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Boundedness of solutions and stability of certain second-order difference equation with quadratic term

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Abstract

We consider the second-order rational difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

where γ, δ are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers. Boundedness along with global attractivity and Neimark–Sacker bifurcation results are established. Furthermore, we give an asymptotic approximation of the invariant curve near the equilibrium point.

Keywords: Difference equation; Birkhoff normal form; Boundedness; Invariant set; Neimark–Sacker bifurcation; Stability

1 Introduction and preliminaries

In this paper we consider the following second-order rational difference equation, where we introduce the quadratic term into the denominator

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}, \quad n = 0, 1, \ldots. \tag{1}$$

We assume that $\gamma, \delta > 0$ and initial conditions x_{-1}, x_0 are positive real numbers. It is convenient to introduce the change of variable

$$y_n = \frac{x_n}{\gamma}. \tag{2}$$

This change of variable transforms Eq. (1) into the equation

$$x_{n+1} = 1 + p \frac{x_n}{x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{3}$$

where $p = \delta/\gamma^2 > 0$.

The linear rational version of Eq. (3),

$$x_{n+1} = a + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{4}$$

was considered in [1], where the authors proved that the unique positive equilibrium $\bar{x} = a + 1$ is globally asymptotically stable. In [2], the authors investigated Eq. (4) but with quadratic terms in both the numerator and denominator. More precisely, they considered the following equation:

$$x_{n+1} = a + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \ldots. \tag{5}$$

They proved that the unique positive equilibrium $\bar{x} = a + 1$ is globally asymptotically stable if $a > \sqrt{2}$. Also, they computed the direction of the Neimark–Sacker bifurcation. We can see that the introduction of quadratic terms into Eq. (4) changes the character of local stability and, consequently, the global dynamics as well.

An elementary calculation shows that Eq. (3) has the unique positive equilibrium point

$$\bar{x} = (\sqrt{4p + 1} + 1)/2. \tag{6}$$

In this paper we prove that the parameter space splits into three regions given by $p < 2$, $p > 2$, and $p = 2$. By using a result from [3], we obtain a global result in the case $p \leq \frac{1+\sqrt{2}}{2}$. We conjecture that the equilibrium point (6) is globally asymptotically stable for $p < 2$. To obtain a sequence of invariant and attractive sets that contain all solutions in the case $1 < p < \infty$, we use the technique that is similar to those used in [4, 5] and [6]. This technique is a special case of the so-called Kobayashi metric mentioned in [6]. Namely, we construct a one-parameter family $\mathcal{S}_t$ of compact sets invariant for all t such that $t_0(p) \leq t < \infty$, for some $t_0(p) > \bar{x}$. Figure 1 shows a typical set $\mathcal{S}_t$. The boundary consists of four linear segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, $\overline{P_4P_5}$ and the curve $\mathcal{C}_{51}$, which is the part of a parabola joining the points P_1 and P_5. By using such a construction, it can be shown that any image of a bounded subset of $\mathbb{R}^2$ under the map T is contained in a T-invariant compact set, see

Figure 1 The boundary of a set $\mathcal{S}_t$ (solid) and its image $T(\partial \mathcal{S}_t)$ (dashed)
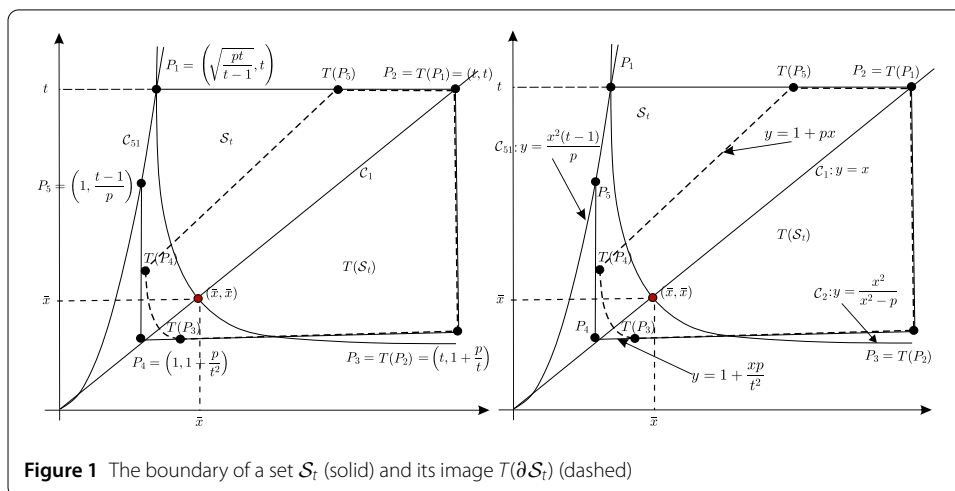
[1, 7, 8]. In the case $p = 2$, Neimark–Sacker bifurcation occurs, and we approximate the invariant curve near the positive equilibrium point. For the case $p > 2$, it is easy to see that the equilibrium point (6) is a repeller.

The paper is structured as follows. In Sect. 2, a local analysis of the positive equilibrium is performed. In Sect. 3 we present some results about semicycle character of solutions of Eq. (3). Section 4 contains the global stability analysis of the positive equilibrium in the case $p \leq \frac{1+\sqrt{2}}{2}$. A proof of boundedness of solutions of Eq. (3) for $1 < p < \infty$ is presented in Sect. 5. Section 6 gives a reduction to the Birkhoff normal form and computation of the coefficients of the Neimark–Sacker bifurcation and the asymptotic approximation of the invariant curve near the positive equilibrium point of Eq. (3).

2 Linearized stability

In this section, we perform a local stability analysis of the positive equilibrium of Eq. (3).

Set

$$f(u,v) = 1 + p\frac{u}{v^2}$$

and observe that

$$f_u(u,v) = \frac{p}{v^2}$$

and

$$f_v(u,v) = -\frac{2pu}{v^3}.$$

If $\bar{x}$ denotes an equilibrium point of Eq. (3), then the linearized equation associated with Eq. (3) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = f_u(\bar{x},\bar{x}) \quad \text{and} \quad q = f_v(\bar{x},\bar{x}). \tag{7}$$

Now, the linearized equation of Eq. (3) at the equilibrium point $\bar{x} = \frac{1}{2}(\sqrt{4p+1} + 1)$ has the following form:

$$z_{n+1} = \frac{p}{\bar{x}^2}z_n - \frac{2p}{\bar{x}^2}z_{n-1}, \quad n = 0, 1, \dots. \tag{8}$$

The characteristic equation associated with Eq. (8) is given by

$$\lambda^2 - \frac{p}{\bar{x}^2}\lambda + \frac{2p}{\bar{x}^2} = 0. \tag{9}$$

By using Theorem 1.1.1 in [1] it is easy to see that the following linearized stability result holds:

Theorem 1 *Assume $p > 0$. Then the equilibrium point $\bar{x} = \frac{1}{2}(\sqrt{4p + 1} + 1)$ of* (3) *is*

   (i) *Locally asymptotically stable if $p < 2$.*

  (ii) *A repeller if and only if $p > 2$.*

 (iii) *A non-hyperbolic equilibrium if and only if $p = 2$.*

3 Semicycle analysis

In this section, we present some results about the semicycle character of solutions of Eq. (3). See [1, 9] for more about semicycle analysis.

Consider the following second order difference equation:

$$x_{n+1} = x_n h(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{10}$$

The following result holds.

Theorem 2 *Assume*

   (i) $h \in [(0, \infty) \times (0, \infty), (0, \infty)]$;

  (ii) $xh(x, y)$ *is increasing in x and decreasing in y;*

 (iii) $h(x, y)$ *is decreasing in each argument;*

 (iv) $xh(x, x)$ *is decreasing in x;*

  (v) *The equation*

$$x_{n+1} = x_n h(x_n, x_{n-1}), \quad n = 0, 1, \ldots,$$

has a unique positive equilibrium $\bar{x}$.

Then the following holds:

   (i) *If $x_{N-1} < \bar{x} \le x_N$ (respectively, $x_N \le \bar{x} < x_{N-1}$), then $\bar{x} \le x_{N+1}$ (respectively, $x_{N+1} \le \bar{x}$).*

  (ii) *If $\bar{x} \le x_{N-1}, x_N$ (respectively, $x_N, x_{N-1} \le \bar{x}$), then $x_{N+1} \le x_N$ (respectively, $x_{N+1} \ge x_N$). Moreover, except in the case $x_{N-1} = x_N = \bar{x}$, the inequalities are strict. Furthermore, if $x_N \le x_{N-1}$ (respectively, $x_{N-1} \le x_N$) then $x_{N+1} < \bar{x}$ (respectively, $\bar{x} < x_{N+1}$).*

 (iii) *If $\bar{x} \le x_{N-1}, x_N, x_{N+1}$ (respectively, $x_N, x_{N-1}, x_{N+1} \le \bar{x}$), then $x_{N+2} \le \bar{x}$ (respectively, $\bar{x} \le x_{N+2}$).*

 (iv) *Except possibly for the first semicycle, every semicycle contains two or three terms.*

  (v) *The extreme in each semicycle occurs at either the first or the second term.*

Proof We only give the proof for positive semicycles whose terms are not equal to the equilibrium $\bar{x}$. The proof for negative semicycles is similar and will be omitted. The proof for the trivial semicycle is obvious.

   (i) Assume that $N \ge 0$ is such that $x_{N-1} < \bar{x} \le x_N$. Then, by using assumption (iv), we have

$$x_{N+1} = x_N h(x_N, x_{N-1}) > \bar{x} f(\bar{x}, \bar{x}) = \bar{x},$$

so x_{N+1} is also in the same semicyle.

  (ii) Assume that $N \ge 0$ such that $\bar{x} \le x_{N-1}, x_N$. Then (iii) implies

$$x_{N+1} = x_N h(x_N, x_{N-1}) < x_N h(\bar{x}, \bar{x}) = x_N.$$

Assume that $x_N \leq x_{N-1}$ then (iii) and (iv) imply

$$x_{N+1} = x_N h(x_N, x_{N-1}) \leq x_N h(x_N, x_N) < \bar{x} h(\bar{x}, \bar{x}) = \bar{x}.$$

(iii) Assume that $N \geq 0$ such that $\bar{x} \leq x_{N-1}, x_N, x_{N+1}$. Then by statement (ii), we have $x_{N+1} \leq x_N$. Now, assumptions (ii) and (iv) imply

$$x_{N+2} = x_{N+1} h(x_{N+1}, x_N) \leq x_{N+1} h(x_{N+1}, x_{N+1}) < \bar{x} h(\bar{x}, \bar{x}) = \bar{x}.$$

The rest of the proof follows from (i)–(iii). □

It is obvious that Eq. (3) can be rewritten in the following form:

$$x_{n+1} = x_n \left(\frac{1}{x_n} + \frac{p}{x_{n-1}^2} \right). \tag{11}$$

Set

$$h(x_n, x_{n-1}) = \frac{1}{x_n} + \frac{p}{x_{n-1}^2}.$$

It is easy to see that this function h satisfies all the conditions of Theorem 2, which implies that all statements hold. Now, we have the following corollary of Theorem 2.

Corollary 1 *Assume that $p > 0$. Then for Eq. (3) the following holds true*:
  (i) *Except possibly for the first semicycle, every semicycle contains two or three terms.*
 (ii) *The extreme in each semicycle occurs at either the first or the second term.*
(iii) *Every nontrivial solution is strictly oscillatory about the equilibrium $\bar{x}$.*
(iv) *Equation (3) does not have any period 2 or period 3 point in $(0, +\infty)$.*

4 Global asymptotic stability analysis

The following lemma details an invariant interval of Eq. (3).

Lemma 1 *Equation (3) possesses an invariant interval only if $0 < p < 1$. The interval $[1, \frac{1}{1-p}]$ is an invariant and attractive set.*

Proof Assume that interval $[L, U]$, $0 < L < U$ is an invariant interval. Then by using the monotonicity of function f, for $u, v \in [L, U]$, we have

$$L < f(L, U) \leq f(u, v) < f(U, L) \leq U$$

which is equivalent to

$$U(p - L^2) + L^2 \leq 0 \quad \text{and} \quad (L - 1)U^2 - Lp \leq 0.$$

It is easy to see that

$$\frac{1}{2} L \big(L + 1 - \sqrt{(L-1)(L+3)} \big) < 1$$

for $1 < L < \frac{3}{2}$, hence $0 < p < 1$. Straightforward calculations show that $[1, \frac{1}{1-p}]$ is an invariant and attractive set. □

Notice that Lemma 1 implies that Eq. (3) does not posses an invariant and attractive set for $p > 1$. Also, it is easy to see that in the interval $[1, \frac{1}{1-p}]$ we can apply Theorem 1.4.5 (see Sect. 1.4. of the monograph [1] or paper [10]) and obtain an attractivity result. Since an invariant interval does not exist for $p > 1$, it is not possible to obtain an attractivity result by applying Theorem 1.4.5.

Because of the above discussion, we apply Lemma 3.3 and Theorem 2.1 in [3] to obtain a global attractivity result of Eq. (3) for $p \leq \frac{1+\sqrt{2}}{2}$. For this purpose, we substitute the following equation:

$$x_n = 1 + p\frac{x_{n-1}}{x_{n-2}^2}$$

into Eq. (3) and get

$$x_{n+1} = 1 + \frac{p}{x_{n-1}^2} + \frac{p^2}{x_{n-2}^2 x_{n-1}}. \tag{12}$$

Observe now that every solution of Eq. (3) is also a solution of Eq. (12), with initial values x_{-2}, x_{-1} and $x_0 = 1 + p\frac{x_0}{x_{-1}^2}$.

Remark 1 Notice that every solution of Eq. (12) satisfies the following inequality:

$$1 \leq x_n \leq 1 + p + p^2, \quad n = 1, 2, \ldots, \text{for } p > 0,$$

which implies that every solution of Eq. (3) is bounded.

Observe also that Eq. (12) is of the form $x_{n+1} = g(x_{n-2}, x_{n-1})$ where

$$g(x, y) = 1 + \frac{p}{y^2} + \frac{p^2}{x^2 y}.$$

Set

$$G(x) = g(x, x) = 1 + \frac{p}{x^2} + \frac{p^2}{x^3} = \frac{p^2 + px + x^3}{x^3}. \tag{13}$$

Notice that equations (3) and (12) have the same unique positive equilibrium $\bar{x} = \frac{1}{2}(\sqrt{4p+1} + 1)$. Hence, to prove the global attractivity of equilibrium point $\bar{x}$ of Eq. (3), it is sufficient to prove the global attractivity of $\bar{x}$ of Eq. (12). The following theorem holds.

Theorem 3 *Assume that $p \leq \frac{1+\sqrt{2}}{2}$. Then the unique positive equilibrium of* (3) *is globally asymptotically stable.*

Proof First, computing the first-order partial derivatives of g, it is easy to see that

$$\frac{\partial g}{\partial x}(x, y) < 0 \quad \text{and} \quad \frac{\partial g}{\partial y}(x, y) < 0, \quad \text{for all } x, y > 0.$$

From

$$G'(\bar{x}) = -\frac{16p(3p + \sqrt{4p+1} + 1)}{(\sqrt{4p+1} + 1)^4}$$

we have $G'(\bar{x}) \geq -1$ if $p \leq \frac{1+\sqrt{2}}{2}$. The Schwarzian derivative,

$$SG(x) = \frac{G'''(x)}{G'(x)} - \frac{3}{2}\left(\frac{G''(x)}{G'(x)}\right)^2,$$

is given by

$$SG(x) = -\frac{6(6p^2 + 4px + x^2)}{x^2(3p + 2x)^2},$$

and it is obvious that

$$SG(x) < 0, \quad \text{for all } x \geq 0.$$

Therefore, by Lemma 3.3 and Theorem 2.1 in [3], equilibrium $\bar{x}$ is a global attractor of all positive solutions of Eq. (12), and hence is also a global attractor of all positive solutions of Eq. (3). □

5 Attractive set for solutions

In this section, we give a construction of a sequence of invariant and attractive sets that contain all solutions. The main result of this section is Theorem 5 concerning attractive sets for solutions of Eq. (3).

Throughout the section we shall assume the inequality

$$1 < p < \infty.$$

The change of variables $u_n = x_{n-1}$ and $v_n = x_n$ transforms Eq. (3) into the system of difference equations:

$$\begin{cases} u_{n+1} = v_n, \\ v_{n+1} = 1 + p\frac{v_n}{u_n^2}. \end{cases} \tag{14}$$

The map T corresponding to system (14) is given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} y \\ 1 + p\frac{y}{x^2} \end{pmatrix}. \tag{15}$$

It is obvious that $T(\mathbb{R}_+^2) \subset (1,\infty) \times (1,\infty)$ and $T((1,\infty) \times (1,\infty)) \subset (1,\infty) \times (1,\infty)$, so it is sufficient to consider the map T on the set $\mathcal{K} = (1,\infty) \times (1,\infty)$. Let $T = (T_1, T_2)$. The equilibrium curves of T are the following sets:

$$\mathcal{C}_1 := \{(x,y) \in \mathcal{K} : T_1(x,y) = x\} \quad \text{and} \quad \mathcal{C}_2 := \{(x,y) \in \mathcal{K} : T_2(x,y) = y\}.$$

It is easy to see that the equilibrium curves are as follows:

$$\mathcal{C}_1 := \left\{ (x,y) \in \mathcal{K} : y = x \right\} \quad \text{and} \quad \mathcal{C}_2 := \left\{ (x,y) \in \mathcal{K} : x = \sqrt{\frac{py}{y-1}} \right\}.$$

Assume that t is any parameter such that $t > \bar{x}$. Since $\bar{x} > 1$, we have $t > 1$. This implies that assumption $t > \bar{x}$ is equivalent to $t^2 - t - p > 0$.

Now, consider the following five points, see Fig. 1:

$$P_1 = \left(\sqrt{\frac{pt}{t-1}}, t \right), \qquad P_2 = (t,t), \qquad P_3 = \left(t, 1 + \frac{p}{t} \right),$$

$$P_4 = \left(1, 1 + \frac{p}{t^2} \right), \qquad P_5 = \left(1, \frac{t-1}{p} \right).$$

Denote with $\preceq_{ne}$ the *northeast* partial ordering on $\mathbb{R}^2$ whose nonnegative cone is the standard first quadrant $\{(u,v) : u, v \geq 0\}$. That is, $(u_1, v_1) \preceq_{ne} (u_2, v_2)$ if and only if $u_1 \leq u_2$ and $v_1 \leq v_2$, see [11]. The following lemma holds.

Lemma 2 *The following statements hold true.*
  (i) $P_1 \preceq_{ne} P_2$, $P_3 \preceq_{ne} P_2$, $P_4 \preceq_{ne} P_3$, $P_4 \preceq_{ne} P_5$.
  (ii) *Assume that* $t^3 - (p+1)t^2 - p^2 > 0$, *then* $P_5 \preceq_{ne} P_1$.

Proof Notice that $P_1 \preceq_{ne} P_2$ is equivalent with $t^2 - t - p > 0$ which is true since $t > \bar{x}$. Since $t > 1$ and $p > 1$ it is obvious that $P_3 \preceq_{ne} P_2$, $P_4 \preceq_{ne} P_3$ and $P_5 \preceq_{ne} P_1$. Only, we have to check that $P_4 \preceq P_5$. This is equivalent to

$$1 + \frac{p}{t^2} \leq \frac{t-1}{p},$$

which is true if and only if $t^3 - t^2(1 + p) - p^2 \geq 0$.   □

Denote by $\overline{P_i P_j}$ the line segment joining points P_i and P_j for $1 \leq i, j \leq 5$, $i \neq j$.
Let

$$\mathcal{C}_{51} := \left\{ \left(x, \frac{x^2(t-1)}{p} \right) : 1 \leq x \leq \sqrt{\frac{pt}{t-1}} \right\}$$

be the part of parabola

$$y = \frac{x^2(t-1)}{p}$$

between P_5 and P_1.

Let $\mathcal{S}_t$ be set consisting of four line segments $\overline{P_i P_j}$ and curve $\mathcal{C}_{51}$, namely of

$$\overline{P_1 P_2}, \overline{P_2 P_3}, \overline{P_3 P_4}, \overline{P_4 P_5}, \mathcal{C}_{51},$$

as in Fig. 1.

To prove some properties of images of the line segments $\overline{P_iP_j}$ and curve $\mathcal{C}_{51}$, we shall employ the resultant of two polynomials, see [12, 13]. Let $f_1(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $g_1(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ be two polynomials of degrees n and m, respectively. Their resultant $\text{Res}(f_1,g_1)$ is the determinant of the $(m+n) \times (m+n)$ Sylvester matrix given by

$$\text{Syl}(f_1,g_1) = \begin{pmatrix} a_n & a_{n-1} & \ldots & a_1 & a_0 & 0 & \ldots & 0 \\ 0 & a_n & a_{n-1} & \ldots & a_1 & a_0 & \ldots & 0 \\ \vdots & & & & & & & \\ 0 & \ldots & a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \\ b_m & b_{m-1} & \ldots & b_1 & b_0 & 0 & \ldots & 0 \\ 0 & b_m & b_{n-1} & \ldots & b_1 & b_0 & \ldots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \ldots & b_m & b_{m-1} & b_{m-2} & \ldots & b_0 \end{pmatrix}$$

or

$$\text{Res}(f_1,g_1) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j),$$

where α_i, $i = 1,2,\ldots,n$ and β_j, $j = 1,2,\ldots,m$ are the zeros of the polynomials $f_1(x)$ and $g_1(x)$, respectively.

The following lemma holds.

Lemma 3 *Let $f_1(x)$ and $f_2(x)$ be two polynomials of degrees $n \geq 1$ and $m \geq 1$, respectively. Further, suppose that $f_1(x)$ and $g_1(x)$ each have exactly one positive root α_n and β_m. Assume that $a_n, b_m > 0$. Then $(-1)^{n-1}(\alpha_n - \beta_m) > 0$ if and only if $\text{res}(f_1,f_2) > 0$. Furthermore, assume that $\text{res}(f_1,f_2) > 0$. If n is odd then $f_1(x) > 0 \Rightarrow f_2(x) > 0$ and if n is even then $f_2(x) > 0 \Rightarrow f_1(x) > 0$.*

Proof Since α_n is the only one positive root of $f_1(x)$, and β_m is the only one positive root of $g_1(x)$, we have

$$f_1(x) = a_n(x - \alpha_n)f_2(x), \quad \text{where } f_2(x) \geq 0 \; \forall x > 0,$$

$$g_1(x) = b_m(x - \beta_m)g_2(x), \quad \text{where } g_2(x) \geq 0 \; \forall x > 0,$$

where the leading coefficient of $f_2(x)$ and $g_2(x)$ is one. We have

$$\text{Res}(f_1,g_1) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j)$$

$$= a_n^m b_m^n \prod_{i=1}^{n}(\alpha_i - \beta_m)g_2(\alpha_i)$$

$$= a_n^m b_m^n \prod_{i=1}^n (\alpha_i - \beta_m) \prod_{i=1}^n g_2(\alpha_i)$$

$$= a_n^m b_m^n (-1)^{n-1} (\alpha_n - \beta_m) f_2(\alpha_n) \prod_{i=1}^n g_2(\alpha_i).$$

This implies

$$\mathrm{Res}(f_1, g_1) > 0 \quad \Leftrightarrow \quad (-1)^{n-1}(\alpha_n - \beta_m) > 0.$$

Assume that $\mathrm{Res}(f_1, g_1) > 0$. Then from the previous conclusion, we have $\alpha_n > \beta_m$. Since $a_n, b_m > 0$, we have that $f_2(x) > 0 \Leftrightarrow x > \beta_m$ and $f_1(x) > 0 \Leftrightarrow x > \alpha_n$, from which the proof follows. $\qquad\square$

Also, we shall need the following two lemmas.

Lemma 4 *Assume that $p > 1$. The equation*

$$t^5 - (p^2 + p + 1)t^4 + 2pt^3 - p(p^2 + 2)t^2 + p^2 t - p^2 = 0$$

has exactly one positive root and two pairs of conjugate complex roots.

Proof Let

$$\tilde{f}(t) = t^5 - (p^2 + p + 1)t^4 + 2pt^3 - p(p^2 + 2)t^2 + p^2 t - p^2.$$

By the Descartes' Rule of Signs, $\tilde{f}(t) = 0$ has no negative roots. The following matrix, called the discrimination matrix of $\tilde{f}(y)$ and $\tilde{f}'(y)$, see [14], is actually the Sylvester matrix of $\tilde{f}(y)$ and $\tilde{f}'(y)$ with some permuted rows, and is given by

$$\mathrm{Discr}(\tilde{f}) = \begin{pmatrix}
1 & -B & 2p & -Ap & p^2 & -p^2 & 0 & 0 & 0 & 0 \\
0 & 5 & -4B & 6p & -2Ap & p^2 & 0 & 0 & 0 & 0 \\
0 & 1 & -B & 2p & -Ap & p^2 & -p^2 & 0 & 0 & 0 \\
0 & 0 & 5 & -4B & 6p & -2Ap & p^2 & 0 & 0 & 0 \\
0 & 0 & 1 & -B & 2p & -Ap & p^2 & -p^2 & 0 & 0 \\
0 & 0 & 0 & 5 & -4B & 6p & -2Ap & p^2 & 0 & 0 \\
0 & 0 & 0 & 1 & -B & 2p & -Ap & p^2 & -p^2 & 0 \\
0 & 0 & 0 & 0 & 5 & -4B & 6p & -2Ap & p^2 & 0 \\
0 & 0 & 0 & 0 & 1 & -B & 2p & -Ap & p^2 & -p^2 \\
0 & 0 & 0 & 0 & 0 & 5 & -4B & 6p & -2Ap & p^2
\end{pmatrix},$$

where $A = p^2 + 2$ and $B = p^2 + p + 1$. Let D_k denote the determinant of the submatrix of $\mathrm{Discr}(\tilde{f})$, formed by the first $2k$ rows and the first $2k$ columns, for $k = 1, \ldots, 5$. So, by straightforward calculation one can see that

$$D_1(p) = 5,$$

$$D_2(p) = 4(p^4 + 2p^3 + 3p^2 - 3p + 1),$$

$$D_3(p) = -p\big(8p^8 + 24p^7 + 64p^6 + 77p^5 + 76p^4 + 100p^3 - 24p^2 + 80p + 16\big),$$

$$D_4(p) = -2p^4\big(4p^{11} + 12p^{10} + 32p^9 + 19p^8 - 20p^7 - 62p^6 - 120p^5$$
$$+ 70p^4 - 246p^3 + 178p^2 - 160p - 32\big)$$

$$D_5(p) = p^8\big(16p^{12} + 44p^{11} + 84p^{10} - 108p^9 - 404p^8 - 348p^7 + 384p^6$$
$$+ 1856p^5 - 427p^4 + 960p^3$$
$$- 480p^2 + 1536p + 256\big)$$

One can see that $D_3(p) < 0$. By using Theorem 1 in [14] for the polynomial $D_5(p)$, one can see that D_5 has five pairs of conjugate complex roots and two negative roots for $p > 1$, which implies $D_5(p) > 0$. Then the sign list of the sequence $\{D_1(p), D_2(p), D_3(p), D_4(p), D_5(p)\}$ is given by

$$\big[1, 1, -1, \mathrm{sign}\big(D_4(p)\big), 1\big], \tag{16}$$

from which it follows that the number of sign changes of the revised sign list of the list (16) is two. Now, the statement follows in view of Theorem 1 in [14]. □

Lemma 5 *Assume that* $1 < p < \infty$. *Let*

$$P_p^{(1)}(t) := t^2 - t - p,$$
$$P_p^{(2)}(t) := t^3 - (p + 1)t^2 - p^2,$$
$$P_p^{(3)}(t) := 2t^3 - p^2 t^2 - 2t^2 + 2pt - 2p,$$
$$P_p^{(4)}(t) := t^5 - \big(p^2 + p + 1\big)t^4 + 2pt^3 - p\big(p^2 + 2\big)t^2 + p^2 t - p^2.$$

Then $P_p^{(i)}$, $1, \ldots, 4$, *have exactly one positive root and the following holds*:
  (i) $P_p^{(2)}(t) > 0 \Rightarrow P_p^{(1)}(t) > 0$;
  (ii) $P_p^{(4)}(t) > 0 \Rightarrow P_p^{(3)}(t) > 0$;
  (iii) $P_p^{(4)}(t) > 0 \Rightarrow P_p^{(2)}(t) > 0$.

Proof By Lemma 4, $P_p^{(4)}(t)$ has exactly one positive root. By the Descartes' Rule of Signs, $P_p^{(1)}(t)$ and $P_p^{(2)}(t)$ have exactly one positive root, and $P_p^{(3)}(t)$ has no negative root. The discriminant of $P_p^{(3)}(t)$ is given by

$$\tilde{g}(p) := \mathrm{Dis}\big(P_p^{(3)}\big) = -4p\big(2p^6 - p^5 + 12p^4 - 40p^3 + 40p^2 + 32p + 16\big).$$

Since

$$\tilde{g}(u + 1) = -4(u + 1)\big(2u^6 + 11u^5 + 37u^4 + 38u^3 + 12u^2 + 47u + 61\big)$$

for $u > 0$, we have $\mathrm{Dis}(P_p^{(3)}) < 0$ for $p > 1$. This implies that $P_p^{(3)}$ has one pair of conjugate complex roots. Hence, $P_p^{(3)}(t)$ has exactly one positive root.

The rest of the proof follows from the facts:

$$\mathrm{res}\left(P_p^{(2)}, P_p^{(1)}\right) = 4p^4 > 0 \quad \text{for } p > 0;$$

$$\mathrm{res}\left(P_p^{(4)}, P_p^{(2)}\right) = p^6\left(p^7 - p^6 + p^5 - 9p^4 + 13p^3 - 6p^2 + 4p - 2\right) > 0 \quad \text{for } p > 1;$$

$$\mathrm{res}\left(P_p^{(4)}, P_p^{(3)}\right) = p^7\left(p^7 + 2p^6 - 8p^4 - 16p^3 + 16p^2 + 16p + 32\right) > 0 \quad \text{for } p > 1,$$

which can be verified in a similar way and by using Lemma 3. □

Lemmas 6 and 7 given below detail the images of line segments $\overline{P_i P_j}$ and curve $\mathcal{C}_{51}$ under the map T.

Lemma 6 *Assume that* $1 < p < \infty <$ *and* $P_p^{(2)}(t) > 0$ *Then the following holds*:
  (i) $T(\overline{P_1 P_2}) = \overline{P_2 P_3}$;
  (ii) $T(\overline{P_2 P_3}) \subset \overline{P_2 P_4} \setminus \{P_4\}$;
  (iii) $T(\mathcal{C}_{51}) \subset \overline{P_1 P_2} \setminus \{P_1\}$.

Proof
  (i) Since $\overline{P_1 P_2} = \{(1-u)P_1 + uP_2 : 0 \le u \le 1\}$, we have

$$T(\overline{P_1 P_2}) = \left\{\left(t, \frac{pt}{((1-u)\sqrt{\frac{pt}{t-1}} + tu)^2} + 1\right) : 0 \le u \le 1\right\} = \overline{P_2 P_3}.$$

  (ii) From $\overline{P_2 P_3} = \{(1-u)P_2 + uP_3 : 0 \le u \le 1\}$ we have

$$T(\overline{P_2 P_3}) = \left\{\left(\frac{pu}{t} - tu + t + u, \frac{p^2 u + pt(t + u - tu) + t^3}{t^3}\right) : 0 \le u \le 1\right\}$$

$$= \left\{(x_1(u), y_1(u) : 0 \le u \le 1\right\}.$$

Solving $x_1(u) = x$ for u, we obtain

$$u = -\frac{t(t-x)}{p - t^2 + t}.$$

Substituting this into $y = y_1(u)$, we get

$$y = \frac{px}{t^2} + 1.$$

So we obtain

$$T(\overline{P_2 P_3}) = \left\{\left(x, \frac{px}{t^2} + 1\right) : 1 + \frac{p}{t} \le x \le t\right\}.$$

Since

$$\overline{P_3 P_4} = \left\{\left(x, \frac{px}{t^2} + 1\right) : 1 \le x \le t\right\} \quad \text{and} \quad t^2 - t - p > 0,$$

we obtain $T(\overline{P_2 P_3}) \subset \overline{P_3 P_4} \setminus \{P_4\}$.

(iii) Since $\mathcal{C}_{51} = \{(u, \frac{u^2(t-1)}{p}) : 1 \le u \le \sqrt{\frac{pt}{t-1}}\}$, we obtain

$$T(\mathcal{C}_{51}) = \left\{ \left(\frac{(t-1)u^2}{p}, t \right) : 1 \le u \le \sqrt{\frac{pt}{t-1}} \right\}$$

$$= \left\{ (x_2(u), y_2(u) : 1 \le u \le \sqrt{\frac{pt}{t-1}} \right\}.$$

Since $x_2(1) = (\frac{t-1}{p}, t)$ and $x_2(\sqrt{\frac{pt}{t-1}}) = (t, t)$, we have that $T(\mathcal{C}_{51}) \subset \overline{P_1 P_2} \setminus \{P_1\}$. $\quad\square$

Lemma 7 *Assume that* $1 < p < \infty$ *and* $P_p^{(4)}(t) > 0$. *Then the following holds*:
  (i) $T(\overline{P_3 P_4} \setminus \{P_3\}) \subset \mathrm{int}(\mathcal{S}_t)$;
  (ii) $T(\overline{P_4 P_5} \setminus \{P_5\}) \subset \mathrm{int}(\mathcal{S}_t)$

Proof From (ii) and (iii) of Lemma 5, we have $2t^3 - p^2 t^2 + 2pt - 2p - 2t^2 > 0$.
  (i) From $\overline{P_3 P_4} = \{(1-u)P_3 + uP_4 : 0 \le u \le 1\}$ we have

$$T(\overline{P_3 P_4}) = \left\{ \left(\frac{p(t+u-tu)+t^2}{t^2}, \frac{p(p(t(-u)+t+u)+t^2)}{t^2(t+u-tu)^2} + 1 \right) : 0 \le u \le 1 \right\}$$

$$= \left\{ (x_3(u), y_3(u)) : 0 \le u \le 1 \right\}.$$

Solving $x_3(u) = x$ for u, we obtain

$$u = \frac{t(p - tx + t)}{p(t-1)}.$$

Substituting this into $y = y_3(u)$, we get

$$y_3(x) = \frac{p^3 x}{t^4(x-1)^2} + 1.$$

So we obtain

$$T(\overline{P_2 P_3}) = \left\{ \left(x, \frac{p^3 x}{t^4(x-1)^2} + 1 \right) : \frac{p}{t^2} + 1 \le x \le \frac{p}{t} + 1 \right\}.$$

It is easy to see

$$y_3'(x) = -\frac{p^3(x+1)}{t^4(x-1)^3} \quad \text{and} \quad y_3''(x) = \frac{2p^3(x+2)}{t^4(x-1)^4}.$$

Since

$$y_3\left(\frac{p}{t^2} + 1 \right) = p\left(\frac{p}{t^2} + 1 \right) + 1,$$

to show $T(\overline{P_2 P_3}) \subset \overline{P_2 P_3}$ it is enough to prove

$$y_3\left(\frac{p}{t^2} + 1 \right) - \frac{(\frac{p}{t^2} + 1)^2(t-1)}{p} < 0.$$

Since

$$y_3\left(\frac{p}{t^2}+1\right) - \frac{(\frac{p}{t^2}+1)^2(t-1)}{p}$$

$$= \frac{p^3t^2 + p^2t^4 - p^2t + p^2 + pt^4 - 2pt^3 + 2pt^2 - t^5 + t^4}{pt^4}$$

$$= -\frac{P_p^{(4)}(t)}{pt^4} < 0,$$

we have $T(\overline{P_3P_4}) \subset S_t$.

(ii) One can see that $T(\overline{P_4P_5}) = \{(x, 1+px) : \frac{p}{t^2} + 1 \leq x \leq \frac{t-1}{p}\}$. Since $\mathcal{C}_{51}$ is the graph of an increasing, concave up function $y = \frac{x^2(t-1)}{p}$, it is enough to show that its slope at $(\frac{p}{t^2}+1, p(\frac{p}{t^2}+1)+1)$ is not smaller than the slope of the line equation $y = 1 + px$, i.e.,

$$\frac{2(t-1)(\frac{p}{t^2}+1)}{p} - p = \frac{2t^3 - p^2t^2 + 2pt - 2p - 2t^2}{pt^2} > 0,$$

which is true by relation (iii) of Lemma 6.

□

In the sequel let $t_0(p)$ be a unique positive root of the equation (see Lemma 4)

$$t^5 - (p^2 + p + 1)t^4 + 2pt^3 - p(p^2 + 2)t^2 + p^2t - p^2 = 0.$$

Lemma 8 details elementary properties of sets S_t. See Fig. 1.

Lemma 8 *Assume that* $1 < p < \infty$.
   (i) $(\bar{x}, \bar{x}) \in S_t$ *for* $t_0(p) \leq t < \infty$.
   (ii) *If* $t_0(p) < t' < t < \infty$, *then* $S_{t'} \subset \text{int}(S_t)$.
   (iii) *If* $(1, \infty)^2 = \cup\{S_t : t_0(p) \leq t < \infty\}$.
   (iv) *For* $(x, y) \notin S_{t_0(p)}$ *there exists* $t' > t_0(p)$ *such that* $(x, y) \in \partial S_{t'}$.

Proof Statement (i) follows from the construction of the set S_t and the point P_1 which belongs to the equilibrium curve $\mathcal{C}_2$. Statement (ii) follows directly from the definition of the set S_t and Lemmas 7, 6, and 2. Notice that the x-coordinate of the point P_1 goes to infinity as t approaches one. Then it is obvious that the union of the sets S_t covers the set $(1, \infty)^2$, from which statement (iii) follows. Statement (iv) follows from (iii) and definition of the set S_t.   □

A set R is said to be invariant under the map T if $T(R) \subset R$.

Lemma 9 *Assume that* $1 < p < \infty$. *The set* S_t *is invariant if* $t_0(p) \leq t < \infty$. *Further* $T^4(S_t) \subset \text{int}(S_t)$.

Proof It is very easy to see that the map T is $1-1$ on $(1, \infty)^2$. From this and from the fact that S_t contains the unique positive fixed point of T, it follows that to prove S_t is invariant,

it suffices to show $T(\partial S_t) \subset S_t$, which follows from Lemmas 6 and 7. The rest of the proof follows from Lemmas 2 and 8. See Fig. 1.   □

For $\mathcal{A} \subset \mathbb{R}_+^2$ let $\omega(\mathcal{A})$ denote the omega-limit of $\mathcal{A}$ under the map T, i.e., the set of all accumulation points of all sequences $\{T^n(x)\}$ for $x \in \mathcal{A}$.

Lemma 4, given below details the property of the omega limit set.

Theorem 4 *Assume that* $1 < p < \infty$. *Then* $\omega((0,\infty) \times (0,\infty)) \subset S_{t_0(p)}$.

Proof Let $(x,y) \in (0,\infty) \times (0,\infty)$. Since $T(x,y) \in (1,\infty) \times (1,\infty)$, we have $T(x,y) \in \mathcal{S}_t$ for some $t > t_0(p)$. By Lemma 9, $T^n(x,y) \in \mathcal{S}_t$ for $n > 1$ and $T^n(x,y)$ has at least one accumulation point (x_1, y_1). Suppose that $(x_1, y_1) \notin S_{t_0(p)}$, then by (iv) of Lemma 8 there exists $t' > t_0(p)$ such that $(x_1, y_1) \in \partial \mathcal{S}_{t'}$. By continuity of T and by Lemma 9, $T^n(x,y) \in \text{int}(\mathcal{S}_{t'})$ for all n sufficiently large. This is not possible since (x_1, y_1) is an accumulation point of $T^n(x,y)$. Hence, $(x_1, y_1) \in S_{t_0(p)}$.   □

Now, we state Theorem 5 concerning the attractivity sets for solutions of Eq. (3). The proof of Theorem 5 directly follows from Lemma 8 and Theorem 4.
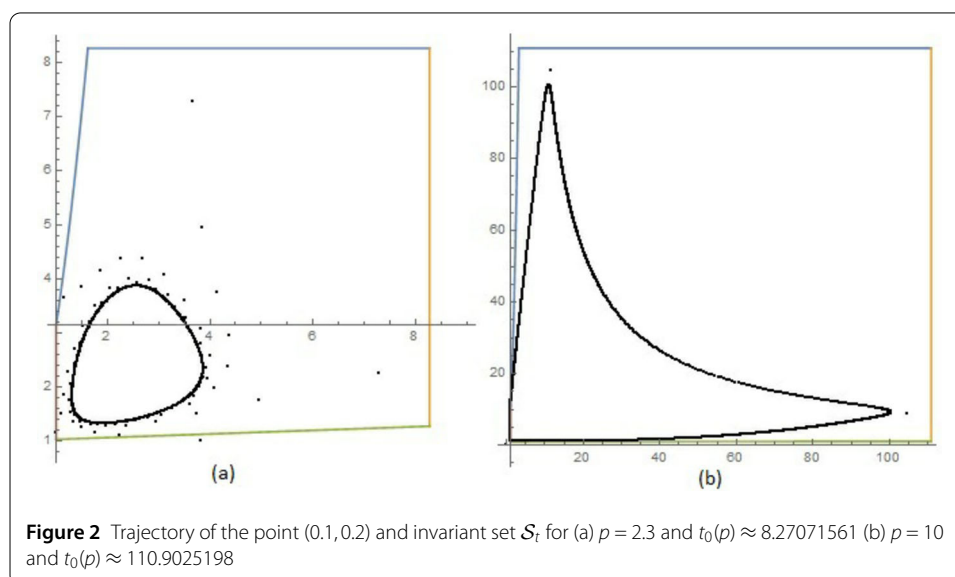
Theorem 5 *For all n sufficiently large,* $T^n(x,y) \in S_{t_0(p)}$.

Based on numerical visualization, we pose the following conjecture. See Fig. 2.

Conjecture 1 *The equilibrium point* $\bar{x} = \frac{1}{2}(\sqrt{4p+1} + 1)$ *of Eq.* (3) *is globally asymptotically stable if* $0 < p < 2$.

6 Neimark–Sacker bifurcation

In this section we discuss the existence of Neimark–Sacker bifurcation (for more details see [15–19]) for the unique positive equilibrium and compute asymptotic approximation of the invariant curve near the positive equilibrium point of Eq. (3), according to the procedure given in [16].

Figure 2 Trajectory of the point $(0.1, 0.2)$ and invariant set $\mathcal{S}_t$ for (a) $p = 2.3$ and $t_0(p) \approx 8.27071561$ (b) $p = 10$ and $t_0(p) \approx 110.9025198$

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = 1 + p\frac{y_n + \bar{x}}{(y_{n-1} + \bar{x})^2} - \bar{x}, \quad n = 0, 1, \dots \tag{17}$$

By using the substitution $u_n = y_{n-1}$ and $v_n = y_n$, we write Eq. (3) in the equivalent form:

$$\begin{cases} u_{n+1} = v_n, \\ v_{n+1} = 1 + p\frac{v_n + \bar{x}}{(u_n + \bar{x})^2} - \bar{x}. \end{cases} \tag{18}$$

Let **F** be the corresponding map defined by

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 1 + p\frac{v + \bar{x}}{(u + \bar{x})^2} - \bar{x} \end{pmatrix}. \tag{19}$$

It is easy to see that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = J_{\mathbf{F}}(0, 0)\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{20}$$

where

$$\mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + p\frac{v + \bar{x}}{(u + \bar{x})^2} + \frac{2pu}{\bar{x}^2} - \frac{pv}{\bar{x}^2} - \bar{x} \end{pmatrix}.$$

To study Neimark–Sacker bifurcation, we need the following lemma.

Lemma 10 *If $p = p_0 = 2$ then eigenvalues of Jacobian matrix of $\mathbf{F}$ at $(0,0)$ are μ and $\bar{\mu}$ where*

$$\mu = \frac{1}{4}(1 + i\sqrt{15}).$$

Moreover, μ satisfies the following:
(a) $\mu^k \neq 1$ *for $k = 1, 2, 3, 4$.*
(b) $d = \frac{d|\mu(p)|}{dp}\big|_{p=p_0} = \frac{1}{12} > 0$.
(c) *Eigenvectors associated to μ are*

$$\mathbf{q} = \left(\frac{1}{4}(1 - i\sqrt{15}), 1\right)^T \quad and \quad \mathbf{p} = \left(\frac{2(\sqrt{15} + i)}{\sqrt{15} - 15i}, \frac{8}{15 + i\sqrt{15}}\right)$$

such that $A\mathbf{q} = \mu\mathbf{q}$, $\mathbf{p}A = \mu\mathbf{p}$, and $\mathbf{p}\mathbf{q} = 1$.

Proof The Jacobian matrix of **F** is given by

$$J_{\mathbf{F}}(u, v) = \begin{pmatrix} 0 & 1 \\ -\frac{2p(v + \bar{x})}{(u + \bar{x})^3} & \frac{p}{(u + \bar{x})^2} \end{pmatrix}. \tag{21}$$

The eigenvalues of (21)) are $\mu_\pm(p)$ where

$$\mu_\pm(p) = \frac{p \pm \sqrt{p(p - 8\bar{x}^2)}}{2\bar{x}^2}. \tag{22}$$

At $(0,0)$, $\mathrm{Jac}_\mathbf{F}(u,v)$ has the form

$$\mathrm{Jac}_\mathbf{F}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2p}{\bar{x}^2} & \frac{p}{\bar{x}^2} \end{pmatrix}. \tag{23}$$

For $p = p_0 = 2$, it is easy to see that $\bar{x} = 2$ and

$$\mathbf{A} = J_\mathbf{F}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are μ and $\overline{\mu}$ where

$$\mu = \frac{1}{4}(1 + i\sqrt{15}).$$

The eigenvectors corresponding to μ and $\overline{\mu}$ are $\mathbf{q}$ and $\overline{\mathbf{q}}$ where

$$\mathbf{q} = \left(\frac{1}{4}(1 - i\sqrt{15}), 1 \right)^T.$$

One can prove that $|\mu| = 1$, $\mu^2 = -\frac{7}{8} + i\frac{\sqrt{15}}{8}$, $\mu^3 = -\frac{11}{16} - i\frac{3\sqrt{15}}{16}$, $\mu^4 = \frac{17}{32} - i\frac{7\sqrt{15}}{32}$, from which follows that $\mu^k \neq 1$ for $k = 1, 2, 3, 4$.

From (22) we have

$$\left|\mu(p)\right|^2 = \mu(p)\overline{\mu(p)} = \frac{2p}{\bar{x}^2} = \frac{8p}{(\sqrt{4p+1} + 1)^2},$$

from which we get

$$\frac{d}{dp}\left|\mu(p)\right| = \frac{\sqrt{2}}{\sqrt{p(4p+1)}(\sqrt{4p+1} + 1)},$$

which, by substituting $p_0 = 2$, simplifies to

$$\left.\frac{d|\mu(p)|}{dp}\right|_{p=p_0} = \frac{1}{12} > 0.$$

It is easy to see that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$. $\qquad\square$

Theorem 6 *Let*

$$x_{n+1} = 1 + p\frac{x_n}{x_{n-1}^2}.$$

Assume that $p = p_0 = 2$. Then there exists a neighborhood $\mathcal{O}$ of the equilibrium point $(\bar{x}, \bar{x}) = (2, 2)$ and sufficiently small parameter $\delta > 0$ such that if $|p - 2| < \delta$ and $x_0, x_{-1} \in O$, then

ω-limit set of solutions of Eq. (3), with initial condition x_0, x_{-1} is $(\bar{x}, \bar{x})$ if $-\delta < p - 2 < 0$ and belongs to a closed invariant C^1 curve $\Gamma(p)$ encircling the equilibrium point $(\bar{x}, \bar{x})$ if $0 < p - 2 < \delta$. Furthermore, $\Gamma(2) = \{(2, 2)\}$ and invariant curve $\Gamma(p)$ can be approximated by

$$
\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix}
$$

$$
+ \begin{pmatrix} \frac{1}{144}(72\sqrt{p-2}(\sqrt{15}\sin(\theta) + \cos(\theta)) + (p-2)(9\sqrt{15}\sin(2\theta) - 95\cos(2\theta) + 120)) \\ \frac{1}{36}(72\sqrt{p-2}\cos(\theta) + (p-2)(\sqrt{15}\sin(2\theta) + 25\cos(2\theta) + 30)) \end{pmatrix},
$$

where $\theta \in \mathbb{R}$.

Proof Let $p = p_0 + \delta$ where δ is a sufficiently small parameter. From Lemma 10 we can transform (18) into the normal form

$$
z_{n+1} = \mu(\delta)z_n + c(\delta)z_n^2\bar{z}_n + O(|z_n|^4).
$$

In the polar coordinates it can be written as

$$
\begin{pmatrix} r_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} |\mu(\delta)|r_n + a(\delta)r_n^3 + O(r_n^4) \\ \theta_n + \arg \mu(\delta) + b(\delta)r_n^2 + O(r_n^3) \end{pmatrix}, \tag{24}
$$

where $a(\delta) = \mathrm{Re}(c(\delta)/\mu(\delta))$ and $b(\delta) = \mathrm{Im}(c(\delta)/\mu(\delta))$. Performing a Taylor expansion of the coefficients of the first equation of (24) about $\delta = 0$, we have

$$
r_{n+1} = (1 + d\delta)r_n + a(0)r_n^3 + O(r_n^4).
$$

Now, we compute $a(0)$ using the following procedure from [16]. First, we compute $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{0,2}$ defined in [16]. Substituting $p = p_0$ and $\bar{x}$ into (20), we get

$$
\mathbf{F}_0 \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}_0 \begin{pmatrix} u \\ v \end{pmatrix}, \tag{25}
$$

where

$$
\mathbf{G}_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2(v+2)}{(u+2)^2} + u - \frac{v}{2} - 1 \end{pmatrix}.
$$

Hence, for $p = p_0$ system (18) is equivalent to

$$
\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}_0 \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{26}
$$

Define the basis of $\mathbb{R}^2$ by $\boldsymbol{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$, then we can represent (u, v) as

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \boldsymbol{\Phi} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{4}(1 - i\sqrt{15})z + \frac{1}{4}(1 + i\sqrt{15})\bar{z} \\ \bar{z} + z \end{pmatrix}.
$$

By using this, we have

$$
\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)
$$
$$
= \begin{pmatrix} 0 \\ \frac{1}{4}(-2(\bar{z}+z)+(1+i\sqrt{15})\bar{z}-\frac{128(\bar{z}+z+2)}{(-(\sqrt{15}-i)\bar{z}+(\sqrt{15}+i)z+8i)^2}-i\sqrt{15}z+z-4) \end{pmatrix}. \qquad (27)
$$

Let

$$
\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}\left(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2\right) + O\left(|z|^3\right).
$$

Then, we have

$$
\mathbf{g}_{20} = \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ -\frac{25-i\sqrt{15}}{16} \end{pmatrix},
$$
$$
\mathbf{g}_{11} = \frac{\partial^2}{\partial z\partial\bar{z}}\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{5}{4} \end{pmatrix}, \qquad (28)
$$
$$
\mathbf{g}_{02} = \frac{\partial^2}{\partial\bar{z}^2}\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ -\frac{25+i\sqrt{15}}{16} \end{pmatrix}.
$$

Let

$$
\mathbf{K}_{20} = \left(\mu^2 I - A\right)^{-1}\mathbf{g}_{20} = \begin{pmatrix} -\frac{95+9i\sqrt{15}}{144} \\ \frac{25-i\sqrt{15}}{36} \end{pmatrix},
$$
$$
\mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} \frac{5}{6} \\ \frac{5}{6} \end{pmatrix}, \qquad (29)
$$
$$
\mathbf{K}_{02} = \left(\bar{\mu}^2 I - A\right)^{-1}\mathbf{g}_{02} = \overline{K}_{20},
$$

and

$$
\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\left(\mathbf{K}_{20}z^2 + 2\mathbf{K}_{11}z\bar{z} + \mathbf{K}_{02}\bar{z}^2\right)\right)
$$
$$
= \frac{1}{2}\left(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2\right)
$$
$$
+ \frac{1}{6}\left(\mathbf{g}_{30}z^3 + 3\mathbf{g}_{21}z^2\bar{z} + 3\mathbf{g}_{12}z\bar{z}^2 + \mathbf{g}_{03}\bar{z}^3\right) + O\left(|z|^4\right). \qquad (30)
$$

Then, we have

$$
\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2\partial\bar{z}}\mathbf{G}\left(\boldsymbol{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20}z^2 + \mathbf{K}_{11}z\bar{z} + \frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\bigg|_{z=0}
$$
$$
= \begin{pmatrix} 0 \\ -\frac{1+3i\sqrt{15}}{36} \end{pmatrix}. \qquad (31)
$$

Consequently,

$$a(0) = \operatorname{Re}\left(\frac{c(0)}{\mu}\right)$$
$$= \frac{1}{2}\operatorname{Re}(\mathbf{p}\mathbf{g}_{21}\bar{\mu}) = -\frac{1}{12} < 0. \tag{32}$$

By Lemma 10, we know that

$$d = \frac{d|\mu(\delta)|}{d\delta}\bigg|_{\delta=0} = \frac{d|\mu(p)|}{dp}\bigg|_{p=p_0} = \frac{1}{12} > 0.$$

By the result in [16], we have an asymptotic approximation of the invariant curve as

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + 2\rho_0 \operatorname{Re}(\mathbf{q}e^{i\theta}) + \rho_0^2(\operatorname{Re}(\mathbf{K}_{20}e^{2i\theta}) + \mathbf{K}_{11}),$$

where

$$d = \frac{d}{dp}|\mu(p)|\bigg|_{p=2},$$
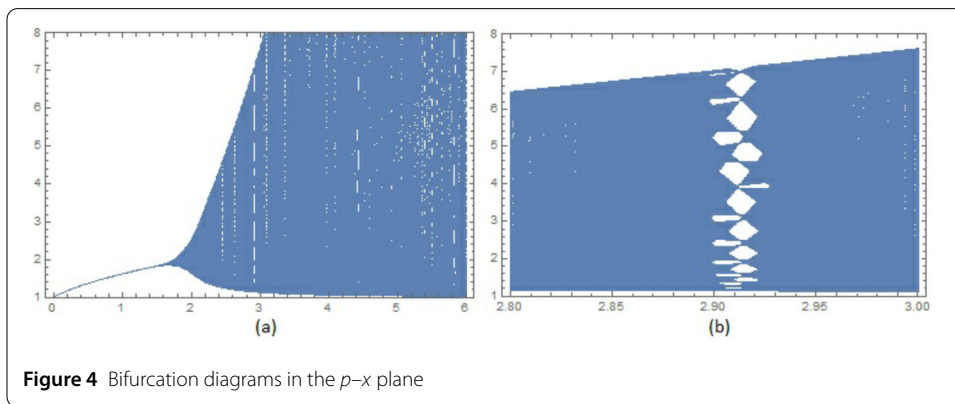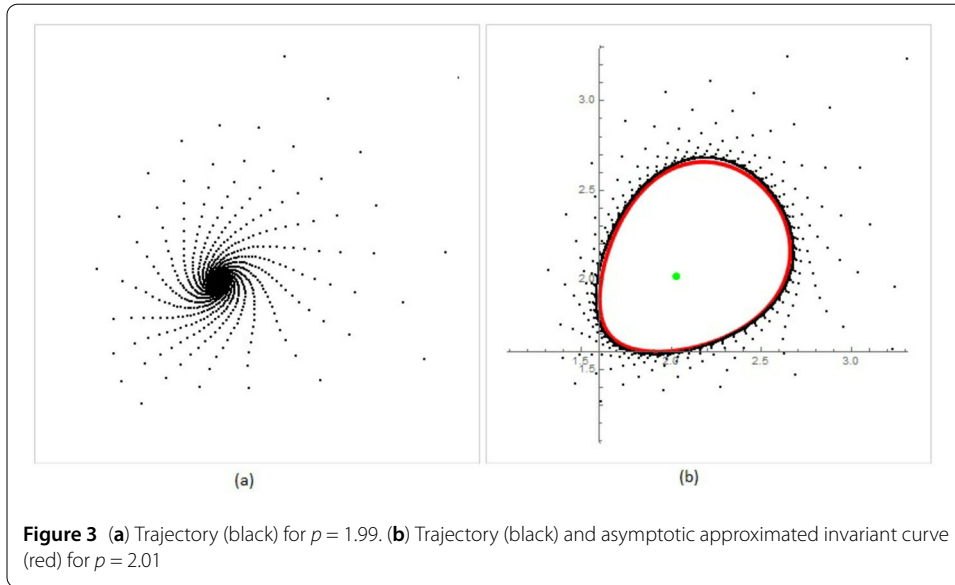$$\rho_0 = \sqrt{-\frac{d}{a(0)}\delta}, \quad \theta \in \mathbb{R}.$$

Since $\rho_0 = \sqrt{p-2}$ for $0 < p - 2 < \delta$, where $\delta > 0$ is a sufficiently small parameter, from the previous calculation we have

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{144}(72\sqrt{p-2}(\sqrt{15}\sin(\theta) + \cos(\theta)) + (p-2)(9\sqrt{15}\sin(2\theta) - 95\cos(2\theta) + 120)) \\ \frac{1}{36}(72\sqrt{p-2}\cos(\theta) + (p-2)(\sqrt{15}\sin(2\theta) + 25\cos(2\theta) + 30)) \end{pmatrix}.$$

This completes the proof. □

7 Conclusion

In this paper, we investigated the stability of the following second-order rational difference equation, $x_{n+1} = 1 + px_n/x_{n-1}^2$, $n = 0, 1, \dots$, with positive initial conditions and $p > 0$. We showed that the parameter space splits into three regions: $p < 2$, $p = 2$, and $p > 2$. We proved a global stability result for $p \leq (1 + \sqrt{2})/2$. Based on our numerical computations, we conjectured that the unique positive equilibrium is globally asymptotically stable for $0 < p < 2$. We proved boundedness of all solutions. In the case $1 < p < +\infty$, to prove the existence of a sequence of invariant and attractive sets that contain all solutions, we constructed compact sets $\mathcal{S}_t$ which are invariant for all t such that $t_0(p) \leq t < +\infty$ for some $t_0(p) > \bar{x}$. In the case $p = 2$, we studied Neimark–Sacker bifurcations for the positive equilibrium point. Also, we computed the approximation of the invariant curve near the positive equilibrium point. The positive equilibrium point is a repeller for $p > 2$. By numerical computations, we confirmed our analytic results. See Figs. 3 and 4.

Figure 3 (**a**) Trajectory (black) for $p = 1.99$. (**b**) Trajectory (black) and asymptotic approximated invariant curve (red) for $p = 2.01$

Figure 4 Bifurcation diagrams in the p–x plane

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors read and approved the final manuscript.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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