for $x \in \mathbb{N}_{a+m-\nu+\mu}$,

$$\begin{aligned}\left[{}^{s}\Delta_{a+m-\nu}^{-\mu}\left({}^{s}\Delta^{\nu}f\right)\right](x) &= \sum_{j=0}^{m}\binom{m}{j}(-p)^{m-js}\Delta^{-(\mu-\nu+m-j)}f(x)\\ &\quad - e_{-p}(x-a+1,0)\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}\\ &\quad \times h_{\mu-j+1}(x,a)^{s}\Delta^{\nu-i+1}f(a+m-\nu).\end{aligned} \tag{8}$$

Proof Case I: Suppose $\mu \notin \mathbb{N}_1^{k-1}$. First note by Lemma 2.12 that $\Delta_\tau[h_{\mu-1}(x,\tau)e_{-p}(x-\tau+1,0)] = h_{\mu-1}(x,\sigma(\tau))[pe_{-p}(x-\tau,0)] - h_{\mu-2}(x,\sigma(\tau))[e_{-p}(x-\tau+1,0)]$. Now, using Definition 3.1 and applying summation by parts (Lemma 2.9), we have

$$\begin{aligned}{}^{s}\Delta_{a}^{-\mu}\left[{}^{s}\Delta^{k}f(x)\right] &= \sum_{\tau=a}^{x-\mu}h_{\mu-1}\big(x,\sigma(\tau)\big)e_{-p}\big(x-\sigma(\tau),0\big)\left[{}^{s}\Delta^{k}f(\tau)\right]\\ &= h_{\mu-1}(x,\tau)e_{-p}(x-\tau+1,0)^{s}\Delta^{k-1}f(\tau)|_{\tau=a}^{x-\mu+1}\\ &\quad - \sum_{\tau=a}^{x-\mu}\Big[ph_{\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0)\\ &\quad - h_{\mu-2}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau+1,0)\Big]^{s}\Delta^{k-1}f(\tau)\\ &= 1.e_{-p}(\mu,0)^{s}\Delta^{k-1}f(x-\mu+1) - h_{\mu-1}(x,a)e_{-p}(x-a+1,0)\\ &\quad \times {}^{s}\Delta^{k-1}f(a) - p\sum_{\tau=a}^{x-\mu}h_{\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0)^{s}\Delta^{k-1}f(\tau)\\ &\quad + \sum_{\tau=a}^{x-\mu}h_{\mu-2}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau+1,0)^{s}\Delta^{k-1}f(\tau).\end{aligned}$$

Combining the first term with last sum, we have

$$\begin{aligned}{}^{s}\Delta_{a}^{-\mu}\left[{}^{s}\Delta^{k}f(x)\right] &= {}^{s}\Delta_{a}^{-(\mu-1)}\left[{}^{s}\Delta^{k-1}f(x)\right] - p^{s}\Delta_{a}^{-\mu}\left[{}^{s}\Delta^{k-1}f(x)\right]\\ &\quad - h_{\mu-1}(x,a)e_{-p}(x-a+1,0)^{s}\Delta^{k-1}f(a)\\ &= \left(-p^{s}\Delta_{a}^{-\mu} + {}^{s}\Delta_{a}^{-(\mu-1)}\right)\left[{}^{s}\Delta^{k-1}f(x)\right]\\ &\quad - e_{-p}(x-a+1,0)h_{\mu-1}(x,a)^{s}\Delta^{k-1}f(a).\end{aligned}$$

Another application of summation by parts gives

$$\begin{aligned}{}^{s}\Delta_{a}^{-\mu}\left[{}^{s}\Delta^{k}f(x)\right] &= \left(p^{2s}\Delta_{a}^{-\mu} - 2p^{s}\Delta_{a}^{-(\mu-1)} + {}^{s}\Delta_{a}^{-(\mu-2)}\right)\left[{}^{s}\Delta^{k-2}f(x)\right]\\ &\quad - e_{-p}(x-a+1,0)h_{\mu-1}(x,a)^{s}\Delta^{k-1}f(a)\\ &\quad - e_{-p}(x-a+1,0)\big\{-ph_{\mu-1}(x,a) + h_{\mu-2}(x,a)\big\}^{s}\Delta^{k-2}f(a).\end{aligned}$$

Again using summation by parts, we get

$$
\begin{aligned}
{}^s\Delta_a^{-\mu}\big[{}^s\Delta^k f(x)\big] = {} & \big(-p^{3s}\Delta_a^{-\mu} + 3p^{2s}\Delta_a^{-(\mu-1)} - 3p^s\Delta_a^{-(\mu-2)} + {}^s\Delta_a^{-(\mu-3)}\big) \\
& \times \big[{}^s\Delta^{k-3}f(x)\big] - e_{-p}(x-a+1,0)h_{\mu-1}(x,a){}^s\Delta^{k-1}f(a) \\
& - e_{-p}(x-a+1,0)\big\{-ph_{\mu-1}(x,a) + h_{\mu-2}(x,a)\big\}{}^s\Delta^{k-2}f(a) \\
& - e_{-p}(x-a+1,0)\big\{p^2h_{\mu-1}(x,a) - 2ph_{\mu-2}(x,a) + h_{\mu-3}(x,a)\big\} \\
& \times {}^s\Delta^{k-3}f(a).
\end{aligned}
$$

Further $(k-3)$ times application of summation by parts gives

$$
\begin{aligned}
\big[{}^s\Delta_a^{-\mu}\big({}^s\Delta^k f\big)\big](x) = {} & \sum_{j=0}^{k}\binom{k}{j}(-p)^{k-js}\Delta^{j-\mu}f(x) - e_{-p}(x-a+1,0) \\
& \times \sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\mu-j+1}(x,a){}^s\Delta^{k-i+1}f(a),
\end{aligned}
$$

where by assumption $h_{\mu-j+1}(x,a)$ is well defined for $\mu \notin \mathbb{N}_1^{k-1}$.

Case II: Now suppose $\mu \in \mathbb{N}_1^{k-1}$. Then $k-\mu \in \mathbb{N}_1$, we have for $x \in \mathbb{N}_{a+\mu}$

$$
\begin{aligned}
\big[{}^s\Delta_a^{-\mu}\big({}^s\Delta^k f\big)\big](x) &= \big\{{}^s\Delta^{k-\mu s}\Delta_{a+\mu}^{-(k-\mu)}\big\}{}^s\Delta_a^{-\mu s}\Delta^k f(x) \\
&= {}^s\Delta^{k-\mu}\big\{{}^s\Delta_{a+\mu}^{-(k-\mu)s}\Delta_a^{-\mu}\big\}{}^s\Delta^k f(x) \\
&= {}^s\Delta^{k-\mu}\big[{}^s\Delta_a^{-ks}\Delta^k f(x)\big].
\end{aligned}
$$

By Case I and Equation (7) we arrive at

$$
\begin{aligned}
\big[{}^s\Delta_a^{-\mu}\big({}^s\Delta^k f\big)\big](x) = {}^s\Delta^{k-\mu}\Bigg[& \sum_{j=0}^{k}\binom{k}{j}(-p)^{k-js}\Delta^{j-k}f(x) - e_{-p}(x-a+1,0) \\
& \times \sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{k-j+1}(x,a){}^s\Delta^{k-i+1}f(a)\Bigg].
\end{aligned}
$$

Using Lemma 3.3 and Lemma 2.6, we get

$$
\begin{aligned}
\big[{}^s\Delta_a^{-\mu}\big({}^s\Delta^k f\big)\big](x) = {} & \sum_{j=0}^{k}\binom{k}{j}(-p)^{k-js}\Delta^{j-\mu}f(x) - e_{-p}(x-a+1,0) \\
& \times \sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\mu-j+1}(x,a){}^s\Delta^{k-i+1}f(a).
\end{aligned}
$$

Now consider Equation (8) for $x \in \mathbb{N}_{a+m-\nu+\mu}$, where m is a positive integer such that $m-1 < \nu \leq m$ and define $g(x) := {}^s\Delta_a^{-(m-\nu)}f(x)$ on $\mathbb{N}_{a+m-\nu}$, then by Lemma 3.4 we have

$$
{}^s\Delta_{a+m-\nu}^{-\mu}\big({}^s\Delta^\nu f\big)(x) = {}^s\Delta_{a+m-\nu}^{-\mu}\,{}^s\Delta^m g(x).
$$

By using Equation (7), we obtain

$$
\begin{aligned}
{}^s\Delta_{a+m-\nu}^{-\mu}\big({}^s\Delta^{\nu}f\big)(x) &= \sum_{j=0}^{m}\binom{m}{j}(-p)^{m-js}\Delta^{j-\mu}g(x) - e_{-p}(x-a+1,0) \\
&\quad\times\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}h_{\mu-j+1}(x,a)^{s}\Delta^{m-i+1}g(a+m-\nu) \\
&= \sum_{j=0}^{m}\binom{m}{j}(-p)^{m-js}\Delta^{j-\mu s}\Delta_a^{-(m-\nu)}f(x) \\
&\quad - e_{-p}(x-a+1,0)\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}h_{\mu-j+1}(x,a) \\
&\quad\times{}^s\Delta^{m-i+1s}\Delta_a^{-(m-\nu)}f(a+m-\nu) \\
&= \sum_{j=0}^{m}\binom{m}{j}(-p)^{m-js}\Delta^{-(\mu-\nu+m-j)}f(x) - e_{-p}(x-a+1,0) \\
&\quad\times\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}h_{\mu-j+1}(x,a)^{s}\Delta^{\nu-i+1}f(a+m-\nu).
\end{aligned}
$$

□

Lemma 3.6 (Relation between RL and substantial fractional operators) *Assume $f:\mathbb{N}_a\to\mathbb{R}$, $m-1<\mu<m$ with positive integer m and a constant $-p\in\mathcal{R}$. Then*

(i) *${}^s\Delta_a^{-\mu}f(x)=e_{-p}(x,0)\Delta_a^{-\mu}[e_{-p}(-x,0)f(x)]$ for $x\in\mathbb{N}_{a+\mu}$, where ${}^s\Delta_a^{-\mu}$ is a substantial fractional sum operator and $\Delta_a^{-\mu}$ is an RL fractional sum operator;*

(ii) *${}^s\Delta^{\mu}f(x)=e_{-p}(x,0)\Delta_{a+\mu}^{\mu}[e_{-p}(-x,0)f(x)]$ for $x\in\mathbb{N}_{a+m-\mu}$, where ${}^s\Delta^{\mu}$ is a substantial $F\Delta$ operator and $\Delta_{a+\mu}^{\mu}$ is an RL $F\Delta$ operator.*

Proof (i) Note that $e_{-p}(x-\tau,0)=e_{-p}(x,0)e_{-p}(-\tau,0)$. For $x\in\mathbb{N}_{a+\mu}$, consider

$$
\begin{aligned}
{}^s\Delta_a^{-\mu}f(x) &= \sum_{\tau=a}^{x-\mu}h_{\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0)f(\tau) \\
&= e_{-p}(x,0)\sum_{\tau=a}^{x-\mu}h_{\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(-\tau,0)f(\tau).
\end{aligned}
$$

By Definition 2.4

$$
{}^s\Delta_a^{-\mu}f(x)=e_{-p}(x,0)\Delta_a^{-\mu}\big[e_{-p}(-x,0)f(x)\big].
$$

(ii) For $x\in\mathbb{N}_{a+m-\mu}$, consider

$$
\begin{aligned}
{}^s\Delta^{\mu}f(x) &= {}^s\Delta^{m}\big[{}^s\Delta_{a+\mu}^{-(m-\mu)}f(x)\big] \\
&= \left(\frac{\Delta_x+p}{1-p}\right)^{m}\sum_{\tau=a+\nu}^{x+\mu-m}h_{m-\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0)f(\tau) \\
&= \left(\frac{\Delta_x+p}{1-p}\right)^{m-1}\left[\left(\frac{\Delta_x+p}{1-p}\right)\sum_{\tau=a+\nu}^{x+\mu-m}h_{m-\mu-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0)f(\tau)\right],
\end{aligned}
$$

$$
\begin{aligned}
{}^{s}\Delta^{\mu} f(x) &= \left(\frac{\Delta_x + p}{1-p}\right)^{m-1} \left[\frac{\Delta_x}{1-p} \left\{ e_{-p}(x,0) \sum_{\tau=a+\nu}^{x+\mu-m} h_{m-\mu-1}\big(x,\sigma(\tau)\big) e_{-p}(-\tau,0) f(\tau) \right\} \right. \\
&\quad \left. + \frac{p}{1-p} {}^{s}\Delta_{a+\mu}^{-(m-\mu)} f(x) \right].
\end{aligned}
$$

By using Lemma 2.12 and Lemma 2.3, we have

$$
\begin{aligned}
{}^{s}\Delta^{\mu} f(x) &= \left(\frac{\Delta_x + p}{1-p}\right)^{m-1} \left[\frac{1}{1-p} \left\{ e_{-p}\big(\sigma(x),0\big) \Delta_x^1 \sum_{\tau=a+\nu}^{x+\mu-m} h_{m-\mu-1}\big(x,\sigma(\tau)\big) \right. \right. \\
&\quad \left. \left. \times e_{-p}(-\tau,0) f(\tau) - p^{s}\Delta_{a+\mu}^{-(m-\mu)} f(x) \right\} + \frac{p}{1-p} {}^{s}\Delta_{a+\mu}^{-(m-\mu)} f(x) \right],
\end{aligned}
$$

using the fact $\frac{e_{-p}(\sigma(x),0)}{1-p} = e_{-p}(x,0)$ and Definition 2.4, we are left with

$$
= \left(\frac{\Delta_x + p}{1-p}\right)^{m-1} \left[e_{-p}(x,0) \Delta_x^1 \Delta_{a+\mu}^{-(m-\mu)} \big\{ e_{-p}(-\tau,0) f(\tau) \big\} \right].
$$

By Lemma 2.7, we get

$$
\begin{aligned}
{}^{s}\Delta^{\mu} f(x) &= \left(\frac{\Delta_x + p}{1-p}\right)^{m-1} \left[e_{-p}(x,0) \Delta_{a+\mu}^{-(m-\mu-1)} \big\{ e_{-p}(-\tau,0) f(\tau) \big\} \right] \\
&= {}^{s}\Delta^{m-1} \left[{}^{s}\Delta_{a+\mu}^{-(m-\mu-1)} f(x) \right].
\end{aligned}
$$

Repeated application of the above process $m-1$ times implies

$$
{}^{s}\Delta^{\mu} f(x) = e_{-p}(x,0) \Delta_{a+\mu}^{\mu} \big[e_{-p}(-x,0) f(x) \big]. \qquad \square
$$

Remark 3 One can find the relation between substantial and Caputo difference by making use of the relation between substantial and RL difference Lemma 3.6, along with the relation given in Theorem 14 [40] for Caputo and RL difference.

Lemma 3.7 *Assume $\mathcal{L}\{f(x)\}(y) = \tilde{F}(y)$, then for $c \in \mathcal{R}$:*

(i) $\mathcal{L}\{e_c(x,a) f(x)\}(y) = \frac{1}{1+c} \tilde{F}(y \ominus c)$, *where* $y \ominus c = \frac{y-c}{1+c}$;

(ii) $\mathcal{L}\{e_c(-x,0) f(x)\}(y) = (1+c) \tilde{F}(y \oplus c)$.

Proof (i) By Definition 2.1 of delta Laplace transform on a,

$$
\mathcal{L}_a \big\{ e_c(x,a) f(x) \big\}(y) = \int_a^{\infty} e_{\ominus y}\big(\sigma(x),a\big) e_c(x,a) f(x) \Delta x.
$$

By Example 2.2 and by additive inverse property,

$$
\mathcal{L}_a \big\{ e_c(x,a) f(x) \big\}(y) = \frac{1}{1+c} \int_a^{\infty} e_{\ominus y}\big(\sigma(x),a\big) e_{\ominus[\ominus c]}\big(\sigma(x),a\big) f(x) \Delta x.
$$

By using Lemma 2.11,

$$
\mathcal{L}_a \big\{ e_c(x,a) f(x) \big\}(y) = \frac{1}{1+c} \int_a^{\infty} e_{\ominus[y \ominus c]}\big(\sigma(x),a\big) f(x) \Delta x.
$$

Again by Definition 2.1 of delta Laplace transform,

$$\mathcal{L}_a\{e_c(x,a)f(x)\}(y) = \frac{1}{1+c}\tilde{F}(y \ominus c).$$

(ii) By using the fact that $e_c(-x,0) = e_{\ominus c}(x,0) = (1+c)e_{\ominus c}(\sigma(x),0)$, one can prove it on the similar line as in part (i). □

Theorem 3.8 *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$ with $\mathcal{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $\mu > 0$. Then, for $|y+1| > r$, we have $\mathcal{L}_{a+\mu}\{{}^s\Delta_a^{-\mu} f\}(y) = (\frac{y+1}{y+p})^\mu \tilde{F}_a(y)$.*

Proof Considering the left-hand side for $-p \in \mathcal{R}$ and using Lemma 3.6(i),

$$\begin{aligned}\mathcal{L}_{a+\mu}\{{}^s\Delta_a^{-\mu} f\}(y) &= \mathcal{L}_{a+\mu}\big[e_{-p}(x,0)\Delta_a^{-\mu}\{e_{-p}(-x,0)f(x)\}\big](y)\\ &= \frac{1}{1-p}\big[\mathcal{L}_{a+\mu}\Delta_a^{-\mu}\{e_{-p}(-x,0)f(x)\}(y)\big]_{y\to\frac{y+p}{1-p}}\\ &= \frac{1}{1-p}\left[\frac{(y+1)^\mu}{y^\mu}\mathcal{L}_a\{e_{-p}(-x,0)f(x)\}(y)\right]_{y\to\frac{y+p}{1-p}}.\end{aligned}$$

In the preceding steps, we used Lemma 3.7(i) and then Lemma 2.13. In the following step we apply Lemma 3.7(ii):

$$\begin{aligned}\mathcal{L}_{a+\mu}\{{}^s\Delta_a^{-\mu} f\}(y) &= \frac{1}{1-p}\left[\frac{(y+1)^\mu}{y^\mu}\{(1-p)\tilde{F}_a(y\oplus(-p))\}\right]_{y\to\frac{y+p}{1-p}}\\ &= \left[\frac{(y+1)^\mu}{y^\mu}\{\tilde{F}_a(y-p-yp)\}\right]_{y\to\frac{y+p}{1-p}}\\ &= \left(\frac{y+1}{y+p}\right)^\mu \tilde{F}_a(y).\end{aligned}$$

□

Theorem 3.9 *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ with $\mathcal{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $m-1 < \mu < m$ with positive integer m. Then, for $|y+1| > r$,*

$$\begin{aligned}\mathcal{L}_{a+m-\mu}\{{}^s\Delta_a^{\mu} f\}(y) &= \frac{(y+p)^\mu}{(1-p)^m}(y+1)^{m-\mu}\{\tilde{F}_a(y)\} - \frac{1}{1-p}\sum_{j=0}^{m-1}\left(\frac{y+p}{1-p}\right)^j\\ &\quad\times\sum_{\tau=a}^{a+m-1-j} h_{-\mu-1}\big(a+m-\mu,\sigma(\tau)\big)e_{-p}(-\tau,0)f(\tau).\end{aligned}$$

Proof Consider the left-hand side for $-p \in \mathcal{R}$ and use Lemma 3.6 (ii) to get

$$\begin{aligned}\mathcal{L}_{a+m-\mu}\{{}^s\Delta_a^{\mu} f\}(y) &= \mathcal{L}_{a+m-\mu}\big[e_{-p}(x,0)\Delta_{a+\mu}^{\mu}\{e_{-p}(-x,0)f(x)\}\big](y)\\ &= \frac{1}{1-p}\big[\mathcal{L}_{a+m-\mu}\Delta_{a+\mu}^{\mu}\{e_{-p}(-x,0)f(x)\}(y)\big]_{y\to\frac{y+p}{1-p}}\\ &= \frac{1}{1-p}\Bigg[y^\mu(y+1)^{m-\mu}\mathcal{L}_a\{e_{-p}(-x,0)f(x)\}(y)\\ &\quad-\sum_{j=0}^{m-1}y^j\{\Delta_a^{\mu-1-j}e_{-p}(-x,0)f(x)\}_{x\to a+m-\mu}\Bigg]_{y\to\frac{y+p}{1-p}}.\end{aligned}$$

In the preceding steps, we used Lemma 3.7(i) and then Lemma 2.14. In the following step we apply Lemma 3.7(ii) and Definition 2.5:

$$\begin{aligned}\mathcal{L}_{a+m-\mu}\{{}^s\Delta_a^\mu f\}(y) = \frac{1}{1-p}\Bigg[& y^\mu (y+1)^{m-\mu}\{(1-p)\tilde{F}_a(y\oplus(-p))\} - \sum_{j=0}^{m-1} y^j \\ & \times \Bigg\{\sum_{\tau=a}^{x+\mu-1-j} h_{-\mu-1}(x,\sigma(\tau))e_{-p}(-\tau,0)f(\tau)\Bigg\}_{x\to a+m-\mu}\Bigg]_{y\to\frac{y+p}{1-p}},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{a+m-\mu}\{{}^s\Delta_a^\mu f\}(y) &= \frac{1}{1-p}\Bigg[y^\mu (y+1)^{m-\mu}\{(1-p)\tilde{F}_a(y-p-yp)\} - \sum_{j=0}^{m-1} y^j \\ &\quad \times \sum_{\tau=a}^{a+m-1-j} h_{-\mu-1}(a+m-\mu,\sigma(\tau))e_{-p}(-\tau,0)f(\tau)\Bigg]_{y\to\frac{y+p}{1-p}} \\ &= \frac{(y+p)^\mu}{(1-p)^m}(y+1)^{m-\mu}\{\tilde{F}_a(y)\} - \frac{1}{1-p}\sum_{j=0}^{m-1}\left(\frac{y+p}{1-p}\right)^j \\ &\quad \times \sum_{\tau=a}^{a+m-1-j} h_{-\mu-1}(a+m-\mu,\sigma(\tau))e_{-p}(-\tau,0)f(\tau).\end{aligned}$$

□

4 Existence uniqueness and stability for Cauchy problem

In order to apply fixed point theory and to set up existence results for a substantial FΔ problem of Cauchy type, to obtain suitable fixed point operators, first we convert the difference equation to the corresponding summation form.

Lemma 4.1 *Assume function $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ and $m-1 < \nu \le m$ for positive integer m. Subsequently, u is the solution of (1) if and only if*

$$\begin{aligned}u(x) &= \frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i \\ &\quad - \frac{1}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}}\,{}^s\Delta_{a+m-\nu}^{-(\nu-m+j)} f(x+\nu-1,u(x+\nu-1)).\end{aligned}$$

The previous lemma can be proved by using Equation (8) of Lemma 3.5. In what follows, we apply Brouwer's theorem [31] to set up sufficient condition for the existence of solutions. The space Z equipped with metric $\|u\| = \sup_{x\in\mathbb{N}_a^T}|u(x)|$ forms a complete norm space, where Z is a collection of all real sequences $u = \{u(x)\}_{x=a}^T$. Define the operator $\mathcal{A} : Z \to Z$ by

$$\begin{aligned}\mathcal{A}u(x) &= \frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i - \frac{1}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} \\ &\quad \times \sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)f(\tau+\nu-1,u(\tau+\nu-1)).\end{aligned}$$

Obviously, the solutions of (1) correspond to the fixed points of $\mathcal{A}$ and vice versa.

Theorem 4.2 *Assume $g : [a, T]_{\mathbb{N}_a} \to \mathbb{R}$ such that $|f(x,u)| \le g(x)|u|$ for a bounded function g and all $u \in Z$. Consequently problem (1) has a solution on Z under the condition*

$$L^* \le (1-p)^m, \tag{9}$$

where $L^* = \sup_{x\in\mathbb{N}_a^T} \sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)g(\tau+\nu-1)$.

Proof For $M > 0$, define the set

$$B = \left\{ u(x) : \left\| u - \frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \binom{i}{j}(-p)^j u_i \right\| \le M \right\}.$$

We start by proving that the operator $\mathcal{A}$ is a self-map on B. For $u \in B$, we have

$$\begin{aligned}
&\left| \mathcal{A}u(x) - \frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \binom{i}{j}(-p)^j u_i \right| \\
&\quad \le \frac{\sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)g(\tau+\nu-1)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} |u-0|.
\end{aligned}$$

Since $\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell} = (1-p)^m$. Taking supremum on both sides, we have

$$\sup_{x\in\mathbb{N}_a^T} \left| \mathcal{A}u(x) - \frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \binom{i}{j}(-p)^j u_i \right| \le \frac{L^* M}{(1-p)^m}.$$

By using inequality (9), we get $\|\mathcal{A}u\| \le M$. From this we deduce that $\mathcal{A}$ maps B into B, and therefore Brouwer's fixed point theorem implies that a fixed point must exist for $\mathcal{A}$. □

Theorem 4.3 *Under assumption (H_1): $|f(x,u) - f(x,v)| \le K|u-v|$ for $K > 0$ and for all $u, v \in Z$ and $x \in [a,T]_{\mathbb{N}_a}$, problem (1) has a unique solution on Z provided*

$$K < \frac{|1-p|^{2m-\nu-j}}{|h_{\nu-m+j}(T, a-\nu+m)|}. \tag{10}$$

Proof For $u, v \in Z$ and $x \in [a,T]_{\mathbb{N}_a}$, we have the estimates

$$\begin{aligned}
\left|\mathcal{A}u(x) - \mathcal{A}v(x)\right| &\le \left| \frac{\sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell}} \right| \\
&\quad \times \left| f\big(\tau+\nu-1, u(\tau+\nu-1)\big) - f\big(\tau+\nu-1, v(\tau+\nu-1)\big) \right| \\
&\le \frac{|h_{\nu-m+j}(x, a-\nu+m)|}{|1-p|^{2m-\nu-j}} K \left| u(\tau+\nu-1) - v(\tau+\nu-1) \right|.
\end{aligned}$$

In the preceding calculation, we have used condition (H_1), $\sum_{\ell=0}^m \binom{m}{\ell}(-p)^{m-\ell} = (1-p)^m$, Lemma 2.10, $\sum_\tau h_{\nu-1}(x,\sigma(\tau)) = -h_\nu(x,\tau)$, and the inequality

$$\left| \sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}\big(x,\sigma(\tau)\big)e_{-p}(x-\tau,0) \right| < \frac{|\sum_{\tau=a+m-\nu}^{x-\nu+m-j} h_{\nu-m+j-1}(x,\sigma(\tau))|}{|1-p|^{m-\nu-j}}.$$

Apply supremum on each part of the inequality

$$\sup_{x\in\mathbb{N}_a^T}\left|\mathcal{A}u(x)-\mathcal{A}v(x)\right| \le \frac{|h_{\nu-m+j}(T,a-\nu+m)|}{|1-p|^{2m-\nu-j}}K\|u-v\|.$$

By taking (10) into account, we obtain $\|\mathcal{A}u-\mathcal{A}v\|\le\|u-v\|$, which proves that $\mathcal{A}$ is a contraction. This implies that the fixed point is unique by the Banach fixed point theorem. □

We next present stability criteria for the Cauchy problem under consideration. Authors also discussed Ulam type stabilities of fractional difference equation with multipoint boundary value problem and for nonlinear Hilfer FΔ Cauchy problem in [30, 41].

Theorem 4.4 *Under assumption* (H_1), *suppose that* $u\in Z$ *satisfies problem* (1) *and* $v\in Z$ *satisfies inequality* (2). *Cauchy problem* (1) *is Ulam–Hyers stable and, therefore, generalized Ulam–Hyers stable for a given value of* K *satisfying inequality* (10).

Proof By Lemma 4.1, for simplicity we can rewrite the solution of IVP (1) as

$$u(x)=w(x)-\sum_{\tau=a+m-\nu}^{x-\nu+m-j}\frac{h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{(1-p)^m}f\bigl(\tau+\nu-1,u(\tau+\nu-1)\bigr), \tag{11}$$

where $w(x)=\frac{e_{-p}(x-a+1,0)h_{\nu-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i$. From inequality (2), for $[a,T]_{\mathbb{N}_a}$, it follows that

$$\begin{aligned}&\left|v(x)-\left(w(x)-\sum_{\tau=a+m-\nu}^{x-\nu+m-j}\frac{h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{(1-p)^m}\right.\right.\\&\qquad\left.\left.\times f\bigl(\tau+\nu-1,v(\tau+\nu-1)\bigr)\right)\right|\le\epsilon.\end{aligned} \tag{12}$$

Using (11) with(12) for $[a,T]_{\mathbb{N}_a}$, we obtain

$$\begin{aligned}\bigl|v(x)-u(x)\bigr| &= \left|v(x)-\left(w(x)-\sum_{\tau=a+m-\nu}^{x-\nu+m-j}\frac{h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{(1-p)^m}\right.\right.\\&\qquad\left.\left.\times f\bigl(\tau+\nu-1,u(\tau+\nu-1)\bigr)\right)\right|\\&\le\left|v(x)-\left(w(x)-\sum_{\tau=a+m-\nu}^{x-\nu+m-j}\frac{h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{(1-p)^m}\right.\right.\\&\qquad\left.\left.\times f\bigl(\tau+\nu-1,v(\tau+\nu-1)\bigr)\right)\right|\\&\qquad+\left|\sum_{\tau=a+m-\nu}^{x-\nu+m-j}\frac{h_{\nu-m+j-1}(x,\sigma(\tau))e_{-p}(x-\tau,0)}{(1-p)^m}\right|\\&\qquad\times\bigl|f\bigl(\tau+\nu-1,v(\tau+\nu-1)\bigr)-f\bigl(\tau+\nu-1,u(\tau+\nu-1)\bigr)\bigr|\\&\le\epsilon+\frac{|h_{\nu-m+j}(x,a-\nu+m)|}{|1-p|^{2m-\nu-j}}K\bigl|v(\tau+\nu-1)-u(\tau+\nu-1)\bigr|.\end{aligned}$$

In the previous calculation, we utilized (H_1) and a similar reasoning applied in Theorem 4.3. Now applying supremum and simplification yields the following:

$$\|v-u\| \leq \frac{\epsilon}{1-\frac{|h_{\nu-m+j}(T,a-\nu+m)|}{|1-p|^{2m-\nu-j}}K} = \epsilon d_f, \quad \text{with } d_f = \frac{1}{1-\frac{|h_{\nu-m+j}(T,a-\nu+m)|}{|1-p|^{2m-\nu-j}}K}.$$

Taking (10) into account, (1) is Ulam–Hyers stable. Making use of $\phi_f(\epsilon) = \epsilon d_f$ and $\phi_f(0) = 0$ suggests that (1) is generalized Ulam–Hyers stable. □

Theorem 4.5 *Under assumption* (H_1), *suppose that* $u \in Z$ *satisfies problem* (1) *and* $v \in Z$ *satisfies inequality* (3). *Cauchy problem* (1) *is Ulam–Hyers–Rassias stable and, therefore, generalized Ulam–Hyers–Rassias stable for a given value of* K *in inequality* (10) *and an arbitrary choice of* $\psi : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$.

The next example is an application of Theorem 4.4.

Example 4.6 Consider the substantial FΔ equation with difference condition:

$$\begin{cases} -{}^{s}\Delta_0^{0.8} u(x) = (x-0.2)u(x-0.2), & x \in [0,10]_{\mathbb{N}_0} \\ {}^{s}\Delta_0^{1.8} u(0.2) = u_0. \end{cases}$$

Since $a = 0$, $\nu = 0.8$, and $T = 10$, therefore $m = 1$, $i = 0$, and $j = 0$. Then, for any $p \neq 1$, we have $K < \frac{|1-p|^{1.2}}{335{,}179.01}$. For instance, if we choose $p = 1001$, then for $K < \frac{1}{84.2}$, the solution to the given problem with inequalities

$$\left|{}^{s}\Delta_0^{0.8} v(x) + (x-0.2)v(x-0.2)\right| \leq \epsilon, \quad x \in [0,10]_{\mathbb{N}_0},$$
$$\left|{}^{s}\Delta_0^{0.8} v(x) + (x-0.2)v(x-0.2)\right| \leq \epsilon\psi(x-0.2), \quad x \in [0,10]_{\mathbb{N}_0}$$

is Ulam–Hyers stable and Ulam–Hyers–Rassias stable for an arbitrary choice of $\psi : [0,10]_{\mathbb{N}_0} \to \mathbb{R}^+$.

5 Conclusion

- Substantial FΔ on a discrete time scale has been presented.
- In its continuous counterpart, this type of derivatives can describe the anomalous diffusion model in a discrete setting.
- The most important delta exponential shift property for delta Laplace transform has been presented and applied.
- Also a relation between Riemann–Liouville and substantial FΔ operator has been constructed and utilized.
- A new class of substantial FΔ equations with initial conditions involving FΔ is investigated.
- The solution existence and stability of Ulam type are studied.

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