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The Green's function of a class of two-term fractional differential equation boundary value problem and its applications

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Abstract

In this paper, we consider a Riemann–Liouville type two-term fractional differential equation boundary value problem. Some positive properties of the Green's function are deduced by using techniques of analysis. As application, we obtain the existence and multiplicity of positive solutions for a fractional boundary value problem under conditions that the nonlinearity $f(t,x)$ may change sign and may be singular at $t = 0, 1$ and $x = 0$, and we also obtain the uniqueness results of positive solution for a singular problem by means of the monotone iterative technique.

Keywords: Two-term fractional differential equation; Boundary value problems; Green's function; Positive solution; Singularity

1 Introduction

In this paper, we study properties of the Green's function of the following two-term fractional differential equation boundary value problem (FBVP):

$$\begin{cases} -D_{0+}^{\alpha}u(t) + au(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = 0, \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $a > 0$, D_{0+}^{α} is the standard Riemann–Liouville derivative.

During the past decades, much attention has been paid to the study of fractional differential equations (FDEs) due to the more accurate effect in describing important phenomena in biology, engineering, and so on. It has been proved that a multi-term FDE can be used to describe various types of visco-elastic damping [1, 2]. Most of the model equations proposed can be expressed by the linear form

$$\left[D^{\alpha_N} + a_{N-1}D^{\alpha_{N-1}} + \cdots + a_1 D^{\alpha_1} + a_0 D^0\right]x(t) = f(t),$$

where $a_i \in \mathbf{R}$, $i = 0, 1, \dots, N-1$, equipped with initial conditions (see [3–7] and the references therein). For example, Elshehawey et al. [5] considered the endolymph equation

$$D^2x(t) + a_1 Dx(t) + a_2 D^{\frac{1}{2}}x(t) + a_3 x(t) = -f(t),$$

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which can be used to describe the response of the semicircular canals to the angular acceleration.

Recently, many authors have focused on the existence of solutions to nonlinear FBVPs by using the techniques of nonlinear analysis such as fixed point theorems, Leray–Schauder theory, etc. (see [8–32]). Since only positive solutions are meaningful in most practical problems, the existence of positive solutions for FBVPs has particularly attracted a great deal of attention, e.g., the nonlocal FBVPs [10, 22, 25], singular FBVPs [16, 21, 28], semipositone FBVPs [15, 18, 27].

It is known that the cone which usually depends on the positive properties of the Green's function plays a very important role in discussing positive solutions. When $1 < \alpha < 2$, Jiang and Yuan [14] obtained some properties of the Green's function for the FDE:

$$D_{0+}^{\alpha} u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \tag{2}$$

with Dirichlet type boundary value condition. Xu and Fei [30] investigated (2) with three-point boundary value condition. In [19], we established some new positive properties of the corresponding Green's function for (2) with multi-point boundary value condition. When $\alpha > 2$, Zhang et al. [27, 28] obtained triple positive solutions for (2) with conjugate type integral conditions by employing height functions on special bounded sets which were derived from properties of the Green's function.

While there are a lot of works dealing with multi-term FDEs with initial conditions, the results dealing with boundary value problems of multi-term FDEs are relatively scarce. For some recent literature on Caputo type multi-term FBVPs, we mention the papers [8, 9] and the references therein. In [20], we established some new positive properties of the Green's function for the Riemann–Liouville type FBVP, in which the linear operator contains two terms:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + bu(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $b > 0$. As application, the existence and uniqueness of positive solution are obtained under singular conditions.

Inspired by the above work, in this paper, we aim to deduce some positive properties of the Green's function for FBVP (1). As application, we investigate the existence and multiplicity of positive solutions for a singular FBVP with changing sign nonlinearity, and we also consider the uniqueness results of positive solution for a singular FBVP. Compared with the existing works, this paper has the following features. Firstly, the fractional derivative discussed in this paper is the standard Riemann–Liouville derivative, which is different from [8, 9], and the linear operator of the FBVP we are considered with contains two terms, which is different from [14, 19, 27, 28, 30]; in other words, we discuss different problem which has been seldom studied before. Secondly, some meaningful properties of the Green's function for the case that $2 < \alpha < 3$ are established; this is different from [20] since Ref. [20] considered the case that $1 < \alpha < 2$. Thirdly, we consider a multiplicity of positive solutions under conditions that the nonlinearity $f(t, x)$ may change sign and possess singularity at $x = 0$; this is different from [15, 18]. It should be noted that there are relatively few results on multiple solutions for FBVPs under this circumstance, not to mention two-term FBVPs. Finally, we obtain the uniqueness results of positive solution for a singular two-term FBVP by means of the monotone iterative technique, and the rate of convergence for the iterative sequence is considered.

2 Basic definitions and preliminaries

Definition 2.1 ([33]) The fractional integral of a function $u:(0,+\infty)\to R$ is given by

$$I_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\,ds$$

provided that the right-hand side is point-wise defined on $(0,+\infty)$.

Definition 2.2 ([33]) The Riemann–Liouville fractional derivative of a function $u:(0,+\infty)\to R$ is given by

$$D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}u(s)\,ds,$$

where $n=[\alpha]+1$, $[\alpha]$denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0,+\infty)$.

For convenience, we introduce the following notations:

$$\begin{aligned}
&h(x)=\sum_{k=0}^{+\infty}\frac{(k\alpha+\alpha-2)(k\alpha+\alpha-3)x^k}{\Gamma(k\alpha+\alpha)},\\
&k(s)=(1-s)^{\alpha-2}-s,\\
&g(t)=t^{\alpha-1}E_{\alpha,\alpha}\bigl(at^{\alpha}\bigr),\\
&g_1(t)=t^{\alpha-2}E_{\alpha,\alpha}\bigl(at^{\alpha}\bigr),
\end{aligned}$$

where

$$E_{\alpha,\alpha}(x)=\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma((k+1)\alpha)}$$

is the Mittag-Leffler function.

It is clear that $h(x)$ is strictly increasing on $[0,+\infty)$, $h(0)<0$, and

$$\lim_{x\to+\infty}h(x)=+\infty.$$

Therefore, $h(x)$ has a unique positive root a^*, that is, $h(a^*)=0$.

Throughout this paper, we always assume that the following assumption holds:

(H_1) $a\in(0,a^*]$ is a constant.

Lemma 2.1 *Let $y\in L^1[0,1]\cap C(0,1)$. Then the unique solution of the two-term FBVP (1) is*

$$u(t)=\int_0^1G(t,s)y(s)\,ds,$$

where

$$G(t,s) = \frac{1}{g(1)} \begin{cases} g(t)g(1-s), & 0 \le t \le s \le 1, \\ g(t)g(1-s) - g(t-s)g(1), & 0 \le s \le t \le 1. \end{cases} \tag{3}$$

Proof It follows from [33] that the general solution of the equation

$$-D_{0+}^{\alpha} u(t) + au(t) = y(t)$$

can be expressed by

$$u(t) = -\int_0^t g(t-s)y(s)\,ds + c_1 g(t) + c_2 g'(t) + c_3 g''(t).$$

By direct calculation, we have

$$g'(t) = \sum_{k=0}^{+\infty} \frac{a^k t^{(k+1)\alpha-2}}{\Gamma((k+1)\alpha-1)} > 0, \quad \forall t > 0, \tag{4}$$

$$g''(t) = \sum_{k=0}^{+\infty} \frac{a^k t^{(k+1)\alpha-3}}{\Gamma((k+1)\alpha-2)} > 0, \quad \forall t > 0. \tag{5}$$

By $u(0) = u'(0) = 0$, there is $c_3 = c_2 = 0$. Then we get

$$u(1) = -\int_0^1 g(1-s)y(s)\,ds + c_1 g(1).$$

It follows from $u(1) = 0$ that

$$c_1 = \frac{\int_0^1 g(1-s)y(s)\,ds}{g(1)}.$$

Therefore, the solution of (1) is

$$\begin{aligned} u(t) &= -\int_0^t g(t-s)y(s)\,ds + \frac{\int_0^1 g(1-s)y(s)\,ds}{g(1)} g(t) \\ &= \frac{\int_0^1 g(t)g(1-s)y(s)\,ds - \int_0^t g(1)g(t-s)y(s)\,ds}{g(1)} \\ &= \int_0^1 G(t,s)y(s)\,ds. \end{aligned}$$

□

Remark 2.1 The unique solution given in Lemma 2.1 satisfies $u \in AC^1[0,1]$, where

$$AC^n[0,1] = \left\{ v : [0,1] \to \mathbf{R} \text{ and } \frac{d^{n-1}v}{dt^{n-1}} \text{ is absolutely continuous on } [0,1] \right\}.$$

Proof In fact, we have

$$u(t) = -\int_0^t g(t-s)y(s)\,ds + c_1 g(t) = c_1 g(t) - I_{0+}^{\alpha} y(t) - \sum_{k=1}^{+\infty} I_{0+}^{(k+1)\alpha} y(t).$$

It follows from [32, Lemma 2.1] that $I_{0+}^{\alpha}y(t) \in AC^1[0,1]$ and

$$I_{0+}^{(k+1)\alpha}y(t) \in AC^{[(k+1)\alpha]-1}[0,1].$$

Notice that $g(t) \in AC^1[0,1]$, we can get $u \in AC^1[0,1]$. □

3 Main results

Lemma 3.1 *For* $0 \leq s \leq t \leq 1$, *we have*

$$g_1(t)g_1(1-s) \geq g_1(t-s)g_1(1).$$

Proof For $t > 0$, we have

$$g_1'(t) = \sum_{k=0}^{+\infty} \frac{(k\alpha+\alpha-2)a^k t^{k\alpha+\alpha-3}}{\Gamma(k\alpha+\alpha)} > 0.$$

Therefore, $g_1(t)$ is strictly increasing on $[0,1]$. By direct calculation, we have

$$g_1''(t) = t^{\alpha-4}h\bigl(at^{\alpha}\bigr) < t^{\alpha-4}h\bigl(a^*\bigr) = 0, \quad t \in (0,1),$$

which implies $g_1'(t)$ is strictly decreasing on $(0,1]$. Thus

$$\begin{aligned}
&\frac{\partial}{\partial s}\bigl[g_1(t)g_1(1-s) - g_1(t-s)g_1(1)\bigr] \\
&\quad = g_1'(t-s)g_1(1) - g_1(t)g_1'(1-s) \\
&\quad \geq g_1'(1-s)\bigl[g_1(1) - g_1(t)\bigr] \geq 0.
\end{aligned}$$

Therefore we can get

$$g_1(t)g_1(1-s) - g_1(t-s)g_1(1) \geq 0,$$

that is,

$$g_1(t)g_1(1-s) \geq g_1(t-s)g_1(1).$$ □

Lemma 3.2 *Assume that* $s^\star \in (0,1)$ *satisfies* $s^\star = (1-s^\star)^{\alpha-2}$, *then*

$$\min\bigl\{s,(1-s)^{\alpha-2}\bigr\} \leq \frac{s(1-s)^{\alpha-2}}{s^\star}, \quad s \in [0,1]. \tag{6}$$

Proof It is clear that $k(s)$ is strictly decreasing on $[0,1]$. Notice that $k(0) = 1$ and $k(1) = -1$, we know $k(s)$ has a unique root $s^\star$ on $(0,1)$, that is, $s^\star = (1-s^\star)^{\alpha-2}$. Therefore,

$$\min\bigl\{s,(1-s)^{\alpha-2}\bigr\} = \begin{cases} s, & s \in [0,s^\star], \\ (1-s)^{\alpha-2}, & s \in [s^\star,1]. \end{cases}$$

Thus

$$\min\bigl\{s,(1-s)^{\alpha-2}\bigr\} \leq \frac{s(1-s)^{\alpha-2}}{s^\star}, \quad s \in [0,1].$$ □

Theorem 3.1 *The Green's function $G(t,s)$ satisfies the following properties*:

(p_1) $G(t,s) > 0, \forall t,s \in (0,1)$;

(p_2) $G(t,s) = G(1-s,1-t), \forall t,s \in [0,1]$;

(p_3) $G(t,s) \geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}, \forall t,s \in [0,1]$;

(p_4) $G(t,s) \leq M_2 s(1-s)^{\alpha-1}, \forall t,s \in [0,1]$, *where*

$$M_1 = \frac{1}{g(1)[\Gamma(\alpha)]^2}, \qquad M_2 = \frac{[g'(1)]^2}{g(1)s^\star}.$$

Proof Since (p_2) is trivially true and (p_1) can be derived from (p_3), it remains to verify (p_3) and (p_4).

For $t \in [0,1]$, it is easy to check that

$$g(t) = \sum_{k=0}^{+\infty} \frac{a^k t^{k\alpha+\alpha-1}}{\Gamma((k+1)\alpha)} \geq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{7}$$

and

$$g(t) \leq t^{\alpha-1} \sum_{k=0}^{+\infty} \frac{a^k}{\Gamma((k+1)\alpha)} = t^{\alpha-1} g(1). \tag{8}$$

Combining the notations of g and g' with

$$\Gamma\big((k+1)\alpha\big) > \Gamma\big((k+1)\alpha - 1\big), \quad k = 0,1,2,\ldots,$$

one has

$$g(1) < g'(1). \tag{9}$$

Case (I): $0 \leq t \leq s \leq 1$.

By (6), one has

$$\begin{aligned} G(t,s) &= \frac{g(t)g(1-s)}{g(1)} \geq \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{g(1)[\Gamma(\alpha)]^2} \\ &\geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}. \end{aligned} \tag{10}$$

By (7), one has

$$\begin{aligned} G(t,s) &\leq g(1)t^{\alpha-1}(1-s)^{\alpha-1} \\ &\leq g(1)s(1-s)^{\alpha-1} \\ &\leq M_2 s(1-s)^{\alpha-1}. \end{aligned} \tag{11}$$

Case (II): $0 < s < t < 1$.

It is obvious that

$$g(t) = tg_1(t).$$

Therefore, it follows from Lemma 3.1 and (7) that

$$\begin{aligned} G(t,s) &= \frac{g(t)g(1-s)-g(t-s)g(1)}{g(1)} \\ &= \frac{t(1-s)g_1(t)g_1(1-s)-(t-s)g_1(t-s)g_1(1)}{g(1)} \\ &\geq \frac{g_1(t)g_1(1-s)[t(1-s)-(t-s)]}{g(1)} \\ &= \frac{g_1(t)g_1(1-s)s(1-t)}{g(1)} \\ &\geq \frac{t^{\alpha-2}(1-s)^{\alpha-2}s(1-t)}{g(1)[\Gamma(\alpha)]^2} \\ &\geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}. \end{aligned} \tag{12}$$

By the monotonicity of $g'(t)$, we have

$$\begin{aligned} \frac{\partial}{\partial s}G(t,s) &= \frac{g'(t-s)g(1)-g(t)g'(1-s)}{g(1)} \\ &\leq \frac{g'(1-s)[g(1)-g(t)]}{g(1)}. \end{aligned}$$

By Lagrange's mean value theorem, there exist $\zeta \in (1-s,1)$ and $\eta \in (t,1)$ such that

$$\begin{aligned} G(t,s) &= \int_0^s \frac{\partial}{\partial \tau}G(t,\tau)\,d\tau \\ &\leq \int_0^s \frac{g'(1-\tau)[g(1)-g(t)]}{g(1)}\,d\tau \\ &= \frac{[g(1)-g(1-s)][g(1)-g(t)]}{g(1)} \\ &= \frac{g'(\zeta)sg'(\eta)(1-t)}{g(1)} \\ &\leq \frac{[g'(1)]^2 s(1-s)}{g(1)}. \end{aligned} \tag{13}$$

On the other hand, it follows from (8) that

$$G(t,s) \leq \frac{g(t)g(1-s)}{g(1)} \leq g(1)t^{\alpha-1}(1-s)^{\alpha-1} \leq g(1)(1-s)^{\alpha-1}. \tag{14}$$

Combining (13) and (14) with (6), we have

$$\begin{aligned} G(t,s) &\leq \min\left\{\frac{[g'(1)]^2 s(1-s)}{g(1)}, g'(1)(1-s)^{\alpha-1}\right\} \\ &= \min\left\{\frac{g'(1)s}{g(1)}, (1-s)^{\alpha-2}\right\} \times g'(1)(1-s) \\ &\leq \min\left\{s,(1-s)^{\alpha-2}\right\} \times \max\left\{1, \frac{g'(1)}{g(1)}\right\} \times g'(1)(1-s) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{s(1-s)^{\alpha-2}}{s^\star} \times \frac{g'(1)}{g(1)} \times g'(1)(1-s) \\
&= M_2 s(1-s)^{\alpha-1}.
\end{aligned} \tag{15}
$$

It follows from (10) and (12) that (p_3) holds. On the other hand, (11) and (15) yield (p_4) holds. □

Corollary 3.1 *It follows from* (p_2) *and* (p_4) *of Theorem* 3.1 *that*

$$G(t,s) \le M_2(1-t)t^{\alpha-1}, \quad \forall t,s \in [0,1].$$

4 Applications

4.1 Semipositone problem

In this section, we consider the existence and multiplicity of positive solutions to the semipositone FBVP:

$$
\begin{cases}
-D_{0+}^{\alpha} u(t) + au(t) = \lambda f(t,u(t)), & 0 < t < 1, \\
u(0) = u'(0) = 0, \qquad u(1) = 0.
\end{cases} \tag{16}
$$

For convenience, we list here the hypotheses to be used in this section:

(H_2) $f \in C((0,1) \times (0,+\infty), (-\infty,+\infty))$ and satisfies

$$f(t,x) \ge -e(t), \quad (t,x) \in (0,1) \times (0,+\infty),$$

where $e \in L^1[0,1] \cap C(0,1)$ is nonnegative and $\int_0^1 e(s)\,ds > 0$.

(H_3) For any $R \ge r > 0$, there exists $\Psi_{r,R} \in L^1[0,1] \cap C(0,1)$ such that

$$f(t,x) + e(t) \le \Psi_{r,R}(t), \quad \forall t \in (0,1), x \in \bigl[r(1-t)t^{\alpha-1}, R\bigr].$$

(H_4) There exists $[c_1,d_1] \subset (0,1)$ such that

$$\liminf_{x\to 0^+} \min_{t\in[c_1,d_1]} f(t,x) = +\infty.$$

(H_5) There exists $[c_2,d_2] \subset (0,1)$ such that

$$\liminf_{x\to +\infty} \min_{t\in[c_2,d_2]} \frac{f(t,x)}{x} = +\infty.$$

Remark 4.1 Condition (H_4) implies that $f(t,x)$ is singular at $x = 0$.

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\le t\le 1} |u(t)|$. Define a cone

$$P = \left\{ u \in E : u(t) \ge \frac{M_1\|u\|}{M_2}(1-t)t^{\alpha-1}, t \in [0,1] \right\}.$$

Denote $B_r = \{u(t) \in E : \|u(t)\| < r\}$ and

$$P_r = P \cap B_r.$$

Lemma 4.1 *The unique solution of the FBVP*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + au(t) = e(t), & 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0 \end{cases}$$

is

$$\omega(t) = \int_0^1 G(t,s)e(s)\,ds$$

with

$$\omega(t) \le M_2(1-t)t^{\alpha-1}\int_0^1 e(s)\,ds.$$

Proof The lemma can be deduced from Lemma 2.1 and Corollary 3.1, so we omit it. □

Next we consider the auxiliary FBVP:

$$\begin{cases} -D_{0+}^{\alpha}u(t) + au(t) = \lambda[f(t,[u(t)-\lambda\omega(t)]^{+}) + e(t)] = 0, & 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases} \tag{17}$$

where $[u(t)-\lambda\omega(t)]^{+} = \max\{u(t)-\lambda\omega(t), 0\}$.

Let

$$Au(t) = \int_0^1 G(t,s)\big[f\big(s,\big[u(s)-\lambda\omega(s)\big]^{+}\big) + e(s)\big]\,ds.$$

Lemma 4.2 *Suppose that* (H_2) *and* (H_3) *hold. Then, for any* $\lambda > 0$ *and*

$$r > \frac{\lambda M_2^2\int_0^1 e(s)\,ds}{M_1},$$

$A : P \setminus P_r \to P$ *is completely continuous.*

Proof For any $u \in P$ with $\|u\| \ge r$, one has

$$u(t) - \lambda\omega(t) \ge \left[\frac{M_1 r}{M_2} - \lambda M_2\int_0^1 e(s)\,ds\right](1-t)t^{\alpha-1} > 0, \quad \forall t \in (0,1).$$

The rest of the proof is similar to Lemma 2.6 in [21], we omit it here. □

By the extension theorem of completely continuous operator (see [34]), there exists an extension operator $\widetilde{A} : P \to P$, which is still completely continuous. Without loss of generality, we still write it as A.

Lemma 4.3 ([34]) *Let* E *be a real Banach space,* $P \subset E$ *be a cone. Assume that* Ω_1 *and* Ω_2 *are two bounded open subsets of* E *with* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *is a completely continuous operator such that either*

(1) $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; *or*
(2) $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 4.1 *Assume that* (H_2)–(H_5) *hold. Then there exists* $\lambda^* > 0$ *such that FBVP* (16) *has at least two positive solutions for any* $\lambda \in (0, \lambda^*)$.

Proof For

$$M' =: \frac{2^{\alpha+1} M_2^2 \int_0^1 e(s)\,ds}{M_1^2 \int_{c_1}^{d_1} s(1-s)^{\alpha-1}\,ds},$$

(H_4) guarantees there exists $X_1 \in (0,1)$ such that

$$f(t,x) > M', \quad \forall (t,x) \in [c_1, d_1] \times (0, X_1].$$

Let

$$\lambda^* = \min\left\{ \frac{M_1 X_1}{2M_2^2 \int_0^1 e(s)\,ds}, \frac{1 + M_2}{M_2 \int_0^1 \Psi_{M_1, 1+M_2}(s)\,ds} \right\}.$$

For any $\lambda \in (0, \lambda^*)$, let

$$r_1 = \frac{2\lambda M_2^2 \int_0^1 e(s)\,ds}{M_1}.$$

It is clear that $r_1 < X_1 < 1$.

$\forall u \in \partial P_{r_1}$, one has

$$u(t) - \lambda\omega(t) \leq r_1 < X_1$$

and

$$\begin{aligned} u(t) - \lambda\omega(t) &\geq \left[\frac{M_1 r_1}{M_2} - \lambda M_2 \int_0^1 e(s)\,ds \right] (1-t) t^{\alpha-1} \\ &= \lambda M_2 \int_0^1 e(s)\,ds (1-t) t^{\alpha-1}. \end{aligned}$$

Then

$$\begin{aligned} Au(t) &= \lambda \int_0^1 G(t,s) \left[f\left(s, \left[u(s) - \lambda\omega(s)\right]^+\right) + e(s) \right] ds \\ &\geq \lambda M_1 (1-t) t^{\alpha-1} \int_{c_1}^{d_1} s(1-s)^{\alpha-1} M'\,ds \\ &= \frac{2^{\alpha+1} \lambda M_2^2 \int_0^1 e(s)\,ds}{M_1} (1-t) t^{\alpha-1} \\ &= 2^\alpha r_1 (1-t) t^{\alpha-1}, \end{aligned}$$

which implies

$$\|Au\| > \|u\|, \quad \forall u \in \partial P_{r_1}. \tag{18}$$

Let $r_2 = 1 + M_2$. For any $u \in \partial P_{r_2}$, one has

$$u(t) - \lambda\omega(t) \le r_2$$

and

$$\begin{aligned} u(t) - \lambda\omega(t) &\ge \left[\frac{M_1 r_2}{M_2} - \lambda M_2 \int_0^1 e(s)\,ds\right](1-t)t^{\alpha-1} \\ &\ge M_1(1-t)t^{\alpha-1}. \end{aligned}$$

This and (H_3) yield

$$f\bigl(t,\bigl[u(t) - \lambda\omega(t)\bigr]^+\bigr) + e(t) \le \Psi_{M_1,1+M_2}(t).$$

Then

$$\begin{aligned} Au(t) &= \lambda \int_0^1 G(t,s)\bigl[f\bigl(s,\bigl[u(s) - \lambda\omega(s)\bigr]^+\bigr) + e(s)\bigr]\,ds \\ &\le \lambda M_2 (1-t)t^{\alpha-1} \int_0^1 \Psi_{M_1,1+M_2}(s)\,ds \\ &< \lambda^* M_2 \int_0^1 \Psi_{M_1,1+M_2}(s)\,ds \\ &\le 1 + M_2 = r_2. \end{aligned}$$

Therefore

$$\|Au\| < \|u\|, \quad \forall u \in \partial P_{r_2}. \tag{19}$$

For

$$M'' =: \frac{2^{\alpha+1} M_2}{\lambda M_1^2 c_2^\alpha (1-d_2)^\alpha (d_2 - c_2)},$$

(H_5) guarantees there exists $X_2 > r_2$ such that

$$f(t,x) > M'' x, \quad \forall (t,x) \in [c_2,d_2] \times [X_2,+\infty).$$

Let

$$r_3 = 1 + \frac{LM_2}{M_1},$$

where

$$L = \frac{X_2}{c_2^{\alpha-1}(1-d_2)}.$$

It is easy to see that

$$r_2 < r_3 < \frac{2LM_2}{M_1}.$$

For any $u \in \partial P_{r_3}$, one has

$$u(t) - \lambda\omega(t) \geq L(1-t)t^{\alpha-1}.$$

Hence

$$u(t) - \lambda\omega(t) > L(1-d_2)c_2^{\alpha-1} = X_2, \quad t \in [c_2, d_2].$$

Thus

$$\begin{aligned} Au(t) &= \lambda \int_0^1 G(t,s)\big[f\big(s,\big[u(s) - \lambda\omega(s)\big]^+\big) + e(s)\big]\,ds \\ &\geq \lambda M_1 (1-t)t^{\alpha-1} \int_{c_2}^{d_2} s(1-s)^{\alpha-1} M'' X_2\,ds. \end{aligned}$$

Then

$$\begin{aligned} \|Au\| &\geq \lambda M_1 \left(\frac{1}{2}\right)^{\alpha} M'' X_2 \int_{c_2}^{d_2} s(1-s)^{\alpha-1}\,ds \\ &> \lambda M_1 \left(\frac{1}{2}\right)^{\alpha} M'' X_2 c_2 (1-d_2)^{\alpha-1}(d_2 - c_2) \\ &= \frac{2LM_2}{M_1} > r_3, \end{aligned}$$

that is,

$$\|Au\| > \|u\|, \quad \forall u \in P_{r_3}. \tag{20}$$

Combining (18)–(20) with Lemma 4.3, we get A has at least two fixed points u_1, u_2 with $r_1 < \|u_1\| < r_2 < \|u_2\| < r_3$, that is, u_1 and u_2 are solutions of the auxiliary FBVP (17). It is clear that $u_i(t) - \lambda\omega(t) > 0$ on $(0,1)$, $i = 1,2$. Let $\bar{u}_i(t) = u_i(t) - \lambda\omega(t)$, $i = 1,2$. Then $\bar{u}_1(t)$ and $\bar{u}_2(t)$ are two positive solutions of the semipositone FBVP (16). □

Corollary 4.1 *Suppose that either* (H_2)–(H_4) *or* (H_2), (H_3), *and* (H_5) *hold. Then FBVP* (16) *has at least one positive solution provided* λ *is small enough.*

Example 4.1 Consider the following problem:

$$\begin{cases} -D_{0+}^{\frac{5}{2}} u(t) + \frac{1}{4} u(t) = \lambda f(t, u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = 0, \end{cases} \tag{21}$$

with

$$f(t,x) = x^2 + x^{-\frac{1}{2}} - t^{-\frac{1}{2}}(1-t)^{-\frac{1}{3}}.$$

It is clear that $f(t,x)$ is singular at $t=0,1$, and $x=0$. For $x\in[0,+\infty)$, notice that $\Gamma(\cdot)$ is strictly increasing on $[2,+\infty)$, we have

$$\begin{aligned}
h(x) &= -\frac{1}{3\Gamma(\frac{1}{2})}+\sum_{k=1}^{+\infty}\frac{x^k}{(\frac{5k}{2}+\frac{3}{2})\Gamma(\frac{5k}{2}-\frac{1}{2})}\\
&\le -\frac{1}{3\Gamma(\frac{1}{2})}+\sum_{k=1}^{+\infty}\frac{x^k}{3\Gamma(\frac{5k}{2}-\frac{1}{2})}\\
&\le -\frac{1}{3\sqrt{\pi}}+\frac{1}{3}\sum_{k=1}^{+\infty}\frac{x^k}{\Gamma(2k)}\\
&= -\frac{1}{3\sqrt{\pi}}+\frac{\sqrt{x}}{6}\bigl[e^{\sqrt{x}}-e^{-\sqrt{x}}\bigr].
\end{aligned}$$

By direct calculation, we have

$$\frac{e^{\frac{1}{2}}-e^{-\frac{1}{2}}}{12}-\frac{1}{3\sqrt{\pi}}\approx -0.1012136<0,$$

that is, $h(\frac{1}{4})<0$. This yields $a^*>\frac{1}{4}$, so (H_1) holds.

Let

$$\begin{aligned}
&e(t)=t^{-\frac{1}{2}}(1-t)^{-\frac{1}{3}},\\
&\Psi_{r,R}(t)=R^2+r^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}t^{-\frac{3}{4}},\\
&[c_1,d_1]=[c_2,d_2]=\left[\frac{1}{4},\frac{3}{4}\right].
\end{aligned}$$

It is easy to check that (H_2)–(H_5) hold. Therefore Theorem 4.1 ensures that FBVP (21) has at least two positive solutions provided λ is small enough.

4.2 Uniqueness results

In this section, we consider the uniqueness results of positive solution to the singular FBVP:

$$\begin{cases} -D_{0+}^{\alpha}u(t)+au(t)=f(t,u(t),u(t)), & 0<t<1,\\ u(0)=u'(0)=0, \qquad u(1)=0. \end{cases} \tag{22}$$

For convenience, we assume that the following assumptions hold in the rest of this paper:

(H_6) $f\in C((0,1)\times[0,+\infty)\times(0,+\infty)\to[0,+\infty))$, $f(t,x,y)$ is nondecreasing on x, nonincreasing on y, and there exists $\mu\in(0,1)$ such that

$$f\left(t,rx,\frac{y}{r}\right)\ge r^{\mu}f(t,x,y),\quad \forall x,y>0, r\in(0,1). \tag{23}$$

(H_7) $0<\int_0^1 f(s,(1-s)s^{\alpha-1},(1-s)s^{\alpha-1})\,ds<+\infty$.

Remark 4.2 Inequality (23) is equivalent to

$$f\left(t,\frac{x}{r},ry\right)\le r^{-\mu}f(t,x,y),\quad \forall x,y>0,r\in(0,1). \tag{24}$$

Define a cone Q by

$$Q=\left\{u\in E:\exists l_u>0,\text{such that } l_u(1-t)t^{\alpha-1}\ge u(t)\ge\frac{M_1\|u\|}{M_2}(1-t)t^{\alpha-1}\right\}.$$

Define a mixed monotone operator T by

$$T(u,v)=\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds.$$

Set $Q_1=Q\setminus\{\theta\}$, where θ is the zero element of E.

Lemma 4.4 $T:Q_1\times Q_1\to Q_1$.

Proof For $u,v\in Q_1$, $\exists l_u,l_v>0$ such that

$$\begin{aligned}
&l_u(1-t)t^{\alpha-1}\ge u(t)\ge\frac{M_1\|u\|}{M_2}(1-t)t^{\alpha-1},\\
&l_v(1-t)t^{\alpha-1}\ge v(t)\ge\frac{M_1\|v\|}{M_2}(1-t)t^{\alpha-1}.
\end{aligned}$$

Denote

$$\delta=\min\left\{\frac{1}{l_u},\frac{M_1\|v\|}{M_2},\frac{1}{2}\right\}.$$

It follows from Corollary 3.1 and Remark 4.2 that

$$\begin{aligned}
T(u,v)&=\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds\\
&\le M_2(1-t)t^{\alpha-1}\int_0^1 f\bigl(s,u(s),v(s)\bigr)\,ds\\
&\le M_2(1-t)t^{\alpha-1}\int_0^1 f\left(s,l_u(1-s)s^{\alpha-1},\frac{M_1\|v\|}{M_2}(1-s)s^{\alpha-1}\right)ds\\
&\le M_2(1-t)t^{\alpha-1}\int_0^1 f\left(s,\frac{(1-s)s^{\alpha-1}}{\delta},\delta(1-s)s^{\alpha-1}\right)ds\\
&\le \delta^{-\mu}M_2(1-t)t^{\alpha-1}\int_0^1 f\bigl(s,(1-s)s^{\alpha-1},(1-s)s^{\alpha-1}\bigr)\,ds\\
&<+\infty.
\end{aligned} \tag{25}$$

By (p_3) and (p_4) of Theorem 3.1, we have

$$T(u,v)(t)\ge M_1(1-t)t^{\alpha-1}\int_0^1 s(1-s)^{\alpha-1}f\bigl(s,u(s),v(s)\bigr)\,ds$$

and

$$T(u,v)(t) \leq M_2 \int_0^1 s(1-s)^{\alpha-1} f\bigl(s,u(s),v(s)\bigr)\,ds,$$

which implies

$$T(u,v)(t) \geq \frac{M_1 \|T(u,v)\|}{M_2}(1-t)t^{\alpha-1}.$$

This and (25) yield $T: Q_1 \times Q_1 \to Q_1$ is well defined. □

Theorem 4.2 *The singular FBVP* (22) *has a unique positive solution.*

Proof Let $w \in Q_1$, it follows from Lemma 4.4 that $T(w,w) \in Q_1$. Then we can select $r_0 \in (0,1)$ such that

$$r_0^{1-\mu} w \leq T(w,w) \leq r_0^{-(1-\mu)} w. \tag{26}$$

Set

$$u_n = T(u_{n-1}, v_{n-1}), \qquad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1,2,\ldots, \tag{27}$$

where

$$u_0 = r_0^{\frac{1}{2}} w, \qquad v_0 = r_0^{-\frac{1}{2}} w.$$

It is easy to see that $u_i, v_i \in Q_1$, $i = 0,1,\ldots$, and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{28}$$

It follows from (23) and (24) that

$$\begin{aligned} u_1 &= T\bigl(r_0^{\frac{1}{2}} w, r_0^{-\frac{1}{2}} w\bigr) \geq r_0^{\frac{\mu}{2}} T(w,w), \\ v_1 &= T\bigl(r_0^{-\frac{1}{2}} w, r_0^{\frac{1}{2}} w\bigr) \leq r_0^{-\frac{\mu}{2}} T(w,w). \end{aligned}$$

Then we have

$$u_1 \geq r_0^{\mu} v_1.$$

By induction, we can get

$$u_n \geq r_0^{\mu^n} v_n, \quad n = 1,2,\ldots. \tag{29}$$

Therefore, (28) and (29) yield

$$0 \leq u_{n+m} - u_n \leq v_n - u_n \leq \bigl(1 - r_0^{\mu^n}\bigr) v_n \leq \bigl(1 - r_0^{\mu^n}\bigr) v_0.$$

Then $\{u_n\}$ is a Cauchy sequence. Similarly, we can get $\{v_n\}$ is a Cauchy sequence. It follows from (28) that there exist $u^*, v^* \in Q_1$ such that $\{u_n\}$ and $\{v_n\}$ converge to u^* and v^* respectively. Moreover,

$$u_n \le u^* \le v^* \le v_n, \quad n = 1, 2, \ldots. \tag{30}$$

This and (29) imply that

$$\|v^* - u^*\| \le \|v_n - u_n\| \le \left(1 - r_0^{\mu^n}\right)\|v_0\|, \quad n = 1, 2, \ldots.$$

Hence

$$u^* = v^*.$$

By (30), we have

$$u_{n+1} = T(u_n, v_n) \le T\left(u^*, v^*\right) = T\left(v^*, u^*\right) \le T(v_n, u_n) = v_{n+1}.$$

Let $n \to +\infty$, we get

$$u^* \le T\left(u^*, v^*\right) = T\left(v^*, u^*\right) \le v^*.$$

Then we have $u^* = T(u^*, u^*)$, that is, u^* is a positive fixed point of T.

Next, we will show that the positive fixed point of T is unique. In fact, if $u \neq u^*$ is a positive fixed point of T, by Lemma 4.4, we have $u \in Q_1$. Denote

$$r_1 = \sup\left\{r \in (0,1) : ru^* \le u \le r^{-1}u^*\right\}.$$

It is clear that $r_1 \in (0,1)$ and

$$r_1 u^* \le u \le r_1^{-1} u^*.$$

Then

$$u = T(u,u) \ge T\left(r_1 u^*, r_1^{-1} u^*\right) \ge r_1^{\mu} T\left(u^*, u^*\right) = r_1^{\mu} u^*$$

and

$$u = T(u,u) \le T\left(r_1^{-1} u^*, r_1 u^*\right) \le r_1^{-\mu} T\left(u^*, u^*\right) = r_1^{-\mu} u^*.$$

Therefore,

$$r_1^{\mu} u^* \le u \le r_1^{-\mu} u^*.$$

This contradicts with the definition of r_1 since $r_1^{\mu} > r_1$. Consequently, the positive fixed point of T is unique, that is, FBVP (22) has a unique positive solution. □

Remark 4.3 The iterative sequence $\{u_n\}$ defined by (27) converges uniformly to the unique positive solution u^*. Moreover, we have the error estimation

$$\left\|u_n - u^*\right\| \leq \left(1 - r_0^{\mu^n}\right)\|v_0\|$$

with the rate of convergence

$$\left\|u_n - u^*\right\| = O\left(1 - r_0^{\mu^n}\right) = O\left(\mu^n\right).$$

Example 4.2 Consider the following problem:

$$\begin{cases} -D_{0+}^{\frac{5}{2}}u(t) + \frac{1}{4}u(t) = f(t, u(t), u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u(1) = 0, \end{cases} \tag{31}$$

with

$$f(t,x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}}.$$

It follows from Example 4.1 that $h(\frac{1}{4}) < 0$, that is, (H_1) holds. Clearly, (H_5) and (H_6) hold. Then Theorem 4.2 ensures that FBVP (31) has a unique positive solution u^*.

By direct calculation, we have

$$\begin{aligned} s^\star &= \frac{\sqrt{5}-1}{2}, \\ g(1) &= \frac{1}{\Gamma(\frac{5}{2})} + \sum_{k=1}^{+\infty} \frac{(\frac{1}{4})^k}{\Gamma(\frac{5}{2}k + \frac{5}{2})} < \frac{1}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{4}}{\Gamma(5)} + \sum_{k=2}^{+\infty} \frac{(\frac{1}{4})^k}{\Gamma(2k+2)} \\ &\approx 0.76339, \\ g(1) &> \frac{1}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{4}}{\Gamma(5)} \approx 0.76286, \\ g'(1) &= \frac{1}{\Gamma(\frac{3}{2})} + \sum_{k=1}^{+\infty} \frac{(\frac{1}{4})^k}{\Gamma(\frac{5}{2}k + \frac{3}{2})} < \frac{1}{\Gamma(\frac{3}{2})} + \sum_{k=1}^{+\infty} \frac{(\frac{1}{4})^k}{\Gamma(2k+2)} \\ &\approx 1.17086. \end{aligned}$$

Therefore,

$$M_1 > 0.741278, \qquad M_2 < 2.90772.$$

Let

$$w(t) = (1-t)t^{\frac{3}{2}}.$$

By Theorem 3.1 and Corollary 3.1, we have

$$T(\omega,\omega) \le M_2\left[B\left(\frac{4}{3},\frac{3}{2}\right)+B\left(\frac{2}{3},\frac{1}{2}\right)\right]\times w \le 8.8541\times w,$$
$$T(\omega,\omega) \ge M_1\left[B\left(\frac{17}{6},\frac{5}{2}\right)+B\left(\frac{13}{6},\frac{3}{2}\right)\right]\times w \ge 0.2196\times w.$$

Set

$$r_0=\frac{1}{27},\qquad u_0=\frac{w}{3\sqrt{3}},\qquad v_0=3\sqrt{3}w,$$
$$u_n=T(u_{n-1},v_{n-1}),\qquad v_n=T(v_{n-1},u_{n-1}),\quad n=1,2,\ldots.$$

Then (26) holds and $\|v_0\|=\frac{54}{25\sqrt{5}}\approx 0.966$. Moreover,

$$\left\|u_n-u^*\right\| \le 0.966\times\left(1-3^{-3^{1-n}}\right),\quad n=1,2,\ldots.$$

Then we have the rate of convergence

$$\left\|u_n-u^*\right\| = O\left(\left(\frac{1}{3}\right)^n\right),$$

and the error estimation

$$\left\|u_1-u^*\right\| < 0.643999,$$
$$\left\|u_2-u^*\right\| < 0.296213,$$
$$\left\|u_3-u^*\right\| < 0.111005,$$
$$\left\|u_4-u^*\right\| < 0.038517,$$
$$\left\|u_5-u^*\right\| < 0.013014,$$
$$\left\|u_6-u^*\right\| < 0.004358,$$
$$\ldots$$

5 Conclusions

In this paper, we establish some positive properties of the Green's function for a class of FBVPs. The interesting point is that the linear operator of the FBVPs contains two terms. As application of the main results, we investigate the existence and multiplicity results of positive solutions for an FBVP under conditions that the nonlinearity may change sign and possess singularity, and we also consider the uniqueness results of positive solution for a singular FBVP.

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Abbreviations
FBVP, Fractional differential equations boundary value problem; FDEs, fractional differential equations.

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