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Degenerate binomial coefficients and degenerate hypergeometric functions

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Abstract

In this paper, we investigate degenerate versions of the generalized p th order Franel numbers which are certain finite sums involving powers of binomial coefficients. In more detail, we introduce degenerate generalized hypergeometric functions and study degenerate hypergeometric numbers of order p . These numbers involve powers of λ -binomial coefficients and λ -falling sequence, and can be represented by means of the degenerate generalized hypergeometric functions. We derive some explicit expressions and combinatorial identities for those numbers. We also consider several related special numbers like λ -hypergeometric numbers of order p and Apostol type λ -hypergeometric numbers of order p , of which the latter reduce in a limiting case to the generalized p th order Franel numbers.

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1 Introduction

First, we study certain finite sums involving powers of binomial coefficients which are called generalized p th order Franel numbers and can be represented in terms of hypergeometric functions. Then, among other things, we find that particular cases of these numbers are connected with many known special numbers and polynomials which include Bernoulli numbers, Euler numbers, Changhee numbers, Daehee numbers, Stirling numbers of the first kind, Catalan numbers, and Legendre polynomials.

In recent years, many mathematicians have devoted their attention to studying various degenerate versions of some special numbers and polynomials [4, 7, 8, 10, 15, 16]. The idea of investigating degenerate versions of some special numbers and polynomials originated from Carlitz's papers [2, 3]. Indeed, he introduced the degenerate Bernoulli and Euler polynomials and numbers, and investigated some arithmetic and combinatorial aspects of them. Here we mention in passing that the degenerate Bernoulli polynomials were later rediscovered by Ustinov under the name of Korobov polynomials of the second kind. Two of the present authors, their colleagues, and some other people have studied quite a few degenerate versions of special numbers and polynomials with their interest

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not only in combinatorial and arithmetic properties but also in differential equations and certain symmetric identities (see [9, 16] and the references therein). It is worth noting that this idea of considering degenerate versions of some special polynomials and numbers is not only limited to polynomials but can also be extended to transcendental functions like gamma functions [11, 12]. We believe that studying some degenerate versions of special polynomials and numbers is a very fruitful and promising area of research in which many things remain yet to be uncovered.

Recently, Dolgy and Kim gave some explicit formulas of degenerate Stirling numbers associated with the degenerate special numbers and polynomials. Motivated by Dolgy and Kim’s paper [4], we would like to investigate degenerate versions of the generalized p th order Franel numbers. In more detail, we introduce degenerate generalized hypergeometric functions and study degenerate hypergeometric numbers of order p . These numbers involve powers of λ -binomial coefficients and λ -falling sequence, and can be represented by means of the degenerate generalized hypergeometric functions. We also consider several related special numbers like λ -hypergeometric numbers of order p and Apostol type λ -hypergeometric numbers of order p , of which the latter reduce in a limiting case to the generalized p th order Franel numbers.

For the rest of this section, we will fix some notations and recall some known results that are needed throughout this paper.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}} = e_\lambda^1(t) \quad (\text{see [2, 3, 11, 12, 14]}). \tag{1}$$

From (1), we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad (\text{see [4, 9–12]}), \tag{2}$$

where $(x)_{n,\lambda}$ is the λ -falling sequence given by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x - \lambda) \cdots (x - (n - 1)\lambda) \quad (n \geq 1). \tag{3}$$

In [10], the degenerate Stirling numbers of the second kind are defined by

$$\frac{1}{k!} (e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \quad (k \geq 0). \tag{4}$$

Let

$$(x)_0 = 1, \quad (x)_n = x(x - 1)(x - 2) \cdots (x - (n - 1)) \quad (n \geq 1). \tag{5}$$

Then $\lim_{\lambda \rightarrow 0} S_{2,\lambda}(n, k) = S_2(n, k)$, where $S_2(n, k)$ are the ordinary Stirling numbers of the second kind given by

$$x^n = \sum_{l=0}^n S_2(n, l)(x)_l \quad (n \geq 0) \quad (\text{see [1–13, 15–22]}). \tag{6}$$

The Stirling numbers of the first kind are defined as

$$\frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!} \quad (k \geq 0) \text{ (see [13, 15]).} \tag{7}$$

Thus, by (7), we get

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l \quad (n \geq 0). \tag{8}$$

In view of (4), the degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!} \left(\frac{(1+t)^\lambda - 1}{\lambda} \right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \quad \text{(see [4, 10]).} \tag{9}$$

Note that $\lim_{\lambda \rightarrow 0} S_{1,\lambda}(n, k) = S_1(n, k)$ ($n, k \geq 0$).

As is well known, the generalized hypergeometric function $F^{(p,q)}$ is defined by

$$F^{(p,q)} \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{\langle a_1 \rangle_k \cdots \langle a_p \rangle_k}{\langle b_1 \rangle_k \langle b_2 \rangle_k \cdots \langle b_q \rangle_k} \frac{x^k}{k!}, \tag{10}$$

where $\langle a \rangle_k = a(a+1) \cdots (a+k-1)$ ($k \geq 1$), $\langle a \rangle_0 = 1$ (see [17, 21]).

For example,

$$F^{(2,1)} \left(\begin{matrix} 1, b \\ b \end{matrix} \middle| -1 \right) = \frac{1}{2}, \quad F^{(2,1)} \left(\begin{matrix} 2, b \\ b \end{matrix} \middle| -1 \right) = \frac{1}{4}, \quad F^{(2,1)} \left(\begin{matrix} 3, b \\ b \end{matrix} \middle| -1 \right) = \frac{1}{8}, \dots$$

The Gauss summation theorem is given by

$$F^{(2,1)} \left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{11}$$

where $R(c) > R(b) > 0$, $R(c-a-b) > 0$, $R(c) > R(a) > 0$.

From (11), we note that

$$F^{(2,1)} \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \tag{12}$$

where $R(c) > R(b) > 0$.

The following are well-known identities related to the binomial coefficients:

$$\sum_{k=0}^n \binom{n}{k} = 2^n \quad (n \geq 0), \tag{13}$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0 \quad (n \neq 0, n \in \mathbb{N}), \tag{14}$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \frac{(2n)!}{(n!)^2} = \binom{2n}{n}, \tag{15}$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{(-1)^{n/2} n!}{((\frac{n}{2})!)^2}, & \text{otherwise,} \end{cases} \tag{16}$$

$$\sum_{k=-n}^n (-1)^k \binom{n+b}{n+k} \binom{n+c}{c+k} \binom{b+c}{b+k} = \frac{\Gamma(b+c+n+1)}{n! \Gamma(b+1) \Gamma(c+1)}. \tag{17}$$

2 Sums of powers of λ -binomial coefficients

The λ -binomial coefficients are defined as

$$\binom{x}{n}_\lambda = \frac{\langle x \rangle_{n,\lambda}}{n!} = \frac{x(x-\lambda) \cdots (x-(n-1)\lambda)}{n!} \quad (n \geq 1), \quad \binom{x}{0}_\lambda = 1 \quad (\lambda \in \mathbb{R}). \tag{18}$$

From (18), we easily get

$$\binom{x+y}{n}_\lambda = \sum_{l=0}^n \binom{x}{l}_\lambda \binom{y}{n-l}_\lambda \quad (n \geq 0). \tag{19}$$

By (1), we easily get

$$B_\lambda^*(n, k) = \left. \frac{d^n}{dt^n} (e_\lambda(t) + 1)^k \right|_{t=0} = \sum_{j=1}^k \binom{k}{j} (j)_{n,\lambda}, \tag{20}$$

where n and k are positive integers.

Note that $\lim_{\lambda \rightarrow 0} B_\lambda^*(n, k) = B(n, k)$, where $B(n, k)$ are defined by Golombek and given by

$$B(n, k) = \sum_{j=1}^k \binom{k}{j} j^n \quad (\text{see [5, 6]}).$$

Now, we define the *degenerate hypergeometric function* as

$$F_\lambda \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \sum_{n=0}^\infty \frac{\langle a \rangle_{n,\lambda} \langle b \rangle_{n,\lambda}}{\langle c \rangle_{n,\lambda}} \frac{z^n}{n!}, \tag{21}$$

where $\langle a \rangle_{n,\lambda} = a(a+\lambda) \cdots (a+(n-1)\lambda)$ ($n \geq 1$), $\langle a \rangle_{0,\lambda} = 1$.

From (21), we note that

$$\begin{aligned} F_\lambda \left(\begin{matrix} -n, -n \\ \lambda \end{matrix} \middle| \lambda e_\lambda(t) \right) &= \sum_{k=0}^\infty \frac{\langle -n \rangle_{k,\lambda} \langle -n \rangle_{k,\lambda} \lambda^k e_\lambda^k(t)}{\langle \lambda \rangle_{k,\lambda} k!} = \sum_{k=0}^\infty \frac{(n)_{k,\lambda} (n)_{k,\lambda} e_\lambda^k(t)}{k! k!} \\ &= \sum_{k=0}^\infty \binom{n}{k}_\lambda^2 e_\lambda^k(t) = \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{n}{k}_\lambda^2 (k)_{m,\lambda} \frac{t^m}{m!}, \end{aligned} \tag{22}$$

where n is a nonnegative integer.

Let us define

$$F_\lambda \left(\begin{matrix} -n, -n \\ \lambda \end{matrix} \middle| \lambda e_\lambda(t) \right) = \sum_{m=0}^\infty Q_\lambda(m, 2) \frac{t^m}{m!}. \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 2.1 For $m \geq 0$, we have

$$Q_\lambda(m, 2) = \sum_{k=0}^\infty \binom{n}{k}_\lambda^2 (k)_{m,\lambda}.$$

We note that $\lim_{\lambda \rightarrow 0} Q_\lambda(m, 2) = \sum_{k=0}^n \binom{n}{k}^2 k^m = Q(m, 2)$, which was introduced by Golombek and Marburg (see [5, 6]).

We observe that

$$F_\lambda \left(\begin{matrix} -a \\ \times \end{matrix} \middle| -z \right) = \sum_{k=0}^\infty (-a)_{k,\lambda} \frac{(-z)^k}{k!} = \sum_{k=0}^\infty \frac{(a)_{k,\lambda}}{k!} z^k = \sum_{k=0}^\infty \binom{a}{k}_\lambda z^k = e_\lambda^a(z).$$

For $n \in \mathbb{N}$, let

$$F_\lambda \left(\begin{matrix} -n \\ \times \end{matrix} \middle| -e_\lambda(t) \right) = (1 + \lambda e_\lambda(t))^\frac{n}{\lambda} = \sum_{m=0}^\infty H_\lambda(n, m) \frac{t^m}{m!}. \tag{24}$$

On the one hand, we have

$$(1 + \lambda e_\lambda(t))^\frac{n}{\lambda} = \sum_{k=0}^\infty \binom{n}{k}_\lambda e_\lambda^k(t) = \sum_{m=0}^\infty \left(\sum_{k=0}^\infty \binom{n}{k}_\lambda (k)_{m,\lambda} \right) \frac{t^m}{m!}. \tag{25}$$

On the other hand, we get

$$\begin{aligned} (1 + \lambda e_\lambda(t))^\frac{n}{\lambda} &= (1 + \lambda + \lambda(e_\lambda(t) - 1))^\frac{n}{\lambda} \\ &= (1 + \lambda)^\frac{n}{\lambda} \left(1 + \frac{\lambda}{1 + \lambda} (e_\lambda(t) - 1) \right)^\frac{n}{\lambda} \\ &= (1 + \lambda)^\frac{n}{\lambda} \sum_{k=0}^\infty \binom{n}{k}_\lambda \left(\frac{1}{1 + \lambda} \right)^k (e_\lambda(t) - 1)^k \\ &= (1 + \lambda)^\frac{n}{\lambda} \sum_{k=0}^\infty (n)_{k,\lambda} \left(\frac{1}{1 + \lambda} \right)^k \frac{1}{k!} (e_\lambda(t) - 1)^k \\ &= \sum_{m=0}^\infty \left((1 + \lambda)^\frac{n}{\lambda} \sum_{k=0}^m (n)_{k,\lambda} \left(\frac{1}{1 + \lambda} \right)^k S_{2,\lambda}(m, k) \right) \frac{t^m}{m!}. \end{aligned} \tag{26}$$

From (24), (25), and (26), we obtain the following theorem.

Theorem 2.2 For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have

$$H_\lambda(n, m) = (1 + \lambda)^\frac{n}{\lambda} \sum_{k=0}^m (n)_{k,\lambda} \left(\frac{1}{1 + \lambda} \right)^k S_{2,\lambda}(m, k) = \sum_{k=0}^\infty \binom{n}{k}_\lambda (k)_{m,\lambda}.$$

As is well known, the degenerate Bell polynomials are defined by

$$e_\lambda^x(e_\lambda(t) - 1) = \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} \quad (\text{see [16]}). \tag{27}$$

By (27), we easily get

$$\text{Bel}_{n,\lambda}(x) = \sum_{k=0}^n (x)_{k,\lambda} S_{2,\lambda}(n, k) \quad (n \geq 0). \tag{28}$$

Now, we define the *degenerate bivariate Bell polynomials* by

$$e_\lambda^x(y(e_\lambda(t) - 1)) = \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(x, y) \frac{t^n}{n!}. \tag{29}$$

Thus, by (29), we get

$$\text{Bel}_{n,\lambda}(x, y) = \sum_{k=0}^n (x)_{k,\lambda} y^k S_{2,\lambda}(n, k) \quad (n \geq 0). \tag{30}$$

From Theorem 2.2 and (30), we obtain the following corollary.

Corollary 2.3 *For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

$$H_\lambda(n, m) = (1 + \lambda)^{\frac{n}{\lambda}} \text{Bel}_{m,\lambda}\left(n, \frac{1}{1 + \lambda}\right).$$

Note that

$$\lim_{\lambda \rightarrow 0} H_\lambda(n, m) = \sum_{k=0}^\infty \frac{n^k}{k!} k^m = e^n \sum_{k=0}^m n^k S_2(m, k).$$

We observe that

$$\begin{aligned} H_\lambda(n, 1) &= \sum_{k=0}^\infty \binom{n}{k}_\lambda (k)_{1,\lambda} = \sum_{k=1}^\infty \frac{(n)_{k,\lambda}}{(k-1)!} \\ &= \sum_{k=0}^\infty \binom{n}{k}_\lambda (n - k\lambda) = n \sum_{k=0}^\infty \binom{n}{k}_\lambda - \lambda \sum_{k=0}^\infty \binom{n}{k}_\lambda k \\ &= n \sum_{k=0}^\infty \binom{n}{k}_\lambda - \lambda H_\lambda(n, 1). \end{aligned} \tag{31}$$

Thus, by (31), we get

$$(1 + \lambda)H_\lambda(n, 1) = n \sum_{k=0}^\infty \binom{n}{k}_\lambda = n(1 + \lambda)^{\frac{n}{\lambda}}. \tag{32}$$

From (32), we have

$$H_\lambda(n, 1) = n(1 + \lambda)^{\frac{n}{\lambda}-1} \quad (\lambda \neq -1).$$

For $m = 2$, we have

$$\begin{aligned}
 \sum_{k=0}^{\infty} \binom{n}{k}_{\lambda} (k)_{2,\lambda} &= \sum_{k=0}^{\infty} \frac{(n)_{k,\lambda}}{k!} k(k-\lambda) = \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} (k-\lambda) \\
 &= \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} (k-1+1-\lambda) = \sum_{k=2}^{\infty} \frac{(n)_{k,\lambda}}{(k-2)!} + (1-\lambda) \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} \\
 &= \sum_{k=2}^{\infty} \frac{(n-2\lambda)_{k-2,\lambda}}{(k-2)!} n(n-\lambda) + (1-\lambda)n \sum_{k=1}^{\infty} \frac{(n-\lambda)_{k-1,\lambda}}{(k-1)!} \\
 &= n(n-\lambda) \sum_{k=2}^{\infty} \binom{n-2\lambda}{k-2}_{\lambda} + n(1-\lambda) \sum_{k=1}^{\infty} \binom{n-\lambda}{k-1}_{\lambda} \\
 &= n(n-\lambda) \sum_{k=0}^{\infty} \binom{n-2\lambda}{k}_{\lambda} + n(1-\lambda) \sum_{k=0}^{\infty} \binom{n-\lambda}{k}_{\lambda} \\
 &= n(n-\lambda)(1+\lambda)^{\frac{n}{\lambda}-2} + n(1-\lambda)(1+\lambda)^{\frac{n}{\lambda}-1} \\
 &= n(n+1-\lambda-\lambda^2)(1+\lambda)^{\frac{n}{\lambda}-2}. \tag{33}
 \end{aligned}$$

By (33), we get

$$H_{\lambda}(n, 2) = n(n+1-2\lambda)(1+\lambda)^{\frac{n}{\lambda}-2}. \tag{34}$$

Let us take $m = 3$. Then we have

$$\begin{aligned}
 H_{\lambda}(n, 3) &= \sum_{k=0}^{\infty} \binom{n}{k}_{\lambda} (k)_{3,\lambda} = \sum_{k=0}^{\infty} \binom{n}{k}_{\lambda} k(k-\lambda)(k-2\lambda) \\
 &= \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} (k-\lambda)(k-2\lambda) = \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} (k-1+1-\lambda)(k-2\lambda) \\
 &= \sum_{k=2}^{\infty} \frac{(n)_{k,\lambda}}{(k-2)!} (k-2\lambda) + (1-\lambda) \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} (k-2\lambda) \\
 &= \sum_{k=3}^{\infty} \frac{(n)_{k,\lambda}}{(k-3)!} + 3(1-\lambda) \sum_{k=2}^{\infty} \frac{(n)_{k,\lambda}}{(k-2)!} + (1-\lambda)(1-2\lambda) \sum_{k=1}^{\infty} \frac{(n)_{k,\lambda}}{(k-1)!} \\
 &= (n)_{3,\lambda} \sum_{k=0}^{\infty} \binom{n-3\lambda}{k}_{\lambda} + 3(1-\lambda)(n)_{2,\lambda} \sum_{k=0}^{\infty} \binom{n-2\lambda}{k}_{\lambda} \\
 &\quad + (1-\lambda)(1-2\lambda)(n)_{1,\lambda} \sum_{k=0}^{\infty} \binom{n-\lambda}{k}_{\lambda} \\
 &= (n)_{3,\lambda}(1+\lambda)^{\frac{n}{\lambda}-3} + 3(1-\lambda)(n)_{2,\lambda}(1+\lambda)^{\frac{n}{\lambda}-2} + (1-\lambda)(1-2\lambda)n(1+\lambda)^{\frac{n}{\lambda}-1}.
 \end{aligned}$$

Note that

$$\lim_{\lambda \rightarrow 1} H_{\lambda}(n, 3) = \sum_{k=0}^n \binom{n}{k} (k)_3 = (n)_3 2^{n-3}$$

and

$$\lim_{\lambda \rightarrow 0} H_\lambda(n, 3) = \sum_{k=0}^\infty \frac{n^k}{k!} k^3 = n(n^2 + 3n + 1)e^n.$$

For $s \in \mathbb{C}$ with $R(s) > 0$, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Let n be a nonnegative integer. Then

$$\frac{\langle b \rangle_{n,\lambda}}{\langle c \rangle_{n,\lambda}} = \frac{\Gamma(\frac{b}{\lambda} + n)\Gamma(\frac{c}{\lambda})}{\Gamma(\frac{c}{\lambda} + n)\Gamma(\frac{b}{\lambda})},$$

where $R(\frac{c}{\lambda}) > 0$ and $R(\frac{b}{\lambda}) > 0$.

For $R(\frac{c}{\lambda}) > R(\frac{b}{\lambda}) > 0$, we have

$$\frac{\Gamma(\frac{b}{\lambda} + n)\Gamma(\frac{c}{\lambda} - \frac{b}{\lambda})}{\Gamma(\frac{c}{\lambda} + n)} = \int_0^1 t^{\frac{b}{\lambda} + n - 1} (1 - t)^{\frac{c}{\lambda} - \frac{b}{\lambda} - 1} dt. \tag{35}$$

From (35), we note that

$$\begin{aligned} F_\lambda \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) &= \sum_{n=0}^\infty \frac{\langle a \rangle_{n,\lambda} \langle b \rangle_{n,\lambda}}{\langle c \rangle_{n,\lambda}} \frac{z^n}{n!} \\ &= \frac{\Gamma(\frac{c}{\lambda})}{\Gamma(\frac{b}{\lambda})\Gamma(\frac{c}{\lambda} - \frac{b}{\lambda})} \int_0^1 t^{\frac{b}{\lambda} - 1} (1 - t)^{\frac{c}{\lambda} - \frac{b}{\lambda} - 1} (1 - \lambda tz)^{-\frac{a}{\lambda}} dt. \end{aligned} \tag{36}$$

In particular, for $z = \frac{1}{\lambda}$ ($\lambda \neq 0$), from (11) we get

$$F_\lambda \left(\begin{matrix} a, b \\ c \end{matrix} \middle| \frac{1}{\lambda} \right) = \frac{\Gamma(\frac{c}{\lambda})\Gamma(\frac{c}{\lambda} - \frac{b}{\lambda} - \frac{a}{\lambda})}{\Gamma(\frac{c}{\lambda} - \frac{b}{\lambda})\Gamma(\frac{c}{\lambda} - \frac{a}{\lambda})}, \quad \text{where } R\left(\frac{c}{\lambda} - \frac{b}{\lambda} - \frac{a}{\lambda}\right) > 0. \tag{37}$$

For $n \in \mathbb{N}$, by (37), we get

$$F_\lambda \left(\begin{matrix} -n, -n \\ \lambda \end{matrix} \middle| \frac{1}{\lambda} \right) = \frac{\Gamma(1)\Gamma(1 + \frac{2n}{\lambda})}{\Gamma(1 + \frac{n}{\lambda})\Gamma(1 + \frac{n}{\lambda})} = \frac{2\lambda}{n} \frac{\Gamma(\frac{2n}{\lambda})}{(\Gamma(\frac{n}{\lambda}))^2}, \tag{38}$$

where λ is a positive real number.

On the other hand,

$$F_\lambda \left(\begin{matrix} -n, -n \\ \lambda \end{matrix} \middle| \frac{1}{\lambda} \right) = \sum_{k=0}^\infty \frac{\langle -n \rangle_{k,\lambda} \langle -n \rangle_{k,\lambda}}{\langle \lambda \rangle_{k,\lambda}} \frac{(\frac{1}{\lambda})^k}{k!} = \sum_{k=0}^\infty \lambda^{-2k} \binom{n}{k}_\lambda. \tag{39}$$

Therefore, by (38) and (39), we obtain the following theorem.

Theorem 2.4 *Let λ be a positive real number. For $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty} \lambda^{-2k} \binom{n}{k}_{\lambda}^2 = \frac{2\lambda}{n} \frac{\Gamma(\frac{2n}{\lambda})}{(\Gamma(\frac{n}{\lambda}))^2}.$$

Note that

$$\lim_{\lambda \rightarrow 1} \sum_{k=0}^{\infty} \lambda^{-2k} \binom{n}{k}_{\lambda}^2 = \sum_{k=0}^n \binom{n}{k}^2 = \frac{(2n)!}{(n!)^2} = \binom{2n}{n}.$$

Now, we define the *degenerate generalized hypergeometric function* as

$$F_{\lambda}^{(p,q)} \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{\langle a_1 \rangle_{k,\lambda} \cdots \langle a_p \rangle_{k,\lambda}}{\langle b_1 \rangle_{k,\lambda} \cdots \langle b_q \rangle_{k,\lambda}} \frac{z^k}{k!}, \quad \text{where } |z| < 1. \tag{40}$$

Let n be a positive integer. Then we define the *degenerate hypergeometric numbers of order p* by

$$F_{\lambda}^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-1)^p \lambda^{p-1} e_{\lambda}(t) \right) = \sum_{m=0}^{\infty} H_{\lambda}^{(p)}(n, m) \frac{t^m}{m!}. \tag{41}$$

From (23) and (24), we note that $H_{\lambda}(n, m) = H_{\lambda}^{(1)}(n, m)$, and $Q_{\lambda}(m, 2) = H_{\lambda}^{(2)}(n, m)$.

In (40), we note that

$$\begin{aligned} & F_{\lambda}^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-1)^p \lambda^{p-1} e_{\lambda}(t) \right) \\ &= \sum_{k=0}^{\infty} \frac{\langle -n \rangle_{k,\lambda} \langle -n \rangle_{k,\lambda} \cdots \langle -n \rangle_{k,\lambda}}{\langle \lambda \rangle_{k,\lambda} \langle \lambda \rangle_{k,\lambda} \cdots \langle \lambda \rangle_{k,\lambda}} \frac{(-1)^{pk} \lambda^{(p-1)k}}{k!} e_{\lambda}^k(t) \\ &= \sum_{k=0}^{\infty} \frac{((n)_{k,\lambda})^p}{(k!)^{p-1}} \frac{e_{\lambda}^k(t)}{k!} = \sum_{k=0}^{\infty} \binom{n}{k}_{\lambda}^p e_{\lambda}^k(t) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{n}{k}_{\lambda}^p (k)_{m,\lambda} \right) \frac{t^m}{m!}. \end{aligned} \tag{42}$$

Therefore, by (41) and (42), we obtain the following theorem.

Theorem 2.5 *For $n, p \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have*

$$H_{\lambda}^{(p)}(n, m) = \sum_{k=0}^{\infty} \binom{n}{k}_{\lambda}^p (k)_{m,\lambda}.$$

Note that

$$\lim_{\lambda \rightarrow 1} H_{\lambda}^{(p)}(n, m) = \sum_{k=0}^n \binom{n}{k}^p (k)_m \quad \text{and} \quad \lim_{\lambda \rightarrow 0} H_{\lambda}^{(p)}(n, m) = \sum_{k=0}^{\infty} \frac{n^{kp}}{(k!)^p} k^m.$$

From (42), we note that

$$\begin{aligned}
 &F_\lambda^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-1)^p \lambda^{p-1} e_\lambda(t) \right) \\
 &= \sum_{k=0}^\infty \binom{n}{k}_\lambda^p e_\lambda^k(t) \\
 &= \sum_{k=0}^\infty \binom{n}{k}_\lambda^p \sum_{l=0}^k \binom{k}{l} (e_\lambda(t) - 1)^l = \sum_{k=0}^\infty \binom{n}{k}_\lambda^p \sum_{l=0}^k (k)_l \sum_{m=l}^\infty S_{2,\lambda}(m, l) \frac{t^m}{m!} \\
 &= \sum_{m=0}^\infty \left(\sum_{k=0}^\infty \sum_{l=0}^k \binom{n}{k}_\lambda^p (k)_l S_{2,\lambda}(m, l) \right) \frac{t^m}{m!}.
 \end{aligned} \tag{43}$$

Therefore, by (41) and (43), we obtain the following theorem.

Theorem 2.6 For $n, p \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have

$$H_\lambda^{(p)}(n, m) = \sum_{k=0}^\infty \sum_{l=0}^k \binom{n}{k}_\lambda^p (k)_l S_{2,\lambda}(m, l).$$

By (41) and (42), we get

$$H_\lambda^{(p)}(n, 0) = \sum_{k=0}^\infty \binom{n}{k}_\lambda^p = F_\lambda^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-1)^p \lambda^{p-1} \right).$$

Note that

$$\sum_{k=0}^n \binom{n}{k}^p = \lim_{\lambda \rightarrow 1} H_\lambda^{(p)}(n, 0) = F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^p \right).$$

From (40), we have

$$\begin{aligned}
 &F_\lambda^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-\lambda)^{p-1} e_\lambda(t) \right) \\
 &= \sum_{k=0}^\infty \binom{n}{k}_\lambda^p (-1)^k e_\lambda^k(t) = \sum_{m=0}^\infty \left(\sum_{k=0}^\infty \binom{n}{k}_\lambda^p (-1)^k (k)_{m,\lambda} \right) \frac{t^m}{m!}.
 \end{aligned} \tag{44}$$

Thus, by (44), we get

$$F_\lambda^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-\lambda)^{p-1} \right) = \sum_{k=0}^\infty \binom{n}{k}_\lambda^p (-1)^k. \tag{45}$$

Note that

$$\begin{aligned} & \lim_{\lambda \rightarrow 1} F_\lambda^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ \lambda, \dots, \lambda \end{matrix} \middle| (-\lambda)^{p-1} \right) \\ &= \sum_{k=0}^n \binom{n}{k}^p (-1)^k = F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^{p-1} \right). \end{aligned}$$

3 Further remarks

Let n be a positive integer. From (10), we have

$$\begin{aligned} F^{(2,1)} \left(\begin{matrix} -n, -n \\ 1 \end{matrix} \middle| e_\lambda(t) \right) &= \sum_{k=0}^\infty \frac{\langle -n \rangle_k \langle -n \rangle_k e_\lambda^k(t)}{\langle 1 \rangle_k k!} \\ &= \sum_{k=0}^n \binom{n}{k}^2 e_\lambda^k(t) = \sum_{m=0}^\infty \left(\sum_{k=0}^n \binom{n}{k}^2 (k)_{m,\lambda} \right) \frac{t^m}{m!}. \end{aligned} \tag{46}$$

Now, we define the λ -hypergeometric numbers of order p by

$$F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^p e_\lambda(t) \right) = \sum_{m=0}^\infty H_{m,\lambda}^{(p)}(n) \frac{t^m}{m!}. \tag{47}$$

By (46) and (47), we get

$$H_{m,\lambda}^{(2)}(n) = \sum_{k=0}^n \binom{n}{k}^2 (k)_{m,\lambda}, \tag{48}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.

The alternating λ -hypergeometric numbers of order p are defined by

$$F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^{p-1} e_\lambda(t) \right) = \sum_{m=0}^\infty T_{m,\lambda}^{(p)}(n) \frac{t^m}{m!}. \tag{49}$$

By (10), we get

$$F^{(2,1)} \left(\begin{matrix} -n, -n \\ 1 \end{matrix} \middle| -e_\lambda(t) \right) = \sum_{m=0}^\infty \left(\sum_{k=0}^\infty \binom{n}{k}^2 (-1)^k (k)_{m,\lambda} \right) \frac{t^m}{m!}. \tag{50}$$

From (49) and (50), we have

$$T_{m,\lambda}^{(2)}(n) = \sum_{k=0}^n \binom{n}{k}^2 (-1)^k (k)_{m,\lambda}, \tag{51}$$

where $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

In general, we have

$$T_{m,\lambda}^{(p)}(n) = \sum_{k=0}^n \binom{n}{k}^p (-1)^k (k)_{m,\lambda} \quad \text{and} \quad H_{m,\lambda}^{(p)}(n) = \sum_{k=0}^n \binom{n}{k}^p (k)_{m,\lambda},$$

where $n, p \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.

We observe that

$$F^{(1,0)} \left(\begin{matrix} -n \\ \times \\ -e_\lambda(t) \end{matrix} \right) = \sum_{k=0}^n \binom{n}{k} e_\lambda^k(t) \tag{52}$$

and

$$F^{(1,0)} \left(\begin{matrix} -n \\ \times \\ e_\lambda(t) \end{matrix} \right) = \sum_{k=0}^n \binom{n}{k} (-1)^k e_\lambda^k(t). \tag{53}$$

Thus, we note that

$$H_{m,\lambda}^{(1)}(n) = \sum_{k=0}^n \binom{n}{k} (k)_{m,\lambda}, \quad T_{m,\lambda}^{(1)} = \sum_{k=0}^n \binom{n}{k} (-1)^k (k)_{m,\lambda},$$

where $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

For example,

$$\begin{aligned} H_{0,\lambda}^{(1)}(n) &= 2^n, & H_{1,\lambda}^{(1)}(n) &= n2^{n-1}, & H_{2,\lambda}^{(1)}(n) &= n(n+1-2\lambda)2^{n-2}, \\ H_{3,\lambda}^{(1)}(n) &= (n)_3 2^{n-3} + 3(n)_2(1-\lambda)2^{n-2} + n(1-\lambda)_{2,\lambda} 2^{n-1}, \dots, \\ T_{0,\lambda}^{(1)}(n) &= 0, & T_{1,\lambda}^{(1)}(n) &= -\delta_{1,n}, \\ T_{2,\lambda}^{(1)}(n) &= n(n-1)\delta_{2,n} + (\lambda-1)\delta_{n,1}, \dots, \end{aligned}$$

where $\delta_{n,k}$ is Kronecker's symbol.

From (49), we note that

$$\begin{aligned} \sum_{m=0}^{\infty} T_{m,\lambda}^{(1)}(n) \frac{t^m}{m!} &= F^{(1,0)} \left(\begin{matrix} -n \\ \times \\ e_\lambda(t) \end{matrix} \right) = (1 - e_\lambda(t))^n = (-1)^n \frac{n!}{n!} (e_\lambda(t) - 1)^n \\ &= (-1)^n n! \sum_{m=n}^{\infty} S_{2,\lambda}(m, n) \frac{t^m}{m!}. \end{aligned} \tag{54}$$

On the other hand,

$$(1 - e_\lambda(t))^n = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} e_\lambda^j(t) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} (-1)^{n-j} (j)_{m,\lambda} \right) \frac{t^m}{m!}. \tag{55}$$

Thus, by (54) and (55), we get

$$T_{m,\lambda}^{(1)}(n) = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} (j)_{m,\lambda} = (-1)^n n! S_{2,\lambda}(m, n) \quad (m \geq n). \tag{56}$$

If $m < n$, then $T_{m,\lambda}^{(1)}(n) = 0$.

Theorem 3.1 For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have

$$T_{m,\lambda}^{(1)}(n) = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} (j)_{m,\lambda} = (-1)^n n! S_{2,\lambda}(m, n) \quad (m \geq n).$$

In particular, if $m < n$, then

$$T_{m,\lambda}^{(1)} = 0.$$

For $k \geq 0$, we have

$$\lim_{\lambda \rightarrow 0} (-1)^n T_{n+k,\lambda}^{(1)}(n) = \sum_{j=0}^n \binom{n}{j} (-1)^j j^{n+k} = n! S_2(n+k, n).$$

Corollary 3.2 For $k \geq 0$ and $n \in \mathbb{N}$, we have

$$\frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (-1)^j j^{n+k} = S_2(n+k, n).$$

It is easy to show that

$$\begin{aligned} (e^t - 1)^n &= \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} e^{jt} \\ &= \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (-1)^{n-j} (j)_l \frac{1}{l!} (e^t - 1)^l \\ &= \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (j)_l (-1)^{n-j} \sum_{m=l}^{\infty} S_2(m, l) \frac{t^m}{m!}. \end{aligned}$$

Thus, we have

$$\sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} = \frac{1}{n!} (e^t - 1)^n = \frac{1}{n!} \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (j)_l (-1)^{n-j} \sum_{m=l}^{\infty} S_2(m, l) \frac{t^m}{m!}. \tag{57}$$

For $k \geq 0$, $n \in \mathbb{N}$, by (57), we get

$$S_2(n+k, n) = \frac{1}{n!} \sum_{j=0}^n \sum_{l=0}^{n+k} \binom{n}{j} (j)_l (-1)^{n-j} S_2(n+k, l). \tag{58}$$

From Corollary 3.2 and (58), we have

$$\sum_{j=0}^n \binom{n}{j} (-1)^j j^{n+k} = \sum_{j=0}^n \sum_{l=0}^{n+k} \binom{n}{j} (j)_l (-1)^{n-j} S_2(n+k, l).$$

For $\lambda, \lambda_1 \in \mathbb{R}$, let us define *Apostol type alternating λ -hypergeometric numbers of order p* by

$$F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^{p-1} \lambda_1 e_\lambda(t) \right) = \sum_{m=0}^{\infty} T_{m,\lambda}^{(p)}(n|\lambda_1) \frac{t^m}{m!}. \tag{59}$$

By (10), we get

$$\begin{aligned}
 F^{(1,0)}\left(\begin{matrix} -n \\ \times \end{matrix} \middle| \lambda_1 e_\lambda(t)\right) &= \sum_{k=0}^n \binom{n}{k} (-\lambda_1 e_\lambda(t))^k = \sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j e_\lambda^j(t) \\
 &= \sum_{m=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j(j)_{m,\lambda} \right) \frac{t^m}{m!}.
 \end{aligned} \tag{60}$$

On the other hand,

$$\begin{aligned}
 F^{(1,0)}\left(\begin{matrix} -n \\ \times \end{matrix} \middle| \lambda_1 e_\lambda(t)\right) &= (1 - \lambda_1 e_\lambda(t))^n = \sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j e_\lambda^j(t) \\
 &= \sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j \sum_{l=0}^j \binom{j}{l} (e_\lambda(t) - 1)^l \\
 &= \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (-1)^j \lambda_1^j(j)_l \frac{(e_\lambda(t) - 1)^l}{l!} \\
 &= \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (-1)^j \lambda_1^j(j)_l \sum_{m=l}^{\infty} S_{2,\lambda}(m, l) \frac{t^m}{m!}.
 \end{aligned} \tag{61}$$

For $k \geq 0$ and $n \in \mathbb{N}$, by (59), (60), and (61), we obtain the following theorem.

Theorem 3.3 For $\lambda, \lambda_1 \in \mathbb{R}$, $n \in \mathbb{N}$, and $k \in \mathbb{N} \cup \{0\}$, we have

$$T_{n+k,\lambda}^{(1)}(n \mid \lambda_1) = \sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j(j)_{n+k,\lambda} = \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (-1)^j \lambda_1^j(j)_l S_{2,\lambda}(n+k, l).$$

Note that

$$\lim_{\lambda \rightarrow 0} T_{n+k,\lambda}^{(1)}(n \mid \lambda_1) = \sum_{j=0}^n \binom{n}{j} (-1)^j \lambda_1^j j^{n+k} = \sum_{j=0}^n \sum_{l=0}^j \binom{n}{j} (-1)^j \lambda_1^j(j)_l S_2(n+k, l).$$

For $\lambda_1 \in \mathbb{R}$, let us define *Apstol–Stirling numbers of the second kind* as

$$\frac{1}{n!} (\lambda_1 e^t - 1)^n = \sum_{m=0}^{\infty} S(m, n \mid \lambda_1) \frac{t^m}{m!} \quad (n \geq 0). \tag{62}$$

Now, we observe that

$$\frac{1}{n!} (\lambda_1 e^t - 1)^n = \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} \lambda_1^j (-1)^{n-j} e^{jt} = \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} \lambda_1^j (-1)^{n-j} \sum_{m=0}^{\infty} j^m \frac{t^m}{m!}. \tag{63}$$

For $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, by (62) and (63), we get

$$S(n+k, n \mid \lambda_1) = \frac{(-1)^n}{n!} \sum_{j=0}^n \binom{n}{j} \lambda_1^j (-1)^j j^{n+k}. \tag{64}$$

From Theorem 3.3 and (64), we have

$$\lim_{\lambda \rightarrow 0} T_{n+k,\lambda}^{(1)}(n | \lambda_1) = (-1)^n n! S(n+k, n | \lambda_1) = \sum_{j=0}^n \binom{n}{j} \lambda_1^j (-1)^j j^{n+k}. \tag{65}$$

Therefore, by (65), we obtain the following corollary.

Corollary 3.4 For $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, we have

$$\sum_{j=0}^n \binom{n}{j} \lambda_1^j (-1)^j j^{n+k} = (-1)^n n! S(n+k, n | \lambda) = \lim_{\lambda \rightarrow 0} T_{n+k,\lambda}^{(1)}(n | \lambda_1).$$

Remarks

- (a) Corollary 3.4 naturally interprets the sum in Corollary 3.4 in terms of Apostol–Stirling numbers of the second kind defined by (62).
- (b) For $\lambda, \lambda_1 \in \mathbb{R}$, let us define *Apostol type λ -hypergeometric numbers of order p* by

$$F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^p \lambda_1 e_\lambda(t) \right) = \sum_{m=0}^{\infty} H_{m,\lambda}^{(p)}(n | \lambda_1) \frac{t^m}{m!}. \tag{66}$$

By (10), we get

$$\begin{aligned} F^{(p,p-1)} \left(\begin{matrix} -n, -n, \dots, -n \\ 1, \dots, 1 \end{matrix} \middle| (-1)^p \lambda_1 e_\lambda(t) \right) &= \sum_{k=0}^{\infty} \binom{n}{k}^p \lambda_1^k e_\lambda^k(t) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}^p \lambda_1^k (k)_{m,\lambda} \right) \frac{t^m}{m!}. \end{aligned} \tag{67}$$

From (66) and (67), we have

$$H_{m,\lambda}^{(p)}(n | \lambda_1) = \sum_{k=0}^n \binom{n}{k}^p \lambda_1^p (k)_{m,\lambda}, \quad \text{where } n, p \in \mathbb{N} \text{ and } m \in \mathbb{N} \cup \{0\}.$$

Note that

$$\lim_{\lambda \rightarrow 0} H_{m,\lambda}^{(p)}(n | \lambda_1) = \sum_{k=0}^n \binom{n}{k}^p \lambda_1^p k^m.$$

4 Conclusion

In this paper, we studied certain finite sums involving powers of binomial coefficients which are called generalized p th order Franel numbers and can be represented in terms of hypergeometric functions. Then, among other things, we found that particular cases of these numbers are connected with many known special numbers and polynomials which include Bernoulli numbers, Euler numbers, Changhee numbers, Daehee numbers, Stirling numbers of the first kind, Catalan numbers, and Legendre polynomials. Recently, Dolgy and Kim gave some explicit formulas of degenerate Stirling numbers associated with the degenerate special numbers and polynomials. Motivated by Dolgy and Kim’s paper [4], we

investigated degenerate versions of the generalized p th order Franel numbers. In more detail, we introduced degenerate generalized hypergeometric functions and studied degenerate hypergeometric numbers of order p . We showed that the degenerate hypergeometric numbers of order p involve powers of λ -binomial coefficients and λ -falling sequence, and can be represented by means of the degenerate generalized hypergeometric functions. We also considered several related special numbers like λ -hypergeometric numbers of order p and Apostol type λ -hypergeometric numbers of order p , of which the latter reduce in a limiting case to the generalized p th order Franel numbers.

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Authors' contributions

TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK paid the article processing charge for the paper; HL typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

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