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On Hilfer fractional difference operator

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Abstract

In this article, a new definition of fractional Hilfer difference operator is introduced. Definition based properties are developed and utilized to construct fixed point operator for fractional order Hilfer difference equations with initial condition. We acquire some conditions for existence, uniqueness, Ulam–Hyers, and Ulam–Hyers–Rassias stability. Modified Gronwall's inequality is presented for discrete calculus with the delta difference operator.

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1 Introduction

In the topics of discrete fractional calculus a variety of results can be found in [1–16], which has helped to construct theory of the subject. A rigorous intrigue in fractional calculus of differences has been exhibited by Atici and Eloe [3, 5]. They explored characteristics of falling function, a new power law for difference operators, and the composition of sums and differences of arbitrary order. Holm presented advance composition formulas for sums and differences in his dissertation [12].

Hilfer fractional order derivative was introduced in [17]. Hilfer's definition is illustrated as follows: the fractional derivative of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ is

$$D_a^{\mu,\nu} f(x) = \left(I_a^{\nu(1-\mu)} \frac{d}{dx} \left(I_a^{(1-\nu)(1-\mu)} f \right) \right)(x).$$

The special cases are Riemann–Liouville fractional derivative for $\nu = 0$ and Caputo fractional derivative for $\nu = 1$. Furati et al. [18, 19] primarily studied the existence theory of Hilfer fractional derivative and also explained the type parameter ν as interpolation between the Riemann–Liouville and the Caputo derivatives. It generates more types of stationary states and gives an extra degree of freedom on the initial condition.

Hilfer fractional calculus has been examined broadly by a lot of researchers. Some recent studies involving Hilfer fractional derivatives can be found in [20–28]. The majority of the work in discrete fractional calculus is developed as analogues of continuous fractional calculus. Extensive work on Hilfer fractional derivative and on its extensions has been done, namely: Hilfer–Hadamard [29–32], K-fractional Hilfer [33], Hilfer–Prabhakar [34], Hilfer–Katugampola [35], and ψ-Hilfer [36] fractional operator. However, to the best of our knowledge no work is available for Hilfer fractional difference operator in the delta fractional setting. Also formation of fractional difference operator is an important aspect of mathematical interest and numerical formulae as well as the applications. It motivated us to generalize the two existing fractional difference operators namely, Riemann–Liouville and Caputo difference operator in Hilfer's sense.

We started by introducing a generalized difference operator analogous to Hilfer fractional derivative [17]. To keep the interpolative property of Hilfer fractional difference operators, we carefully chose the starting points of fractional sums. Some important composition properties were developed and utilized to construct fixed point operator for a new class of Hilfer fractional nonlinear difference equations with initial condition involving Riemann–Liouville fractional sum. An application of Brouwer's fixed point theorem gave us conditions for the existence of solution for a new class of Hilfer fractional nonlinear difference equations. For the uniqueness of solution, we applied the Banach contraction principle. To solve linear fractional Hilfer difference equation, we used successive approximation method and then defined the discrete Mittag-Leffler function in the delta difference setting. Gronwall's inequality for discrete calculus with the delta difference operator has been modified. An application of Gronwall's inequality has been given for the stability of solution to fractional order Hilfer difference equation with different initial conditions.

In the continuous setting extensive work on Ulam–Hyers–Rassias stability for noninteger order differential equation has been done. The idea of Ulam–Hyers type stability is important for both pure and applied problems; especially in biology, economics, and numerical analysis. Rassias [37] introduced the continuity condition, which produced acceptable stronger results. However, in discrete fractional setting a limited work can be found [38–40]. For Hilfer delta difference equation, conditions have been acquired for Ulam–Hyers and Ulam–Hyers–Rassias stability with illustrative example. Interested reader may find some details on Ulam–Hyers–Rassias stability in [37, 41–43].

In this article, we shall study initial value problem (IVP) for the following Hilfer fractional difference equation. Let $\eta = \mu + \nu - \mu\nu$, then for $0 < \mu < 1$ and $0 \leq \nu \leq 1$, we have

$$\begin{cases} \Delta_a^{\mu,\nu} u(x) + g(x+\mu-1, u(x+\mu-1)) = 0, & \text{for } x \in \mathbb{N}_{a+1-\mu}, \\ \Delta_a^{-(1-\eta)} u(a+1-\eta) = \zeta, \quad \zeta \in \mathbb{R}. \end{cases} \tag{1}$$

In Sect. 2, we state a few basic but important results from discrete calculus. In the third section, a new fractional Hilfer difference operator is introduced which interpolates Riemann–Liouville and Caputo fractional differences; we also develop some important properties of a newly defined operator. Conditions for existence, uniqueness, and Ulam–Hyers stability are obtained in Sect. 4. The last section comprises modification and application of discrete Gronwall's inequality in delta setting.

2 Preliminaries

Some basics from discrete fractional calculus are given for later use in the following sections. The functions we consider are usually defined on the set $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$, where $a \in \mathbb{R}$ is fixed. Sometimes the set $\mathbb{N}_a$ is called isolated time scale. Similarly, the sets $\mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}$ and $[a,b]_{\mathbb{N}_a} := [a,b] \cap \mathbb{N}_a$ [44] for $b = a+k$, $k \in \mathbb{N}_0$. The jump operators $\sigma(t) = t + 1$ and $\rho(t) = t - 1$ are forward and backward, respectively, for $t \in \mathbb{N}_a$. Furthermore, the set $\mathcal{R} = \{p_i : 1 + p_i(x) \neq 0\}$ contains regressive functions.

Definition 2.1 ([45]) Assume that $f : \mathbb{N}_a \to \mathbb{R}$ and $b \le c$ are in $\mathbb{N}_a$, then the delta definite integral is defined by

$$\int_b^c f(x)\Delta x = \sum_{x=b}^{c-1} f(x).$$

Note that the value of integral $\int_b^c f(x)\Delta x$ depends on the set $\{b, b+1, \dots, c-1\}$. Also we adopt the empty sum convention $\sum_{x=b}^{b-k} f(x) = 0$, whenever $k \in \mathbb{N}_1$.

Definition 2.2 ([9]) Assume $\mu > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$. Then the delta fractional sum of f is defined by $\Delta_a^{-\mu} f(x) := \sum_{\tau=a}^{x-\mu} h_{\mu-1}(x, \sigma(\tau)) f(\tau)$ for $x \in \mathbb{N}_{a+\mu}$, where $h_\mu(t,s) = \frac{(t-s)^{\underline{\mu}}}{\Gamma(\mu+1)}$ is μth fractional Taylor monomial based at s and $t^{\underline{\mu}}$ is the generalized falling function.

Lemma 2.3 ([9]) *Assume $\nu \ge 0$ and $\mu > 0$. Then $\Delta_{a+\nu}^{-\mu}(x-a)^{\underline{\nu}} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(x-a)^{\underline{\mu+\nu}}$ for $x \in \mathbb{N}_{a+\mu+\nu}$.*

Definition 2.4 ([3, 46]) Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$, and $m - 1 < \mu \le m$ for $m \in \mathbb{N}_1$. Then the Riemann–Liouville fractional difference of f at a is defined by

$$\Delta_a^\mu f(x) = \Delta^m \Delta_a^{-(m-\mu)} f(x) = \sum_{\tau=a}^{x+\mu} h_{-\mu-1}\bigl(x, \sigma(\tau)\bigr) f(\tau) \quad \text{for } x \in \mathbb{N}_{a+m-\mu}.$$

Definition 2.5 ([1, 47]) Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$, and $m - 1 < \mu \le m$ for $m \in \mathbb{N}_1$. Then the Caputo fractional difference of f at a is defined by

$${}^c\Delta_a^\mu f(x) = \Delta_a^{-(m-\mu)} \Delta^m f(x) = \sum_{\tau=a}^{x-(m-\mu)} h_{m-\mu-1}\bigl(x, \sigma(\tau)\bigr) \Delta^m f(\tau)$$

for $x \in \mathbb{N}_{a+m-\mu}$.

Definition 2.6 ([9]) Assume $p \in \mathcal{R}$ and $x, y \in \mathbb{N}_a$. Then the delta exponential function is given by

$$e_{p(x)}(x,y) = \begin{cases} \prod_{t=y}^{x-1}[1+p(t)], & \text{if } x \in \mathbb{N}_y, \\ \prod_{t=x}^{y-1}[1+p(t)]^{-1}, & \text{if } x \in \mathbb{N}_a^{y-1}. \end{cases}$$

By empty product convention $\prod_{t=y}^{y-1}[h(t)] := 1$ for any function h.

Definition 2.7 ([45]) Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then the delta Laplace transform of f based at a is defined by

$$\mathscr{L}_a\{f\}(y) = \int_a^\infty e_{\ominus y}\bigl(\sigma(x), a\bigr) f(x)\Delta x$$

for all complex numbers $y \neq -1$ such that this improper integral converges.

Lemma 2.8 ([9]) *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r > 1$ *and* $\mu > 0$. *Then, for* $|y + 1| > r$, *we have*

$$\mathscr{L}_{a+\mu}\left\{\Delta_a^{-\mu} f\right\}(y) = \frac{(y+1)^\mu}{y^\mu} \tilde{F}_a(y).$$

Lemma 2.9 ([9]) *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r > 0$ *and* m *is a positive integer. Then, for* $|y + 1| > r$,

$$\mathscr{L}_a\left\{\Delta^m f\right\}(y) = y^m \tilde{F}_a(y) - \sum_{j=0}^{m-1} y^j \Delta^{m-1-j} f(a).$$

Lemma 2.10 ([9] Fundamental theorem for the difference calculus) *Assume* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and* F *is an antidifference of* f *on* $\mathbb{N}_a^{b+1}$. *Then* $\sum_{t=a}^b f(t) = \sum_{t=a}^b \Delta F(t) = F(b + 1) - F(a)$.

The definition of Ulam stability for fractional difference equations is discussed in [38, 40]. Consider system (1) and the following inequalities:

$$\left|\Delta_a^{\mu,\nu} v(y) + g\big(y + \mu - 1, v(y + \mu - 1)\big)\right| \le \epsilon, \quad y \in [a, T]_{\mathbb{N}_a}, \tag{2}$$

$$\left|\Delta_a^{\mu,\nu} v(y) + g\big(y + \mu - 1, v(y + \mu - 1)\big)\right| \le \epsilon \psi\big(\rho(y) + \nu\big), \quad y \in [a, T]_{\mathbb{N}_a}, \tag{3}$$

where $\psi : [a, T]_{\mathbb{N}_a} \to \mathbb{R}^+$.

Definition 2.11 ([38]) A solution $u \in Z$ of system (1) is Ulam–Hyers stable if there exists a real number $d_f > 0$ such that, for each $\epsilon > 0$ and for every solution $v \in Z$ of inequality (2), it satisfies

$$\|v - u\| \le \epsilon d_f. \tag{4}$$

A solution of system (1) is generalized Ulam–Hyers stable if we substitute the function $\phi_f(\epsilon)$ for the constant ϵd_f in inequality (4), where $\phi_f(\epsilon) \in C(R^+, R^+)$ and $\phi_f(0) = 0$.

Definition 2.12 ([38]) A solution $u \in Z$ of system (1) is Ulam–Hyers–Rassias stable with respect to function ψ if there exists a real number $d_{f,\psi} > 0$ such that, for each $\epsilon > 0$ and for every solution $v \in Z$ of inequality (3), it satisfies

$$\|v - u\| \le \epsilon \psi(y) d_{f,\psi}, \quad y \in [a, T]_{\mathbb{N}_a}. \tag{5}$$

The solution of system (1) is generalized Ulam–Hyers–Rassias stable if we substitute the function $\Phi(y)$ for the function $\epsilon\psi(y)$ in inequalities (3) and (5).

3 Hilfer-like fractional difference

In this section, we generalize the definition of fractional difference operators. Motivated by the concept of Hilfer fractional derivative [17], and to keep the interpolative property, we introduce the following definition. Assume $f : \mathbb{N}_a \to \mathbb{R}$, then the fractional difference of order $m - 1 < \mu < m$ for $m \in \mathbb{N}_1$ is given by $\Delta_a^{\mu,\nu} f(x) = \Delta_{a+(1-\nu)(m-\mu)}^{-\nu(m-\mu)} \Delta^m \Delta_a^{-(1-\nu)(m-\mu)} f(x)$ for $x \in \mathbb{N}_{a+m-\mu}$, where $0 \le \nu \le 1$ is the type of difference operator. Observe that domain of $\Delta_a^{-(1-\nu)(m-\mu)}f(x)$ is $a+(1-\nu)(m-\mu)$, whereas integer order differences keep the same domain [12]. The starting point of the last sum is compatible with the starting point for the domain of the function $\Delta^m\Delta_a^{-(1-\nu)(m-\mu)}f(x)$, which is $a+(1-\nu)(m-\mu)$. This allows us the successive composition of operators in the above expression, and the final domain of $\Delta_a^{\mu,\nu}f(x)$ is $\mathbb{N}_{a+m-\mu}$. To get some nice properties, we restrict $0<\mu<1$ throughout the article.

Definition 3.1 Assume $f:\mathbb{N}_a\to\mathbb{R}$, then the fractional difference of order $0<\mu<1$ and type $0\le\nu\le1$ is defined by

$$\Delta_a^{\mu,\nu}f(x)=\Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)$$

for $x\in\mathbb{N}_{a+1-\mu}$.

The special cases are Riemann–Liouville fractional difference [3, 46] for $\nu=0$ and Caputo fractional difference [1, 47] for $\nu=1$.

First of all we will develop some composition properties to use them in the next section and to construct a fixed point operator for a new class of Hilfer fractional nonlinear difference equations with initial condition involving Riemann–Liouville fractional sum. Also we will present the delta Laplace transform for newly defined Hilfer fractional difference operator.

Lemma 3.2 *Assume $f:\mathbb{N}_a\to\mathbb{R}$, $0<\mu<1$, and $0\le\nu\le1$, then for $x\in N_{a+1}$:*

(i) $\Delta_{a+1-\mu}^{-\mu}[\Delta_a^{\mu,\nu}f(x)]=\Delta_{a+(1-\nu)(1-\mu)}^{-(\mu+\nu-\mu\nu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)$,
(ii) $\Delta_{a+1-\mu}^{-\mu}[\Delta_a^{\mu,\nu}f(x)]=\Delta_{a+(1-\nu)(1-\mu)}^{-(\mu+\nu-\mu\nu)}\Delta_a^{\mu+\nu-\mu\nu}f(x)$,
(iii) $\Delta_{a+\mu}^{\mu,\nu}[\Delta_a^{-\mu}f(x)]=\Delta_{a+(1-\nu+\mu\nu)}^{-\nu(1-\mu)}\Delta_a^{\nu(1-\mu)}f(x)$,
(iv) $\Delta_{a+\mu}^{\mu,\nu}[\Delta_a^{-\mu}f(x)]=f(x)-\Delta_a^{-(1-\nu(1-\mu))}f(a+1-\nu(1-\mu))\times h_{\nu(1-\mu)-1}(x,a+1-\nu(1-\mu))$.

Proof (i) On the left-hand side we use Definition 3.1 and (Theorem 5 [12]) to obtain

$$\begin{aligned}\Delta_{a+1-\mu}^{-\mu}\big[\Delta_a^{\mu,\nu}f(x)\big]&=\Delta_{a+1-\mu}^{-\mu}\big[\Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)\big]\\&=\Delta_{a+(1-\nu)(1-\mu)}^{-(\mu+\nu-\mu\nu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x).\end{aligned}$$

(ii) On the left-hand side, use (i) and the first part of (Lemma 6 [12]) to get

$$\begin{aligned}\Delta_{a+1-\mu}^{-\mu}\big[\Delta_a^{\mu,\nu}f(x)\big]&=\Delta_{a+(1-\nu)(1-\mu)}^{-(\mu+\nu-\mu\nu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)\\&=\Delta_{a+(1-\nu)(1-\mu)}^{-(\mu+\nu-\mu\nu)}\Delta_a^{\mu+\nu-\mu\nu}f(x).\end{aligned}$$

(iii) Using Definition 3.1 and (Theorem 5 [12]), we obtain

$$\begin{aligned}\Delta_{a+\mu}^{\mu,\nu}\big[\Delta_a^{-\mu}f(x)\big]&=\Delta_{a+\mu+(1-\nu)(1-\mu)}^{-\nu(1-\mu)}\Delta\Delta_{a+\mu}^{-(1-\nu)(1-\mu)}\big[\Delta_a^{-\mu}f(x)\big]\\&=\Delta_{a+(1-\nu+\mu\nu)}^{-\nu(1-\mu)}\Delta\Delta_a^{-(1-\nu+\mu\nu)}f(x)\\&=\Delta_{a+(1-\nu+\mu\nu)}^{-\nu(1-\mu)}\Delta_a^{\nu(1-\mu)}f(x).\end{aligned}$$

In the preceding step we also used the first part of (Lemma 6 [12]).

(iv) Consider the left-hand side, use (iii) and the second part of (Theorem 8 [12]),

$$\begin{aligned}\Delta_{a+\mu}^{\mu,\nu}\left[\Delta_a^{-\mu}f(x)\right] &= \Delta_{a+(1-\nu+\mu\nu)}^{-\nu(1-\mu)}\Delta_a^{\nu(1-\mu)}f(x)\\ &= \Delta_{a+1-\nu(1-\mu)}^{-\nu(1-\mu)}\Delta_a^{\nu(1-\mu)}f(x)\\ &= f(x)-\Delta_a^{-(1-\nu(1-\mu))}f\big(a+1-\nu(1-\mu)\big)\\ &\quad\times h_{\nu(1-\mu)-1}\big(x,a+1-\nu(1-\mu)\big).\end{aligned}$$ □

For a nonempty set N_a^T, the set of all real-valued bounded functions $B(N_a^T)$ is a norm space with $\|f\| = \sup_{x\in\mathbb{N}_a^T}\{f(x)\}$. We consider a weighted space of bounded functions $B_\lambda(N_a^T) := \{f : N_a^T \to \mathbb{R}; |(x-a-\mu)^{\underline{\lambda}}f(x)| < M\}$, with $0 \leq \lambda < \mu$ and $M > 0$. The weighted space of bounded functions is considered for finding left inverse property, however analysis in the following sections is not influenced by this space.

Lemma 3.3 *Let $f \in B_\lambda(N_a^T)$ be given and $0 < \lambda \leq 1$. Then $\Delta_a^{-\mu}f(a+\mu) = 0$ for $0 \leq \lambda < \mu$.*

Proof Since $f \in B_\lambda(N_a^T)$, thus for some positive integer M, we have $|(x-a-\mu)^{\underline{\lambda}}f(x)| < M$ for each $x \in N_a^T$. Therefore

$$\begin{aligned}\left|\Delta_a^{-\mu}f(x)\right| &< M\left[\Delta_a^{-\mu}(y-a-\mu)^{\underline{-\lambda}}\right](x)\\ &\leq M\Gamma(1-\lambda)\frac{(x-a-\mu)^{\underline{\mu-\lambda}}}{\Gamma(\mu-\lambda+1)}.\end{aligned}$$

In the preceding step we used the fact $\Delta_a^{-\mu}(x-a)^{\underline{-\lambda}} = (x-a)^{\underline{\mu-\lambda}}\frac{\Gamma(1-\lambda)}{\Gamma(\mu-\lambda+1)}$. The desired result is achieved by applying limit process $x \to a+\mu$. □

Next we will state the left inverse property.

Lemma 3.4 *Assume $0 < \mu < 1$, $0 \leq \nu \leq 1$, and $\eta = \mu+\nu-\mu\nu$, then for $f \in B_{1-\eta}(N_a^T)$,*

$$\Delta_{a+\mu}^{\mu,\nu}\left[\Delta_a^{-\mu}f(x)\right] = f(x).$$

Proof Since $0 \leq 1-\eta < 1-\nu(1-\mu)$. Thus Lemma 3.3 gives $\Delta_a^{-(1-\nu+\mu\nu)}f(a+1-\nu+\mu\nu) = 0$. Hence the result follows from part (iv) of Lemma 3.2. □

Theorem 3.5 *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$ with $\mathscr{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $0 < \mu < 1$, $0 \leq \nu \leq 1$. Then, for $|y+1| > r$, we have the delta Laplace transform given as*

$$\begin{aligned}\mathscr{L}_{a+1-\mu}\left\{\Delta_a^{\mu,\nu}f\right\}(y) &= y^\mu(y+1)^{1-\mu}\tilde{F}_a(y)\\ &\quad-\frac{(y+1)^{\nu(1-\mu)}}{y^{\nu(1-\mu)}}\Delta_a^{-(1-\nu)(1-\mu)}f\big(a+(1-\nu)(1-\mu)\big).\end{aligned}$$

Proof Considering the left-hand side and using Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
\mathscr{L}_{a+1-\mu}\{\Delta_a^{\mu,\nu}f\}(y) &= \mathscr{L}_{a+1-\mu}\big[\Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)}\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)\big](y)\\
&= \frac{(y+1)^{\nu(1-\mu)}}{y^{\nu(1-\mu)}}\mathscr{L}_{a+(1-\nu)(1-\mu)}\big[\Delta\Delta_a^{-(1-\nu)(1-\mu)}f(x)\big](y)\\
&= \frac{(y+1)^{\nu(1-\mu)}}{y^{\nu(1-\mu)}}\big[y\mathscr{L}_{a+(1-\nu)(1-\mu)}\big[\Delta_a^{-(1-\nu)(1-\mu)}f(x)\big](y)\\
&\quad - \Delta_a^{-(1-\nu)(1-\mu)}f\big(a+(1-\nu)(1-\mu)\big)\big]\\
&= \frac{(y+1)^{\nu(1-\mu)}}{y^{\nu(1-\mu)}}\bigg[y\frac{(y+1)^{(1-\nu)(1-\mu)}}{y^{(1-\nu)(1-\mu)}}\mathscr{L}_a\big[f(x)\big](y)\\
&\quad - \Delta_a^{-(1-\nu)(1-\mu)}f\big(a+(1-\nu)(1-\mu)\big)\bigg]\\
&= y^{\mu}(y+1)^{1-\mu}\tilde{F}_a(y)\\
&\quad - \frac{(y+1)^{\nu(1-\mu)}}{y^{\nu(1-\mu)}}\Delta_a^{-(1-\nu)(1-\mu)}f\big(a+(1-\nu)(1-\mu)\big). \qquad \square
\end{aligned}$$

Remark 1 Notice that, if in Theorem 3.5 we set $\nu = 0$, then we recover Theorem 2.70 in [9]. Further, if we set $\nu = 1$, we obtain the delta Laplace transform for the Caputo fractional difference.

4 Fixed point operators for initial value problem

To establish existence theory for Hilfer fractional difference equation with initial conditions, we transform the problem to an equivalent summation equation which in turn defines an appropriate fixed point operator.

Lemma 4.1 *Let $g : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ be given and $0 < \mu < 1$, $0 \le \nu \le 1$. Then u solves system* (1) *if and only if*

$$u(x) = \zeta h_{\eta-1}(x, a+1-\eta) - \Delta_{a+1-\mu}^{-\mu}g\big(x+\mu-1, u(x+\mu-1)\big)$$

for all $x \in \mathbb{N}_{a+1}$.

The proof of the above lemma is an implication of Lemma 3.2 (i) and (ii) and the second part of Theorem 8 in [12]. In next result Brouwer's fixed point theorem [38] is utilized for establishing existence conditions. The set Z of all real sequences $u = \{u(x)\}_{x=a}^{T}$, with $\|u\| = \sup_{x\in\mathbb{N}_a^T}|u(x)|$ is a Banach space.

Using Definition 2.2 and Lemma 4.1 we define an operator $\mathcal{A} : Z \to Z$ by

$$\mathcal{A}u(x) = \zeta h_{\eta-1}(x, a+1-\eta) - \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\big(x,\sigma(\tau)\big)g\big(\tau+\mu-1, u(\tau+\mu-1)\big). \tag{6}$$

The fixed points of $\mathcal{A}$ coincide with the solutions of problem (1).

Theorem 4.2 *Let $f : [a,T]_{\mathbb{N}_a} \to \mathbb{R}$ be a bounded function in such a way that $|g(x,u)| \leq f(x)|u|$ for all $u \in Z$. Then IVP (1) has at least one solution on Z, provided*

$$L^* \leq \frac{\Gamma(\mu+1)}{(T-a-1+\mu)^{\underline{\mu}}}, \tag{7}$$

where $L^* = \sup_{x\in\mathbb{N}_{a+1-\mu}^T} f(x+\mu-1)$.

Proof For $M > 0$, define the set

$$W = \{u : \|u - \zeta h_{\eta-1}(x, a+1-\eta)\| \leq M, \text{ for } x \in \mathbb{N}_{a+1-\mu}^T\}.$$

To prove this theorem we just have to show that $\mathcal{A}$ maps W into itself. For $u \in W$, we have

$$\begin{aligned}
&\left|\mathcal{A}u(x) - \zeta h_{\eta-1}(x, a+1-\eta)\right| \\
&\quad\leq f(x+\mu-1) \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)\left|u(\tau+\mu-1) - 0\right| \\
&\quad\leq L^* \sup_{x\in\mathbb{N}_{a+1-\mu}^T} \left|u(x+\mu-1) - 0\right| \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr) \\
&\quad= L^*\|u-0\|\left[\frac{(x-a-1+\mu)^{\underline{\mu}}}{\Gamma(\mu+1)} - 0\right] \\
&\quad\leq L^* M \frac{(T-a-1+\mu)^{\underline{\mu}}}{\Gamma(\mu+1)} \leq M.
\end{aligned}$$

We have $\|\mathcal{A}u\| \leq M$, which implies that $\mathcal{A}$ is a self map. Therefore, by Brouwer's fixed point theorem, $\mathcal{A}$ has at least one fixed point. □

Theorem 4.3 *For $K > 0$ and $u, v \in Z$, assume that $|g(x,u) - g(x,v)| \leq K|u-v|$ for all $x \in [a,T]_{\mathbb{N}_a}$. Then IVP (1) has a unique solution on Z, provided*

$$K < \frac{\Gamma(\mu+1)}{(T-a-1+\mu)^{\underline{\mu}}}. \tag{8}$$

Proof Let $u, v \in Z$ and $x \in [a,T]_{\mathbb{N}_a}$, we have by assumption

$$\begin{aligned}
\left|\mathcal{A}u(x) - \mathcal{A}v(x)\right| &\leq \left|\sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)\right| \\
&\quad\times \left|g\bigl(\tau+\mu-1, u(\tau+\mu-1)\bigr) - g\bigl(\tau+\mu-1, v(\tau+\mu-1)\bigr)\right| \\
&\leq \frac{|0-(x-a-1+\mu)^{\underline{\mu}}|}{\Gamma(\mu+1)} K\left|u(\tau+\mu-1) - v(\tau+\mu-1)\right|.
\end{aligned}$$

In the preceding step, we used $\sum_\tau h_{\nu-1}(x,\sigma(\tau)) = -h_\nu(x,\tau)$ and Lemma 2.10. Now taking supremum on both sides, we have

$$\sup_{x\in\mathbb{N}_a^T} \left|\mathcal{A}u(x) - \mathcal{A}v(x)\right| \leq \frac{K(T-a-1+\mu)^{\underline{\mu}}}{\Gamma(\mu+1)}\|u-v\|.$$

Using inequality (8), we get $\|\mathcal{A}u - \mathcal{A}v\| \leq \|u - v\|$, which implies $\mathcal{A}$ is a contraction. Therefore, by Banach's fixed point theorem, $\mathcal{A}$ has a unique fixed point. □

Theorem 4.4 *For $K > 0$, assume that $|g(x,u) - g(x,v)| \leq K|u - v|$ for all $x \in [a,T]_{\mathbb{N}_a}$. Let $u \in Z$ be a solution of system (1) and $v \in Z$ be a solution of inequality (2). Then, for K in inequality (8), nonlinear IVP (1) is Ulam–Hyers stable and, consequently, generalized Ulam–Hyers stable.*

Proof For simplicity the solution of IVP (1) can be rewritten by using equation (6) as follows:

$$u(x) = w(x) - \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)g\bigl(\tau+\mu-1, u(\tau+\mu-1)\bigr), \tag{9}$$

where $w(x) = \zeta h_{\eta-1}(x, a+1-\eta)$. Now, for $[a,T]_{\mathbb{N}_a}$, it follows from inequality (2) that

$$\left|v(x) - \left(w(x) - \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)g\bigl(\tau+\mu-1, v(\tau+\mu-1)\bigr)\right)\right| \leq \epsilon. \tag{10}$$

For $[a,T]_{\mathbb{N}_a}$, making use of equation (9) and inequality (10) together for $[a,T]_{\mathbb{N}_a}$, we have

$$\begin{aligned}
\bigl|v(x) - u(x)\bigr| &= \left|v(x) - \left(w(x) - \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)g\bigl(\tau+\mu-1, u(\tau+\mu-1)\bigr)\right)\right| \\
&\leq \left|v(x) - \left(w(x) - \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)g\bigl(\tau+\mu-1, v(\tau+\mu-1)\bigr)\right)\right| \\
&\quad + \left|\sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x,\sigma(\tau)\bigr)\right| \\
&\quad \times \bigl|g\bigl(\tau+\mu-1, v(\tau+\mu-1)\bigr) - g\bigl(\tau+\mu-1, u(\tau+\mu-1)\bigr)\bigr| \\
&\leq \epsilon + \bigl|0 - h_\mu(x, a+1-\mu)\bigr|K\bigl|v(\tau+\nu-1) - u(\tau+\nu-1)\bigr|.
\end{aligned}$$

In the preceding step, we used assumption and the same argument used in Theorem 4.3. Now taking supremum on both sides and simplifying, we have

$$\|v - u\| \leq \frac{\epsilon}{1 - h_\mu(T, a+1-\mu)K} = \epsilon d_f, \quad \text{with } d_f = \frac{1}{1 - h_\mu(T, a+1-\mu)K}.$$

Therefore by inequality (8), (1) is Ulam–Hyers stable. Further by using $\phi_f(\epsilon) = \epsilon d_f$, $\phi_f(0) = 0$, which implies that (1) is generalized Ulam–Hyers stable. □

Theorem 4.5 *For $K > 0$, assume that $|g(x,u) - g(x,v)| \leq K|u - v|$ for all $x \in [a,T]_{\mathbb{N}_a}$. Let $u \in Z$ be a solution of system (1) and $v \in Z$ be a solution of inequality (3). Then, for K in inequality (8), nonlinear IVP (1) is Ulam–Hyers–Rassias stable with respect to function $\psi : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$ and, consequently, generalized Ulam–Hyers–Rassias stable.*

To illustrate the usefulness of Theorem 4.4, we present the following example.

Example 4.6 Consider the following fractional Hilfer difference equation with initial condition involving Riemann–Liouville fractional sum:

$$\begin{cases} -\Delta_{0.3}^{0.7,0.5}u(x) = (x-0.3)u(x-0.3), & x \in [0.3, 9.3]_{\mathbb{N}_{0.3}}, \\ \Delta_{0.3}^{-(0.15)}u(0.45) = \zeta. \end{cases}$$

Here, $a = 0.3$, $T = 9.3$, $\mu = 0.7$, and $\nu = 0.5$. Therefore $\eta = 0.85$. Thus, for $K < 0.1974$, the solution to the given problem with inequalities

$$\left|\Delta_{0.3}^{0.7,0.5}v(x) + (x-0.3)v(x-0.3)\right| \leq \epsilon, \quad x \in [0.3, 9.3]_{\mathbb{N}_{0.3}},$$
$$\left|\Delta_{0.3}^{0.7,0.5}v(x) + (x-0.3)v(x-0.3)\right| \leq \epsilon\psi(x-0.3), \quad x \in [0.3, 9.3]_{\mathbb{N}_{0.3}}$$

is Ulam–Hyers stable and Ulam–Hyers–Rassias stable with respect to function $\psi : [0.3, 9.3]_{\mathbb{N}_{0.3}} \to \mathbb{R}^+$.

To solve the linear Hilfer fractional difference IVP, we use the successive approximation method.

Example 4.7 Let $\eta = \mu + \nu - \mu\nu$ with $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Consider the IVP for linear Hilfer fractional difference equation:

$$\begin{cases} \Delta_a^{\mu,\nu}u(x) - \lambda u(x+\mu-1) = 0, \\ \Delta_a^{-(1-\eta)}u(a+1-\eta) = \zeta, \quad \zeta \in \mathbb{R}. \end{cases} \tag{11}$$

The solution of (11) is given by

$$u(x) = \zeta h_{\eta-1}(x, a+1-\eta) + \lambda\Delta_{a+1-\mu}^{-\mu}u(x+\mu-1).$$

Definition 2.2 and successive approximation yield the following:

$$u_k(x) = u_0(x) + \lambda \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\big(x, \sigma(\tau)\big)u_{k-1}(\tau+\mu-1) \tag{12}$$

for $k = 1, 2, 3, \ldots$, where $u_0(x) = \zeta h_{\eta-1}(x, a+1-\eta)$.

Initially, for $k = 1$ and by Lemma 2.3, we have

$$u_1(x) = \zeta h_{\eta-1}(x, a+1-\eta) + \lambda\zeta h_{\eta-1+\mu}(x+\mu-1, a+1-\eta).$$

Similarly, for $k = 2$,

$$\begin{aligned} u_2(x) &= \zeta\big[h_{\eta-1}(x, a+1-\eta) + \lambda h_{\eta-1+\mu}(x+\mu-1, a+1-\eta) \\ &\quad + \lambda^2 h_{\eta-1+2\mu}\big(x+2(\mu-1), a+1-\eta\big)\big] \\ &= \zeta\left[\lambda^0\frac{(x+\eta-a-1)^{\underline{0.\mu+\eta-1}}}{\Gamma(\eta)} + \lambda^1\frac{(x+\eta-a-1+(\mu-1))^{\underline{1.\mu+\eta-1}}}{\Gamma(\mu+\eta)}\right. \\ &\quad \left. + \lambda^2\frac{(x+\eta-a-1+2(\mu-1))^{\underline{2.\mu+\eta-1}}}{\Gamma(2\mu+\eta)}\right]. \end{aligned}$$

Proceed inductively and let $k \to \infty$

$$u(x) = \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\mu-1))^{\underline{k\mu+\eta-1}}}{\Gamma(k\mu+\eta)} \right].$$

By using the property $x^{\underline{\mu+\nu}} = (x-\nu)^{\underline{\mu}} x^{\underline{\nu}}$, we obtain

$$u(x) = \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+(k-1)(\mu-1))^{\underline{k\mu}}(x+\eta-a-1+k(\mu-1))^{\underline{\eta-1}}}{\Gamma(k\mu+\eta)} \right].$$

Now, from discrete form (12), we have the numerical formula

$$u(a+n) = u(a) + \frac{\lambda}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} u(a+j-1), \tag{13}$$

with $u(a) = \zeta \frac{\Gamma(n+\eta)}{\Gamma(\eta)\Gamma(n+1)}$. From (13), we can have

$$y(n) = \zeta \frac{\Gamma(n+\eta)}{\Gamma(\eta)\Gamma(n+1)} + \frac{\lambda}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} y(j-1).$$

For different values of ν, the numerical solutions for $\mu = 0.8$ and $\mu = 0.5$ are shown in Fig. 1 and Fig. 2, respectively. Figure 1 and Fig. 2 show the interpolative behavior of Hilfer difference operator between the Riemann–Liouville [7] and the Caputo difference operator [48].

Remark 2 If we set $\nu = 1$ in Example 4.7 above (hence $\eta = 1$) and take $a = \mu - 1$, then we recover Example 17 in [1]. In fact, the solution of the initial Caputo difference equation

$$^{C}\Delta_a^{\mu} x(t) = \lambda x(t+\mu-1), \quad x(a) = x_0, \mu \in (0,1] \tag{14}$$

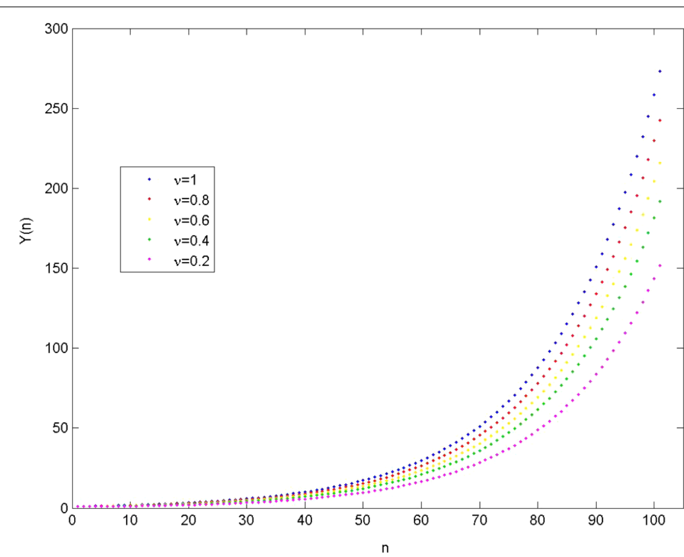

Figure 1 Solutions for $\lambda = 0.1$, $\mu = 0.8$ and different values of ν

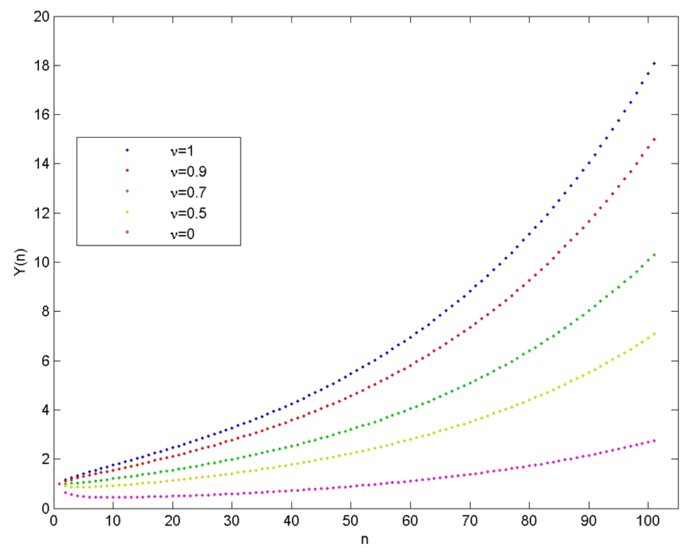

Figure 2 Solutions for $\lambda = 0.1$, $\mu = 0.5$ and different values of ν

will be given by

$$x(t) = x_0 E_{\underline{\mu}}(\lambda, t-a) = x_0 \sum_{k=0}^{\infty} \frac{\lambda^k (t-a+k(\mu-1))^{\underline{k\mu}}}{\Gamma(\mu k+1)}. \tag{15}$$

Observe that the case $a = \mu - 1$ will result in (66) in [1]. That is, formula (66) in [1] represents $E_{\underline{\mu}}(\lambda, t-(\alpha-1))$. Also, one can see that the substitution $\mu = 1$ will give the delta discrete Taylor expansion of the delta discrete exponential function.

The observations in Remark 2 suggest the following modified definitions which are different from those appearing in [1].

Definition 4.8 For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\mu, \eta, \gamma, z \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$, the discrete Mittag-Leffler functions are defined by

$$E^{\gamma}_{\underline{\mu,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\mu-1))^{\underline{\mu k+\eta-1}} (\gamma)_k}{\Gamma(\mu k+\eta) k!}, \quad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1),$$

$$E_{\underline{\mu,\eta}}(\lambda, z) = E^{1}_{\underline{\mu,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\mu-1))^{\underline{\mu k+\eta-1}}}{\Gamma(\mu k+\eta)}, \tag{16}$$

$$E_{\underline{\mu}}(\lambda, z) = E_{\underline{\mu,1}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\mu-1))^{\underline{\mu k}}}{\Gamma(\mu k+1)}. \tag{17}$$

By help of the fact $x^{\underline{\mu+\nu}} = (x-\nu)^{\underline{\mu}} x^{\underline{\nu}}$, we note that

$$\begin{aligned} E^{\gamma}_{\underline{\mu,\mu}}(\lambda, z) &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\mu-1))^{\underline{\mu k+\mu-1}} (\gamma)_k}{\Gamma(\mu k+\mu) k!} \\ &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\mu-1))^{\underline{k\mu}} (z+k(\mu-1))^{\underline{\mu-1}} (\gamma)_k}{\Gamma(k\mu+\mu) k!}. \end{aligned} \tag{18}$$

Definition 4.9 For $\lambda \in \mathbb{R}$, $|\lambda| < 1$, and $\mu, \eta, \gamma, z \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$, the discrete Mittag-Leffler functions are defined by

$$\mathbf{E}^{\gamma}_{\underline{\mu,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + k(\mu - 1) + \eta - 1)^{\underline{\mu k + \eta - 1}} (\gamma)_k}{\Gamma(\mu k + \eta) k!},$$

$$\mathbf{E}_{\underline{\mu,\eta}}(\lambda, z) = \mathbf{E}^{1}_{\underline{\mu,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + k(\mu - 1) + \eta - 1)^{\underline{\mu k + \eta - 1}}}{\Gamma(\mu k + \eta)}, \tag{19}$$

$$\mathbf{E}_{\underline{\mu}}(\lambda, z) = \mathbf{E}_{\underline{\mu,1}}(\lambda, z) = E_{\underline{\mu}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + k(\mu - 1))^{\underline{\mu k}}}{\Gamma(\mu k + 1)}. \tag{20}$$

Next we solve the non-homogeneous Hilfer fractional difference IVP, which shows that the definition is useful.

Example 4.10 Let $\eta = \mu + \nu - \mu\nu$, with $0 < \mu < 1$ and $0 \le \nu \le 1$. Consider Hilfer non-homogeneous fractional difference equation

$$\begin{cases} \Delta_a^{\mu,\nu} u(x) - \lambda u(x + \mu - 1) = f(x), \\ \Delta_a^{-(1-\eta)} u(a + 1 - \eta) = \zeta, \quad \zeta \in \mathbb{R}. \end{cases} \tag{21}$$

The solution of (21) is given by

$$u(x) = \zeta h_{\eta-1}(x, a + 1 - \eta) + \lambda \Delta_{a+1-\mu}^{-\mu} u(x + \mu - 1) + \Delta_{a+1-\mu}^{-\mu} f(x).$$

Then Definition 2.2 and successive approximation yield the following:

$$u_k(x) = u_0(x) + \lambda \sum_{\tau=a+1-\mu}^{x-\mu} h_{\mu-1}\bigl(x, \sigma(\tau)\bigr) u_{k-1}(\tau + \mu - 1) + \Delta_{a+1-\mu}^{-\mu} f(x)$$

for $k = 1, 2, 3, \dots$, where $u_0(x) = \zeta h_{\eta-1}(x, a + 1 - \eta)$.

Initially, for $k = 1$ and by Lemma 2.3, we get

$$u_1(x) = \zeta h_{\eta-1}(x, a + 1 - \eta) + \lambda \zeta h_{\eta-1+\mu}(x + \mu - 1, a + 1 - \eta) + \Delta_{a+1-\mu}^{-\mu} f(x).$$

Similarly, for $k = 2$, we obtain

$$\begin{aligned} u_2(x) &= \zeta \bigl[h_{\eta-1}(x, a + 1 - \eta) + \lambda h_{\eta-1+\mu}(x + \mu - 1, a + 1 - \eta) \\ &\quad + \lambda^2 h_{\eta-1+2\mu}\bigl(x + 2(\mu - 1), a + 1 - \eta\bigr)\bigr] + \lambda \Delta_{a+1-\mu}^{-2\mu} f(x + \mu - 1) + \Delta_{a+1-\mu}^{-\mu} f(x) \\ &= \zeta \biggl[\lambda^0 \frac{(x + \eta - a - 1)^{\underline{0.\mu + \eta - 1}}}{\Gamma(\eta)} + \lambda^1 \frac{(x + \eta - a - 1 + (\mu - 1))^{\underline{1.\mu + \eta - 1}}}{\Gamma(\mu + \eta)} \\ &\quad + \lambda^2 \frac{(x + \eta - a - 1 + 2(\mu - 1))^{\underline{2.\mu + \eta - 1}}}{\Gamma(2\mu + \eta)}\biggr] + \lambda \Delta_{a+1-\mu}^{-2\mu} f(x + \mu - 1) \\ &\quad + \Delta_{a+1-\mu}^{-\mu} f(x). \end{aligned}$$

Proceed inductively and let $k \to \infty$

$$\begin{aligned}
u(x) &= \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\mu-1))^{\underline{k\mu+\eta-1}}}{\Gamma(k\mu+\eta)} \right] \\
&\quad + \sum_{k=1}^{\infty} \lambda^{k-1} \Delta_{a+1-\mu}^{-k\mu} f\bigl(x+(k-1)(\mu-1)\bigr) \\
&= \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\mu-1))^{\underline{k\mu+\eta-1}}}{\Gamma(k\mu+\eta)} \right] \\
&\quad + \sum_{k=1}^{\infty} \lambda^{k-1} \sum_{\tau=a+1-\mu}^{x-k\mu} h_{k\mu-1}\bigl(x, \sigma\bigl(\tau+(k-1)(\mu-1)\bigr)\bigr) f(\tau) \\
&= \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\mu-1))^{\underline{k\mu+\eta-1}}}{\Gamma(k\mu+\eta)} \right] \\
&\quad + \sum_{k=0}^{\infty} \lambda^{k} \sum_{\tau=a+1-\mu}^{x-k\mu-\mu} \frac{(x-\sigma(\tau)+k(\mu-1))^{\underline{k\mu+\mu-1}}}{\Gamma(k\mu+\mu)} f(\tau), \\
u(x) &= \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\mu-1))^{\underline{k\mu+\eta-1}}}{\Gamma(k\mu+\eta)} \right] \\
&\quad + \sum_{\tau=a+1-\mu}^{x-\mu} \sum_{k=0}^{\infty} \lambda^{k} \frac{(x-\sigma(\tau)+k(\mu-1))^{\underline{k\mu+\mu-1}}}{\Gamma(k\mu+\mu)} f(\tau).
\end{aligned}$$

In the preceding step, we have interchanged summation of the second expression. Now we use the property $x^{\underline{\mu+\nu}} = (x-\nu)^{\underline{\mu}}\, x^{\underline{\nu}}$ in the following step:

$$\begin{aligned}
u(x) &= \zeta \left[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+(k-1)(\mu-1))^{\underline{k\mu}}(x+\eta-a-1+k(\mu-1))^{\underline{\eta-1}}}{\Gamma(k\mu+\eta)} \right] \\
&\quad + \sum_{\tau=a+1-\mu}^{x-\mu} \sum_{k=0}^{\infty} \lambda^{k} \frac{(x-\sigma(\tau)+(k-1)(\mu-1))^{\underline{k\mu}}(x-\sigma(\tau)+k(\mu-1))^{\underline{\mu-1}}}{\Gamma(k\mu+\mu)} f(\tau).
\end{aligned}$$

Using Definition 4.8, we have

$$u(x) = \zeta E_{\underline{\mu,\eta}}(\lambda, x+\eta-a-1) + \sum_{\tau=a+1-\mu}^{x-\mu} \bigl[E_{\underline{\mu,\mu}}\bigl(\lambda, x-\sigma(\tau)\bigr)\bigr] f(\tau).$$

Alternatively, by using Definition 4.9, we get

$$u(x) = \zeta \mathbf{E}_{\underline{\mu,\eta}}(\lambda, x-a) + \sum_{\tau=a+1-\mu}^{x-\mu} \bigl[\mathbf{E}_{\underline{\mu,\mu}}\bigl(\lambda, x-\sigma(\tau)+\mu-1\bigr)\bigr] f(\tau).$$

Note that above is the generalization of Caputo fractional difference IVP [1], it can prevail for $\nu = 1$.

5 Modified Gronwall's inequality and its application in delta difference setting

First we develop a Gronwall's inequality for the delta difference operator. Then a simple utilization of Gronwall's inequality leads to stability for Hilfer difference equation. For this purpose, choose u and w such that

$$u(x) \leq u(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu}\phi(x+\mu)u(x+\mu), \tag{22}$$

$$w(x) \geq w(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu}\phi(x+\mu)w(x+\mu). \tag{23}$$

Lemma 5.1 *Assume u and w respectively satisfy* (22) *and* (23). *If $w(a) \geq u(a)$, then $w(x) \geq u(x)$ for $x \in \mathbb{N}_a$.*

Proof We give the proof by induction principle. Assume $w(\tau) - u(\tau) \geq 0$ is valid for $\tau = a, a+1, \dots, x-1$. Then we have

$$\begin{aligned} w(x) - u(x) &\geq h_{\eta-1}(x, a+1-\eta)\bigl(w(a) - u(a)\bigr) + \Delta_{a+1-\mu}^{-\mu}\phi(x+\mu)w(x+\mu) \\ &\quad - \Delta_{a+1-\mu}^{-\mu}\phi(x+\mu)u(x+\mu) \\ &= h_{\eta-1}(x, a+1-\eta)\bigl(w(a) - u(a)\bigr) \\ &\quad + \sum_{\tau=a+1-\mu}^{x-\mu} \frac{(x-\sigma(\tau))^{\underline{\mu-1}}}{\Gamma(\mu)}\phi(\tau+\mu)\bigl(w(\tau+\mu) - u(\tau+\mu)\bigr), \end{aligned}$$

where the last summation is valid for $x \in \mathbb{N}_{a+\mu}$. Now we shift the domain of summation to $\mathbb{N}_a$:

$$\begin{aligned} w(x) - u(x) &\geq h_{\eta-1}(x, a+1-\eta)\bigl(w(a) - u(a)\bigr) \\ &\quad + \sum_{\tau=a+1}^{x} \frac{(x+\mu-\sigma(\tau))^{\underline{\mu-1}}}{\Gamma(\mu)}\phi(\tau)\bigl(w(\tau) - u(\tau)\bigr). \end{aligned}$$

By assumption, for $\tau = a, a+1, \dots, x-1$, we have

$$w(x) - u(x) \geq \phi(x)\bigl(w(x) - u(x)\bigr).$$

This implies that $(1 - \phi(x))(w(x) - u(x)) \geq 0$ and for $|\phi(x)| < 1$, which is the desired result. □

Following the approach for nabla fractional difference in [49], let $E_\nu\phi = \Delta_{a+1-\mu}^{-\mu}\nu(x)\phi(x)$. For constant ϕ one can use $E_\nu\phi$ to express the Mittag-Leffler function.

Theorem 5.2 *Assume $\eta = \mu + \nu - \mu\nu$, with $0 < \mu < 1$ and $0 \leq \nu \leq 1$. The solution of summation equation*

$$u(x) = u(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu}\nu(x+\mu-1)u(x+\mu-1)$$

is given by

$$u(x) = \frac{u(a)}{\Gamma(\eta)}\sum_{\ell=0}^{\infty} E_\nu^\ell\bigl(x+\eta-a-1+\ell(\mu-1)\bigr)^{\underline{\eta-1}}.$$

Proof By the method of successive approximation, the following is obtained:

$$u_k(x) = u_0(x) + \Delta_{a+1-\mu}^{-\mu} \nu(x+\mu-1)u_{k-1}(x+\mu-1), \quad k = 1,2,3,\dots,$$

where $u_0(x) = u(a)h_{\eta-1}(x, a+1-\eta)$.

For $k = 1$,

$$\begin{aligned} u_1(x) &= u(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu} \nu(x+\mu-1)u_0(x+\mu-1) \\ &= \frac{u(a)}{\Gamma(\eta)} E_\nu^0 (x+\eta-a-1)^{\underline{\eta-1}} + \frac{u(a)}{\Gamma(\eta)} E_\nu^1 (x+\eta-a-1+\mu-1)^{\underline{\eta-1}}. \end{aligned}$$

Proceeding inductively, we obtain

$$u_k(x) = \frac{u(a)}{\Gamma(\eta)} \sum_{\ell=0}^{k} E_\nu^\ell \bigl(x+\eta-a-1+\ell(\mu-1)\bigr)^{\underline{\eta-1}}, \quad k = 1,2,3,\dots,$$

and let $k \to \infty$,

$$u(x) = \frac{u(a)}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_\nu^\ell \bigl(x+\eta-a-1+\ell(\mu-1)\bigr)^{\underline{\eta-1}}.$$

□

Next we derive a Gronwall's inequality in delta discrete setting.

Theorem 5.3 *Let $\eta = \mu+\nu-\mu\nu$, with $0 < \mu < 1$ and $0 \le \nu \le 1$. Assume $|\nu(x)| < 1$ for $x \in \mathbb{N}_a$. If u and v are nonnegative real-valued functions with*

$$u(x) \le u(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu} \nu(x+\mu-1)u(x+\mu-1).$$

Then

$$u(x) \le \frac{u(a)}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_\nu^\ell \bigl(x+\eta-a-1+\ell(\mu-1)\bigr)^{\underline{\eta-1}}.$$

Proof Consider $w(x) = \frac{u(a)}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_\nu^\ell (x+\eta-a-1+\ell(\mu-1))^{\underline{\eta-1}}$. The proof of theorem follows from Lemma 5.1 and Theorem 5.2. □

For $\eta = 1$, a special case is obtained as follows.

Corollary 5.4 *Let $0 < \mu < 1$ and $0 \le \nu \le 1$. Assume $0 < \nu(x) < 1$ for $x \in \mathbb{N}_a$. If u is a nonnegative real-valued function with*

$$u(x) \le u(a) + \Delta_{a+1-\mu}^{-\mu} \nu(x+\mu-1)u(x+\mu-1).$$

Then

$$u(x) \le u(a)e_\nu(x,a),$$

where $e_\nu(x,a)$ is the delta exponential function.

Proof It follows from Theorem 5.3 that

$$u(x) \leq u(a)\sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1).$$

We claim that $\sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1) = e_{\nu}(x,a)$. To justify our claim, we utilize the uniqueness of solution of the following IVP: $\Delta u(x) = \nu(x)u(x)$, $u(a) = 1$. A unique solution $u(x) = e_{\nu}(x,a)$ of IVP is given in [9] for regressive function $\nu(x)$. Thus, we have to show that $\sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1)$ satisfies the IVP $\Delta u(x) = \nu(x)u(x)$, $u(a) = 1$. Indeed,

$$\begin{aligned}
\Delta \sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1) &= \sum_{\ell=0}^{\infty} \Delta E_{\nu}^{\ell}(1) \\
&= \sum_{\ell=1}^{\infty} \Delta E_{\nu}\bigl(E_{\nu}^{\ell-1}(1)\bigr) \\
&= \sum_{\ell=1}^{\infty} \Delta \Delta_a^{-1}\bigl(\nu(x)E_{\nu}^{\ell-1}(1)\bigr) = \nu(x)\sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1).
\end{aligned}$$

Also, by Definition 2.2 and empty sum convention, we have $\sum_{\ell=0}^{\infty} E_{\nu}^{\ell}(1)(a) = 1 + \sum_{\ell=1}^{\infty} E_{\nu}^{\ell}(1)(a) = 1$. Then the result follows. □

Let $\eta = \mu + \nu - \mu\nu$, then for $0 < \mu < 1$ and $0 \leq \nu \leq 1$, we have $0 < \eta \leq 1$. The following result illustrates the application of Gronwall's inequality for the system

$$\begin{cases} \Delta_a^{\mu,\nu} v(x) + g(x+\mu-1, v(x+\mu-1)) = 0, & \text{for } x \in \mathbb{N}_{a+1-\mu}, \\ \Delta_a^{-(1-\eta)} v(a+1-\eta) = \xi, \quad \xi \in \mathbb{R}. \end{cases} \tag{24}$$

Theorem 5.5 *Assume that the Lipschitz condition $|g(x,u) - g(x,v)| \leq K|u-v|$ holds for function g. Then the solution to Hilfer fractional difference system is stable.*

Proof Let $u \in Z$ be a solution of system (1) and $v \in Z$ be a solution of system (24). Then the corresponding summation equations are

$$\begin{aligned}
u(x) &= \zeta h_{\eta-1}(x, a+1-\eta) - \Delta_{a+1-\mu}^{-\mu} g\bigl(x+\mu-1, u(x+\mu-1)\bigr), \\
v(x) &= \xi h_{\eta-1}(x, a+1-\eta) - \Delta_{a+1-\mu}^{-\mu} g\bigl(x+\mu-1, v(x+\mu-1)\bigr).
\end{aligned}$$

For the absolute value of the difference, we have

$$\begin{aligned}
\bigl|u(x) - v(x)\bigr| &\leq |\zeta - \xi|\bigl|h_{\eta-1}(x, a+1-\eta)\bigr| \\
&\quad + \bigl|\Delta_{a+1-\mu}^{-\mu}\bigl(g\bigl(x+\mu-1, u(x+\mu-1)\bigr) - g\bigl(x+\mu-1, v(x+\mu-1)\bigr)\bigr)\bigr| \\
&\leq |\zeta - \xi| h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\mu}^{-\mu} K\bigl|u(x+\mu-1) - v(x+\mu-1)\bigr|.
\end{aligned}$$

Then it follows from Theorem 5.3 that

$$\bigl|u(x) - v(x)\bigr| \leq \frac{|\zeta - \xi|}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_K^{\ell}\bigl(x+\eta-a-1+\ell(\mu-1)\bigr)^{\underline{\eta-1}}.$$

By using Lemma 2.3, we obtain $E_K^{\ell}(x+\eta-a-1+\ell(\mu-1))^{\underline{\eta-1}} = \frac{K^{\ell}\Gamma(\eta)}{\Gamma(\eta+\mu\ell)}(x+\eta-a-1+\ell(\mu-1))^{\underline{\eta+\mu\ell-1}}$. To shape in the form of a discrete Mittag-Leffler function, we use the property $x^{\underline{\mu+\nu}} = (x-\nu)^{\underline{\mu}} x^{\underline{\nu}}$,

$$\begin{aligned}
\bigl|u(x)-v(x)\bigr| &\le |\zeta-\xi| \sum_{\ell=0}^{\infty} \frac{K^{\ell}}{\Gamma(\eta+\mu\ell)} \bigl(x+\eta-a-1+(k-1)(\mu-1)\bigr)^{\underline{k\mu}} \\
&\quad \times \bigl(x+\eta-a-1+k(\mu-1)\bigr)^{\underline{\eta-1}} \\
&= |\zeta-\xi| E_{\underline{\mu,\eta}}(K, x+\eta-a-1),
\end{aligned}$$

where $E_{\underline{\mu,\eta}}(\lambda, x)$ is the discrete Mittag-Leffler functions defined in [1]. Replace system (24) with

$$\begin{cases} \Delta_a^{\mu,\nu} v(x) + g(x+\mu-1, v(x+\mu-1)) = 0, \\ \Delta_a^{-(1-\eta)} v(a+1-\eta) = \zeta_n \end{cases} \tag{25}$$

for $x \in \mathbb{N}_{a+1-\mu}$ and $\zeta_n \to \zeta$. The solutions are denoted by v_n. Now we have

$$\bigl|u(x)-v_n(x)\bigr| \le |\zeta-\zeta_n| E_{\underline{\mu,\eta}}(K, x+\eta-a-1).$$

This leads to $|u(x)-v_n(x)| \to 0$, when $\zeta_n \to \zeta$ for $n \to \infty$. This completes the proof. □

6 Conclusion

We finish by concluding the following:

- A new definition of Hilfer-like fractional difference on discrete time scale has been introduced.
- The delta Laplace transform has been developed for newly defined Hilfer fractional difference operator.
- We have investigated a new class of Hilfer-like fractional nonlinear difference equations with initial condition involving Riemann–Liouville fractional sum.
- In particular, conditions for the existence, uniqueness, and two types of stabilities, called Ulam–Hyers stability and Ulam–Hyers–Rassias stability, have been obtained.
- The linear Hilfer fractional difference equation with initial conditions has been solved and alternative versions of discrete Mittag-Leffler functions are presented in comparison to [1].
- A Gronwall's inequality has been modified and applied for discrete calculus with the delta operator.

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