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Global exponential stability and existence of periodic solutions of fuzzy wave equations

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Abstract

In this paper, the global exponential stability and the existence of periodic solutions of fuzzy wave equations are investigated. By variable substitution the system of partial differential equations (PDEs) is transformed from second order to first order. Some sufficient conditions that ensure the global exponential stability and the existence of periodic solution of the system are obtained by an analysis that uses a suitable Lyapunov functional. In addition, a concrete example is given to show the effectiveness of the results.

Keywords: Fuzzy; Wave equations; Periodic solution; Exponential stability

1 Introduction

Fuzzy differential equations (FDEs) are those whose parameters, initial conditions and solutions are fuzzy numbers, while all these are real- or complex-valued functions for standard differential equations. Recently, the theory of FDEs has been developed by the community, since it is not only of interest but also one of the useful tools for modeling dynamic systems, in which uncertainties or vagueness pervade. For basic theory of FDEs one may refer to [1]. Some relevant applications that uses FDEs for demographic and life expectancy modeling problems have been proposed in [2], which are also problems that present high parametric uncertainty. In [3], the authors prove the existence and uniqueness of the solution of the FDEs with the right-hand side satisfying the Lipschitz condition by the concept of Hukuhara derivative. For more examples one can refer to the significant results in [4–16]. Therefore, the construction of a theory that combines appropriately the theory of impulsive differential equations with that of FDEs is essential. In [17], the authors consider the exact solutions of fuzzy wave-like equations with variable coefficients by a variational iteration method; see [18]. A systematic spectral-τ method for the solution of fuzzy fractional diffusion and fuzzy fractional wave equations are investigated. In [19] the authors study the calculus of fuzzy-valued functions of two variables and some properties are discussed, and the solutions of the fuzzy wave equations are given. In [20] the homotopy analysis method is proposed to obtain a semi-analytical solution of the fuzzy wave-like equations with variable coefficients. In [21] the authors study a geometric approach for solving the density-dependent diffusion Nagumo equation. In [22], a numerical solution of the time-fractional diffusion-wave equation with the fictitious time integration method are investigated.

On the other hand, nonlinear hyperbolic equations are applied widely to the modeling of many phenomena arising from experiments in physics and engineering, such as electromagnetic waves, elastic vibration in nonlinear media, and water waves. Indeed, as one studies the long-time behavior of any dynamical system, the periodic solutions are very important for a deep understanding of the dynamics of the system of fuzzy wave equations. To the best of our knowledge, there are no research results on the global exponential stability of the equilibrium point and the existence of periodic solutions of the fuzzy wave equations, which will be a more innovative research topic in both theory and applications. In this work we therefore investigate the global exponential stability and the existence of periodic solutions of the fuzzy wave equations.

The model equation we study has the following form:

$$\begin{cases} \frac{\partial^2 u_i(t,x)}{\partial t^2} = \sum_{k=1}^n \frac{\partial}{\partial x_k}(a_{ik}\frac{\partial u_i(t,x)}{\partial x_k}) - b_i u_i(t,x) - e_i\frac{\partial u_i(t,x)}{\partial t} + \bigwedge_{j=1}^n \alpha_{ij} f_j(u_j(t-\tau_{ij},x)) \\ \qquad + \bigvee_{j=1}^n \beta_{ij} f_j(u_j(t-\tau_{ij},x)) + \bigwedge_{j=1}^n T_{ij}\mu_j + \bigvee_{j=1}^n H_{ij}\mu_j, \quad t \ge 0, x \in \Omega, \end{cases} \tag{1}$$

for $i = 1,2,\ldots,n$, $u_i(t,x)$ corresponds to the state of the ith unit at time t and space x; $b_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state; $f_j(u_j(t,x))$ denotes the activation function of the jth unit at time t and space x; $e_i > 0$ represents the friction along the ith unit; τ_{ij} corresponds to the transmission delay along the axon of the jth unit from the ith unit and satisfies $0 \le \tau_{ij} \le \tau$; $a_{ik} > 0$ corresponds to the transmission diffusion operator along the ith unit; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operation, respectively; α_{ij} and β_{ij} are elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively; T_{ij} and H_{ij} are fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $x_i (i = 1,2,\ldots,n)$ corresponds to the ith coordinate in the space; Ω is a compact set with smooth boundary and $\operatorname{mes}\Omega > 0$ in space R^n.

The initial and boundary conditions of system (1) are

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial n} = 0, \quad t \ge 0, x \in \partial\Omega, \\ u_i(s,x) = \varphi_i(s,x), \quad -\infty < s < 0, \qquad u_i(0,x) = \varphi_i(0,x) = u_0, \quad x \in \Omega, \\ \frac{\partial u_i(s,x)}{\partial t} = \psi_i(s,x), \quad -\infty < s \le 0, x \in \Omega, \end{cases} \tag{2}$$

for $i = 1,2,\ldots,n$, where u_0 is constant, $\varphi_i(s,x)$ and $\psi_i(t,x) (i = 1,2,\ldots,n)$ are bounded and continuous functions on $(-\infty,0] \times \Omega$; $\frac{\partial u_i(t,x)}{\partial n} = (\frac{\partial u_i(t,x)}{\partial x_1}, \frac{\partial u_i(t,x)}{\partial x_2}, \ldots, \frac{\partial u_i(t,x)}{\partial x_n})^T$.

With appropriate parameters, system (1) can be reduced to:

1. One-dimensional wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a\frac{\partial^2 u(t,x)}{\partial x^2} + f(x,t).$$

2. Two-dimensional wave equation

$$\frac{\partial^2 u(t,x,y)}{\partial t^2} = a\frac{\partial^2 u(t,x)}{\partial x^2} + b\frac{\partial^2 u(t,x)}{\partial y^2} + f(x,y,t).$$

3. Two-dimensional dissipative wave equation

$$\frac{\partial^2 u(t,x,y)}{\partial t^2} = a\frac{\partial^2 u(t,x,y)}{\partial x^2} + b\frac{\partial^2 u(t,x,y)}{\partial y^2} + c\frac{\partial u(t,x,y)}{\partial t} + f(x,y,t)$$

etc., which have real-world applications.

This paper is organized as follows. Some preliminaries are given in Sect. 2. In Sect. 3, the sufficient conditions are derived for the global exponential stability and the existence of periodic solutions of the fuzzy wave equations, by the construction of a suitable Lyapunov functional and using some analytical techniques, respectively. In Sect. 4, an illustrative example is given to show the effectiveness of the proposed theory.

2 Preliminaries

Throughout this paper, we make the following assumptions.

(H) The activation functions f_i $(i = 1,2,\dots,n)$ satisfy Lipschitz condition, i.e., there exists a constant $l_i > 0$ $(i = 1,2,\dots,n)$ such that

$$\left|f_i(v_1) - f_i(v_2))\right| \le l_i|v_1 - v_2|, \quad i = 1,2,\dots,n,$$

for all $v_1, v_2 \in R$.

By introducing variable transformation

$$v_i(t,x) = \frac{\partial u_i(t,x)}{\partial t} + u_i(t,x), \quad i = 1,2,\dots,n,$$

the systems (1) and (2) can be rewritten as

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = -u_i(t,x) + v_i(t,x), \\ \frac{\partial v_i(t,x)}{\partial t} = \sum_{k=1}^n \frac{\partial}{\partial x_k}(a_{ik}\frac{\partial u_i(t,x)}{\partial x_k}) - (b_i - e_i + 1)u_i(t,x) - (e_i - 1)v_i(t,x) \\ \qquad + \bigwedge_{j=1}^n \alpha_{ij} f_j(u_j(t-\tau_{ij},x) + \bigvee_{j=1}^n \beta_{ij} f_j(u_j(t-\tau_{ij},x) \\ \qquad + \bigwedge_{j=1}^n T_{ij}\mu_j + \bigvee_{j=1}^n H_{ij}\mu_j, \quad t \ge 0, x \in \Omega, \end{cases} \tag{3}$$

for $i = 1,2,\dots,n$,

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial n} = 0, \quad t \ge 0, x \in \partial\Omega, \\ u_i(s,x) = \varphi_i(s,x), \quad -\infty < s < 0, \qquad u_i(0,x) = \varphi_i(0,x) = u_0, \quad x \in \Omega, \\ v_i(s,x) = \varphi_i(s,x) + \psi_i(s,x), \quad -\infty < s \le 0, x \in \Omega, \end{cases} \tag{4}$$

for $i = 1,2,\dots,n$.

For convenience, we denote $\bar{\psi}_i(s,t) = \varphi_i(s,x) + \psi_i(s,x)$.

Let $w_i(t,x) = (u_i(t,x), v_i(t,x))^T$, system (3) can be written in the form

$$\begin{aligned} \frac{\partial w_i(t,x)}{\partial t} &= P\begin{pmatrix} \sum_{k=1}^n \frac{\partial}{\partial x_k}(a_{ik}\frac{\partial u_i(t,x)}{\partial x_k}) \\ 0 \end{pmatrix} - B_i w_i(t,x) + P\begin{pmatrix} \bigwedge_{j=1}^n \alpha_{ij} f_j(u_j(t-\tau_{ij},x)) \\ 0 \end{pmatrix} \\ &\quad + P\begin{pmatrix} \bigvee_{j=1}^n \beta_{ij} f_j(u_j(t-\tau_{ij},x)) \\ 0 \end{pmatrix} + P\begin{pmatrix} \bigwedge_{j=1}^n T_{ij}\mu_j \\ 0 \end{pmatrix} + P\begin{pmatrix} \bigvee_{j=1}^n H_{ij}\mu_j \\ 0 \end{pmatrix}, \end{aligned} \tag{5}$$

for $t \geq 0$, $x \in \Omega$, $i = 1, 2, \ldots, n$, where

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B_i = \begin{pmatrix} 1 & -1 \\ b_i + 1 - e_i & e_i - 1 \end{pmatrix}.$$

Definition 1 The point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ is called an equilibrium point of system (1), if the following equation holds true:

$$-b_i u_i^* + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(u_j^*) + \bigvee_{j=1}^{n} \beta_{ij} f_j(u_j^*) + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n} H_{ij}\mu_j = 0, \tag{6}$$

for $i = 1, 2, \ldots, n$.

The point (u^*, v^*) is called an equilibrium point of system (3), if the following equations hold true:

$$\begin{cases} -u_i^* + v_i^* = 0, \\ -b_i u_i^* + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(u_j^*) + \bigvee_{j=1}^{n} \beta_{ij} f_j(u_j^*) + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n} H_{ij}\mu_j = 0, \end{cases} \tag{7}$$

for $i = 1, 2, \ldots, n$, where $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$, $v^* = (v_1^*, v_2^*, \ldots, v_n^*)^T$.

Definition 2 Let $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$, $v^* = (v_1^*, v_2^*, \ldots, v_n^*)^T$ be the equilibrium points of system (3), then we may define the following norms:

$$\begin{aligned}
&\|u_i(t,x) - u_i^*\|_{L^2}^2 = \int_\Omega (u_i(t,x) - u_i^*)^2\, dx, \\
&\|v_i(t,x) - v_i^*\|_{L^2}^2 = \int_\Omega (v_i(t,x) - v_i^*)^2\, dx, \\
&\|\varphi - u^*\|_{L^2} = \sup_{-\infty < t \leq 0} \sum_{i=1}^{n} \|\varphi_i(t,x) - u_i^*\|_{L^2}^2, \\
&\|\bar{\psi} - v^*\|_{L^2} = \sup_{-\infty < t \leq 0} \sum_{i=1}^{n} \|\bar{\psi}_i(t,x) - v_i^*\|_{L^2}^2, \\
&\left\| \begin{pmatrix} \varphi \\ \bar{\psi} \end{pmatrix} - \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{L^2} = \|\varphi - u^*\|_{L^2} + \|\bar{\psi} - v^*\|_{L^2},
\end{aligned}$$

where $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T$ and $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_n)^T$ are initial values.

Definition 3 The equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of system (1) is said to be globally exponentially stable, if there exist two constants $\sigma > 0$ and $M > 1$ such that

$$\sum_{i=1}^{n} \|u_i(t,x) - u_i^*\|_{L^2}^2 \leq M e^{-\sigma t} \|\varphi - u^*\|_{L^2},$$

for all $t \geq 0$, where $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))^T$ is a solution of system (1) with initial value

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial n} = 0, \quad t \geq 0, x \in \partial\Omega, \\ u_i(s,x) = \varphi_i(s,x), \quad -\infty < s < 0, \qquad u_i(0,x) = \varphi_i(0,x) = u_0, \quad x \in \Omega, \\ \frac{\partial u_i(s,x)}{\partial t} = \psi_i(s,x), \quad -\infty < s \leq 0, x \in \Omega, \end{cases}$$

for $i = 1, 2, \ldots, n$.

Lemma 1 (Yang and Yang [23]) *Suppose $u(t,x)$ and $\bar{u}(t,x)$ are two states of system* (1), *then we have*

$$\left| \bigwedge_{j=1}^{n} \alpha_{ij} f_j\big(u_j(t,x)\big) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j\big(\bar{u}_j(t,x)\big) \right| \leq \sum_{i=1}^{n} |\alpha_{ij}| \big| f_j\big(u_j(t,x)\big) - f_j\big(\bar{u}_j(t,x)\big)\big|,$$

$$\left| \bigvee_{j=1}^{n} \beta_{ij} f_j\big(u_j(t,x)\big) - \bigvee_{j=1}^{n} \beta_{ij} f_j\big(\bar{u}_j(t,x)\big) \right| \leq \sum_{i=1}^{n} |\beta_{ij}| \big| f_j\big(u_j(t,x)\big) - f_j\big(\bar{u}_j(t,x)\big)\big|,$$

for $i = 1, 2, \ldots, n$.

Lemma 2 (Forti and Tesi [24]) *If $H(u) \in C^0$, and it satisfies the following conditions*:

(1) *$H(u)$ is injective on R^n*,

(2) *$\|H(u)\| \to +\infty$, as $\|u\| \to +\infty$*,

then $H(u)$ is a homeomorphism of R^n.

Lemma 3 *For any $\alpha > 0$ and x, y, there exists a constant $0 \leq \beta \leq 1$, then we have $\alpha|x||y| \leq \frac{1}{2}[(\alpha^{\beta})^2 x^2 + (\alpha^{1-\beta})^2 y^2]$.*

Proof of Lemma 3 Using the inequality $a^2 + b^2 \geq 2ab$, we have

$$\alpha|x||y| = \big(\alpha^{\beta}|x|\big)\big(\alpha^{1-\beta}|y|\big) \leq \frac{1}{2}\big[\big(\alpha^{\beta}\big)^2 x^2 + \big(\alpha^{1-\beta}\big)^2 y^2\big]. \qquad \square$$

3 Main results

In this section, we can derive some sufficient conditions which ensure the existence and globally exponential stability of periodic solution of system (1) by constructing a suitable Lyapunov functional and using some analysis techniques.

Theorem 1 *For system* (1), *under the hypothesis* (*H*), *system* (1) *has a unique equilibrium point, which is globally exponentially stable if there exist constants $0 \leq \beta_i \leq 1$, $0 \leq \gamma_i \leq 1$ ($i = 1, 2, \ldots, n$), such that*

$$\sum_{j=1}^{n} \frac{|\alpha_{ji}|}{2} \big(l_i^{1-\beta_i}\big)^2 + \sum_{j=1}^{n} \frac{|\beta_{ji}|}{2} \big(l_i^{1-\gamma_i}\big)^2 + \frac{|b_i - e_i|}{2} - 1 < 0,$$

$$\sum_{j=1}^{n} \frac{|\alpha_{ij}|}{2} \big(l_j^{\beta_j}\big)^2 + \sum_{j=1}^{n} \frac{|\beta_{ij}|}{2} \big(l_j^{\gamma_j}\big)^2 + 1 + \frac{|b_i - e_i|}{2} - e_i < 0,$$

for $i = 1, 2, \ldots, n$.

Proof of Theorem 1 We shall prove Theorem 1 in two steps.

Step 1: We prove the existence and uniqueness of the equilibrium point.

From Definition 1, we know that equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of system (1) satisfies the following equation:

$$-b_i u_i^* + \bigwedge_{j=1}^{n} \alpha_{ij} f_j\bigl(u_j^*\bigr) + \bigvee_{j=1}^{n} \beta_{ij} f_j\bigl(u_j^*\bigr) + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n} H_{ij}\mu_j = 0,$$

for $i = 1, 2, \ldots, n$. Let $\Phi(u) = (\Phi_1(u), \Phi_2(u), \ldots, \Phi_n(u))^T$, where

$$\Phi_i(u) = -b_i u_i + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(u_j) + \bigvee_{j=1}^{n} \beta_{ij} f_j(u_j) + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n} H_{ij}\mu_j,$$

for $i = 1, 2, \ldots, n$.

It is known that the solutions of $\Phi(u) = 0$ are equilibria of system (1). If the mapping $\Phi(u)$ is a homeomorphism on R^n, then there exists a unique point u^* such that $\Phi(u^*) = 0$, i.e., system (1) has a unique equilibrium point u^* (see [25, 26]).

Next we prove that $\Phi(u)$ is a homeomorphism.

First, we prove that $\Phi(u)$ is an injective mapping on R^n.

In fact, if there exist $u = (u_1, u_2, \ldots, u_n)^T$, $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)^T \in R^n$ and $u \neq \bar{u}$ such that $\Phi(u) = \Phi(\bar{u})$, then

$$-b_i(u_i - \bar{u}_i) + \bigwedge_{j=1}^{n} \alpha_{ij}\bigl(f_j(u_j) - f_j(\bar{u}_j)\bigr) + \bigvee_{j=1}^{n} \beta_{ij}\bigl(f_j(u_j) - f_j(\bar{u}_j)\bigr) = 0,$$

for $i = 1, 2, \ldots, n$. We have

$$(u_i - \bar{u}_i)\left[-b_i(u_i - \bar{u}_i) + \bigwedge_{j=1}^{n} \alpha_{ij}\bigl(f_j(u_j) - f_j(\bar{u}_j)\bigr) + \bigvee_{j=1}^{n} \beta_{ij}\bigl(f_j(u_j) - f_j(\bar{u}_j)\bigr)\right] = 0,$$

for $i = 1, 2, \ldots, n$. From assumptions (H), Lemma 1 and Lemma 3, we obtain

$$\sum_{i=1}^{n}\left[-b_i|u_i - \bar{u}_i|^2 + \sum_{j=1}^{n} |\alpha_{ij}| l_j |u_i - \bar{u}_i| \cdot |u_j - \bar{u}_j| + \sum_{j=1}^{n} |\beta_{ij}| l_j |u_i - \bar{u}_i| \cdot |u_j - \bar{u}_j|\right] \geq 0,$$

$$\sum_{i=1}^{n}\left\{-b_i|u_i - \bar{u}_i|^2 + \sum_{j=1}^{n} \frac{|\alpha_{ij}|}{2}\bigl[\bigl(l_j^{\beta_j}\bigr)^2 |u_i - \bar{u}_i|^2 + \bigl(l_j^{1-\beta_j}\bigr)^2 |u_j - \bar{u}_j|^2\bigr]\right.$$
$$\left. + \sum_{j=1}^{n} \frac{|\beta_{ij}|}{2}\bigl[\bigl(l_j^{\gamma_j}\bigr)^2 |u_i - \bar{u}_i|^2 + \bigl(l_j^{1-\gamma_j}\bigr)^2 |u_j - \bar{u}_j|^2\bigr]\right\} \geq 0.$$

From the above, since

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\beta_{ij}|}{2}\bigl(l_j^{\gamma_j}\bigr)^2 |u_j - \bar{u}_j|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\beta_{ji}|}{2}\bigl(l_i^{1-\gamma_i}\bigr)^2 |u_i - \bar{u}_i|^2,$$

we obtain

$$\sum_{i=1}^{n}\Biggl\{-b_i+\sum_{j=1}^{n}\biggl(\frac{|\alpha_{ij}|}{2}\bigl(l_j^{\beta_j}\bigr)^2+\frac{|\alpha_{ji}|}{2}\bigl(l_i^{1-\beta_i}\bigr)^2\biggr)$$
$$+\sum_{j=1}^{n}\biggl(\frac{|\beta_{ij}|}{2}\bigl(l_j^{\gamma_j}\bigr)^2+\frac{|\beta_{ji}|}{2}\bigl(l_i^{1-\gamma_i}\bigr)^2\biggr)\Biggr\}|u_i-\bar{u}_i|^2\geq 0. \tag{8}$$

From the condition of Theorem 1, we have

$$-b_i+\sum_{j=1}^{n}\biggl(\frac{|\alpha_{ij}|}{2}\bigl(l_j^{\beta_j}\bigr)^2+\frac{|\alpha_{ji}|}{2}\bigl(l_i^{1-\beta_i}\bigr)^2\biggr)+\sum_{j=1}^{n}\biggl(\frac{|\beta_{ij}|}{2}\bigl(l_j^{\gamma_j}\bigr)^2+\frac{|\beta_{ji}|}{2}\bigl(l_i^{1-\gamma_i}\bigr)^2\biggr)$$
$$< -b_i+e_i-|b_i-e_i|\leq 0, \tag{9}$$

for $i=1,2,\ldots,n$. From (8) and (9), we obtain $u_i=\bar{u}_i$ for $i=1,2,\ldots,n$, which is in contradiction to $u\neq\bar{u}$. So $\Phi(u)$ is an injective mapping on R^n.

Second, we prove that $\|\Phi(u)\|\to+\infty$ as $\|u\|\to+\infty$.

Let $\tilde{\Phi}(u)=\Phi(u)-\Phi(0)=(\tilde{\Phi}_1(u),\tilde{\Phi}_2(u),\ldots,\tilde{\Phi}_n(u))^T$, then

$$\tilde{\Phi}_i(u)=-b_iu_i+\bigwedge_{j=1}^{n}\alpha_{ij}\bigl(f_j(u_j)-f_j(0)\bigr)+\bigvee_{j=1}^{n}\beta_{ij}\bigl(f_j(u_j)-f_j(0)\bigr),$$

for $i=1,2,\ldots,n$. Calculating $u^T\tilde{\Phi}(u)$, we obtain

$$\begin{aligned}
u^T\tilde{\Phi}(u) &= \sum_{i=1}^{n}\Biggl[-b_iu_i^2+\bigwedge_{j=1}^{n}\alpha_{ij}u_i\bigl(f_j(u_j)-f_j(0)\bigr)+\bigvee_{j=1}^{n}\beta_{ij}u_i\bigl(f_j(u_j)-f_j(0)\bigr)\Biggr]\\
&\leq \sum_{i=1}^{n}\Biggl[-b_iu_i^2+\sum_{j=1}^{n}|\alpha_{ij}|l_j|u_i||u_j|+\sum_{j=1}^{n}|\beta_{ij}|l_j|u_i||u_j|\Biggr]\\
&\leq -\min_{1\leq i\leq n}\Biggl\{b_i-\sum_{j=1}^{n}\biggl(\frac{|\alpha_{ij}|}{2}\bigl(l_j^{\beta_j}\bigr)^2+\frac{|\alpha_{ji}|}{2}\bigl(l_i^{1-\beta_i}\bigr)^2\biggr)\\
&\qquad -\sum_{j=1}^{n}\biggl(\frac{|\beta_{ij}|}{2}\bigl(l_j^{\gamma_j}\bigr)^2+\frac{|\beta_{ji}|}{2}\bigl(l_i^{1-\gamma_i}\bigr)^2\biggr)\Biggr\}\|u\|_{L^2}^2.
\end{aligned}$$

By the Schwartz inequality $-X^TY\leq|X^TY|\leq\|X\|\|Y\|$ $(X,Y\in R^n)$, where $\|X\|=(\sum_{i=1}x_i^2)^{\frac{1}{2}}$, we get

$$\|u\|\cdot\|\tilde{\Phi}\|\geq\min_{1\leq i\leq n}\Biggl\{b_i-\sum_{j=1}^{n}\biggl(\frac{|\alpha_{ij}|}{2}\bigl(l_j^{\beta_j}\bigr)^2+\frac{|\alpha_{ji}|}{2}\bigl(l_i^{1-\beta_i}\bigr)^2\biggr)$$
$$-\sum_{j=1}^{n}\biggl(\frac{|\beta_{ij}|}{2}\bigl(l_j^{\gamma_j}\bigr)^2+\frac{|\beta_{ji}|}{2}\bigl(l_i^{1-\gamma_i}\bigr)^2\biggr)\Biggr\}\|u\|^2.$$

When $\|u\| \neq 0$, we have

$$\|\tilde{\Phi}\| \geq \min_{1\leq i\leq n}\left\{b_i - \sum_{j=1}^{n}\left(\frac{|\alpha_{ij}|}{2}\left(l_j^{\beta_j}\right)^2 + \frac{|\alpha_{ji}|}{2}\left(l_i^{1-\beta_i}\right)^2\right)\right.$$
$$\left. - \sum_{j=1}^{n}\left(\frac{|\beta_{ij}|}{2}\left(l_j^{\gamma_j}\right)^2 + \frac{|\beta_{ji}|}{2}\left(l_i^{1-\gamma_i}\right)^2\right)\right\}\|u\|.$$

Therefore $\|\tilde{\Phi}(u)\| \to +\infty$ as $\|u\| \to +\infty$, which implies that $\|H(u)\| \to +\infty$ as $\|u\| \to +\infty$.

From Lemma 2, we know that $\Phi(u)$ is a homeomorphism on R^n, then system (1) has a unique equilibrium point.

Step 2: We prove that the unique equilibrium point $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$ of system (1) is globally exponential stable.

Let $Z_i = (u_i - u_i^*, v_i - v_i^*)^T$, $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$, $v^* = (v_1^*, v_2^*, \dots, v_n^*)^T$ be the equilibrium of system (3). From (5), (6) and (7), we have

$$\begin{aligned}\frac{\partial Z_i(t,x)}{\partial t} &= P\begin{pmatrix}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}(a_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k})\\ 0\end{pmatrix}\\ &\quad - B_i Z_i(t,x) + P\begin{pmatrix}\bigwedge_{j=1}^{n}\alpha_{ij}(f_j(u_j(t-\tau_{ij},x)) - f_j(u_j^*))\\ 0\end{pmatrix}\\ &\quad + P\begin{pmatrix}\bigvee_{j=1}^{n}\beta_{ij}(f_j(u_j(t-\tau_{ij},x)) - f_j(u_j^*))\\ 0\end{pmatrix},\end{aligned} \tag{10}$$

for $t \geq 0$, $x \in \Omega$, $i = 1, 2, \dots, n$.

By multiplying both sides of (10) with $Z_i^T = (u_i - u_i^*, v_i - v_i^*)$, and Lemma 1, we obtain

$$\begin{aligned}Z_i^T\frac{\partial Z_i}{\partial t} &= \left(v_i - v_i^*\right)\sum_{k=1}^{n}a_{ik}\frac{\partial^2(u_i - u_i^*)}{\partial x_k^2}\\ &\quad - \left[\left(u_i - u_i^*\right)^2 + (e_i - 1)\left(v_i - v_i^*\right)^2 + (b_i - e_i)\left(u_i - u_i^*\right)\left(v_i - v_i^*\right)\right]\\ &\quad + \bigwedge_{j=1}^{n}\alpha_{ij}\left(v_i - v_i^*\right)\left(f_j\left(u_j(t-\tau_{ij},x)\right) - f_j\left(u_j^*\right)\right)\\ &\quad + \bigvee_{j=1}^{n}\beta_{ij}\left(v_i - v_i^*\right)\left(f_j\left(u_j(t-\tau_{ij},x)\right) - f_j\left(u_j^*\right)\right)\\ &\leq \left(v_i - v_i^*\right)\sum_{k=1}^{n}a_{ik}\frac{\partial^2(u_i - u_i^*)}{\partial x_k^2}\\ &\quad - \left[\left(1 - \frac{|b_i - e_i|}{2}\right)\left(u_i - u_i^*\right)^2 + \left(e_i - 1 - \frac{|b_i - e_i|}{2}\right)\left(v_i - v_i^*\right)^2\right]\\ &\quad + \sum_{j=1}^{n}|\alpha_{ij}|l_j\left|v_i - v_i^*\right|\left|u_j(t-\tau_{ij},x) - u_j^*\right| + \sum_{j=1}^{n}|\beta_{ij}|l_j\left|v_i - v_i^*\right|\left|u_j(t-\tau_{ij},x) - u_j^*\right|\\ &\leq \left(v_i - v_i^*\right)\sum_{k=1}^{n}a_{ik}\frac{\partial^2(u_i - u_i^*)}{\partial x_k^2}\end{aligned}$$

$$
\begin{aligned}
&-\left[\left(1-\frac{|b_i-e_i|}{2}\right)\left(u_i-u_i^*\right)^2+\left(e_i-1-\frac{|b_i-e_i|}{2}\right)\left(v_i-v_i^*\right)^2\right]\\
&+\sum_{j=1}^{n}\frac{|\alpha_{ij}|}{2}\left[\left(l_j^{\beta_j}\right)^2\left|v_i-v_i^*\right|^2+\left(l_j^{1-\beta_j}\right)^2\left|u_j(t-\tau_{ij},x)-u_j^*\right|^2\right]\\
&+\left(l_j^{1-\gamma_j}\right)^2\left|u_j(t-\tau_{ij},x)-u_j^*\right|^2\right],
\end{aligned}
$$

for $i=1,2,\ldots,n$.

We consider the Lyapunov functional

$$
\begin{aligned}
V(t)=\int_{\Omega}\sum_{i=1}^{n}\Bigg\{&\frac{\|Z_i(t,x)\|_{L^2}^2}{2}e^{2\varepsilon t}\\
&+\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\int_{t-\tau_{ij}}^{t}\left(u_j(s,x)-u_j^*\right)^2e^{2\varepsilon(s+\tau_{ij})}\,ds\\
&+\sum_{k=1}^{n}\frac{a_{ik}}{2}e^{2\varepsilon t}\left[\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right]^2\Bigg\}\,dx,
\end{aligned}
\tag{11}
$$

where $\varepsilon>0$ is sufficiently small.

By calculating the upper right Dini derivative $D^+V(t)$ of $V(t)$ along the solution of (10), with some analysis techniques, we have

$$
\begin{aligned}
D^+V(t)={}&\int_{\Omega}\sum_{i=1}^{n}\Bigg\{Z_i^T(t,x)\frac{\partial Z_i(t,x)}{\partial t}e^{2\varepsilon t}+\varepsilon e^{2\varepsilon t}\left\|Z_i(t,x)\right\|_{L^2}^2\\
&+\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\\
&\times\left[\left(u_j(t,x)-u_j^*\right)^2e^{2\varepsilon(t+\tau_{ij})}-\left(u_j(t-\tau_{ij},x)-u_j^*\right)^2e^{2\varepsilon t}\right]\\
&+\sum_{k=1}^{n}\frac{a_{ik}}{2}\frac{\partial}{\partial t}\left[e^{2\varepsilon t}\left(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)^2\right]\Bigg\}\,dx\\
\le{}&\int_{\Omega}\sum_{i=1}^{n}e^{2\varepsilon t}\Bigg\{\varepsilon\left\|Z_i(t,x)\right\|_{L^2}^2+\left(v_i-v_i^*\right)\sum_{k=1}^{n}a_{ik}\frac{\partial^2(u_i-u_i^*)}{\partial x_k^2}\\
&-\left[\left(1-\frac{|b_i-e_i|}{2}\right)\left(u_i-u_i^*\right)^2+\left(e_i-1-\frac{|b_i-e_i|}{2}\right)\left(v_i-v_i^*\right)^2\right]\\
&+\sum_{j=1}^{n}\frac{|\alpha_{ij}|}{2}\left[\left(l_j^{\beta_j}\right)^2\left|v_i-v_i^*\right|^2+\left(l_j^{1-\beta_j}\right)^2\left|u_j(t-\tau_{ij},x)-u_j^*\right|^2\right]\\
&+\sum_{j=1}^{n}\frac{|\beta_{ij}|}{2}\left[\left(l_j^{\gamma_j}\right)^2\left|v_i-v_i^*\right|^2+\left(l_j^{1-\gamma_j}\right)^2\left|u_j(t-\tau_{ij},x)-u_j^*\right|^2\right]\\
&+\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\left[\left(u_j(t,x)-u_j^*\right)^2e^{2\varepsilon\tau_{ij}}-\left(u_j(t-\tau_{ij},x)-u_j^*\right)^2\right]\\
&+e^{-2\varepsilon t}\sum_{k=1}^{n}\frac{a_{ik}}{2}\frac{\partial}{\partial t}\left[e^{2\varepsilon t}\left(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)^2\right]\Bigg\}\,dx.
\end{aligned}
\tag{12}
$$

Since $v_i - v_i^* = \frac{\partial u_i}{\partial t} + u_i - u_i^* = \frac{\partial (u_i - u_i^*)}{\partial t} + (u_i - u_i^*)$, $a_{ik} > 0$, we have

$$\begin{aligned}
& e^{2\varepsilon t} \int_\Omega (v_i - v_i^*) \sum_{k=1}^n a_{ik} \frac{\partial^2 (u_i - u_i^*)}{\partial x_k^2}\, dx \\
&= e^{2\varepsilon t} \sum_{k=1}^n a_{ik} \int_\Omega \left\{ \frac{\partial}{\partial x_k} \left[(v_i - v_i^*) \frac{\partial (u_i - u_i^*)}{\partial x_k} \right] - \frac{\partial (v_i - v_i^*)}{\partial x_k} \frac{\partial (u_i - u_i^*)}{\partial x_k} \right\} dx \\
&= e^{2\varepsilon t} \sum_{k=1}^n a_{ik} [\int_{\partial\Omega} (v_i - v_i^*) \frac{\partial (u_i - u_i^*)}{\partial x_k}\, dx - \int_\Omega \frac{\partial (v_i - v_i^*)}{\partial x_k} \frac{\partial (u_i - u_i^*)}{\partial x_k}\, dx \\
&= -e^{2\varepsilon t} \sum_{k=1}^n a_{ik} \int_\Omega \frac{\partial (v_i - v_i^*)}{\partial x_k} \frac{\partial (u_i - u_i^*)}{\partial x_k}\, dx \\
&= -e^{2\varepsilon t} \sum_{k=1}^n a_{ik} \int_\Omega \left[\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 + \left(\frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 \right] dx \\
&\le -\sum_{k=1}^n \frac{a_{ik}}{2} \int_\Omega \left[e^{2\varepsilon t} \frac{\partial}{\partial t} \left(\frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 + 2\varepsilon e^{2\varepsilon t} \left(\frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 \right] dx \\
&= -\sum_{k=1}^n \frac{a_{ik}}{2} \int_\Omega \frac{\partial}{\partial t} \left[e^{2\varepsilon t} \left(\frac{\partial (u_i - u_i^*)}{\partial x_k} \right)^2 \right] dx. \qquad (13)
\end{aligned}$$

It follows from (12) and (13) that

$$\begin{aligned}
D^+ V(t) \le{}& \sum_{i=1}^n e^{2\varepsilon t} \Bigg\{ \varepsilon \left(\|u_i - u_i^*\|_{L^2}^2 + \|v_i - v_i^*\|_{L^2}^2 \right) \\
& - \left(1 - \frac{|b_i - e_i|}{2} \right) \|u_i - u_i^*\|_{L^2}^2 - \left(e_i - 1 - \frac{|b_i - e_i|}{2} \right) \|v_i - v_i^*\|_{L^2}^2 \\
& + \sum_{j=1}^n \frac{|c_{ji}|}{2} \left(l_i^{1-\alpha_i} \right)^2 \|u_i - u_i^*\|_{L^2}^2 + \sum_{j=1}^n \frac{|\alpha_{ij}| (l_j^{\beta_j})^2 + |\beta_{ij}| (l_j^{\gamma_j})^2}{2} \|v_i - v_i^*\|_{L^2}^2 \\
& + \sum_{j=1}^n \frac{|\alpha_{ji}| (l_i^{1-\beta_i})^2 + |\beta_{ji}| (l_i^{1-\gamma_i})^2}{2} \|u_i - u_i^*\|_{L^2}^2 e^{2\varepsilon \tau_{ji}} \Bigg\} \\
\le{}& e^{2\varepsilon t} \sum_{i=1}^n \Bigg\{ - \Bigg[1 - \frac{|b_i - e_i|}{2} - \varepsilon \\
& - \sum_{j=1}^n \frac{|\alpha_{ji}| (l_i^{1-\beta_i})^2 + |\beta_{ji}| (l_i^{1-\gamma_i})^2}{2} e^{2\varepsilon \tau_{ji}} \Bigg] \|u_i - u_i^*\|_{L^2}^2 \\
& - \left[e_i - 1 - \frac{|b_i - e_i|}{2} - \varepsilon - \sum_{j=1}^n \frac{|\alpha_{ij}| (l_j^{\beta_j})^2 + |\beta_{ij}| (l_j^{\gamma_j})^2}{2} \right] \|v_i - v_i^*\|_{L^2}^2 \Bigg\}, \qquad (14)
\end{aligned}$$

for $i = 1, 2, \dots, n$. From the condition of Theorem 1, we can choose a sufficiently small $\varepsilon > 0$ such that

$$\sum_{j=1}^n \frac{|\alpha_{ji}|}{2} \left(l_i^{1-\beta_i} \right)^2 e^{2\varepsilon \tau_{ij}} + \sum_{j=1}^n \frac{|\beta_{ji}|}{2} \left(l_i^{1-\gamma_i} \right)^2 e^{2\varepsilon \tau_{ij}} + \frac{|b_i - e_i|}{2} + \varepsilon - 1 \le 0,$$

$$\sum_{j=1}^{n}\frac{|\alpha_{ij}|}{2}\left(l_j^{\beta_j}\right)^2+\sum_{j=1}^{n}\frac{|\beta_{ij}|}{2}\left(l_j^{\gamma_j}\right)^2+\frac{|b_i-e_i|}{2}+1+\varepsilon-e_i\le 0,$$

for $i=1,2,\ldots,n$. From (14), we have $D^+V(t)\le 0$, and so $V(t)\le V(0)$, for all $t\ge 0$. From (11), we have

$$V(t)\ge\int_\Omega\sum_{i=1}^{n}\frac{\|Z_i(t,x)\|_{L^2}^2}{2}e^{2\varepsilon t}\,dx=\sum_{i=1}^{n}\frac{e^{2\varepsilon t}}{2}\left(\left\|u_i-u_i^*\right\|_{L^2}^2+\left\|v_i-v_i^*\right\|_{L^2}^2\right). \tag{15}$$

If we assume the initial values $u_i(0,x)=u_0$, $x\in\Omega$, then $\frac{\partial(u_i(0,x)-u_i^*)}{\partial x_k}=0$, thus from (11) we obtain

$$\begin{aligned}
V(0) &= \int_\Omega\sum_{i=1}^{n}\left\{\frac{\|Z_i(0,x)\|_{L^2}^2}{2}\right.\\
&\quad\left.+\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\int_{-\tau_{ij}}^{0}\left(u_j(s,x)-u_j^*\right)^2e^{2\varepsilon(s+\tau_{ij})}\,ds\right\}dx\\
&= \int_\Omega\sum_{i=1}^{n}\left\{\frac{\|\varphi_i(0,x)-u_i^*\|_{L^2}^2}{2}+\frac{\|\bar\psi_i(0,x)-u_i^*\|_{L^2}^2}{2}\right.\\
&\quad\left.+\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\int_{-\tau_{ij}}^{0}\left(\varphi_j(s,x)-u_j^*\right)^2e^{2\varepsilon(s+\tau_{ij})}\,ds\right\}dx\\
&\le \frac{\|\varphi-u^*\|_{L^2}}{2}+\frac{\|\bar\psi-u^*\|_{L^2}}{2}\\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\int_\Omega\left[\int_{-\tau_{ij}}^{0}\left(\varphi_j(s,x)-u_j^*\right)^2e^{2\varepsilon(s+\tau_{ij})}\,ds\right]dx\\
&\le \frac{\|\varphi-u^*\|_{L^2}}{2}+\frac{\|\bar\psi-u^*\|_{L^2}}{2}\\
&\quad+\tau\sum_{j=1}^{n}\max_{1\le i\le n}\left\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\right\}e^{2\varepsilon\tau}\left\|\varphi-u^*\right\|_{L^2}\\
&= \left[\frac{1}{2}+\tau\sum_{j=1}^{n}\max_{1\le i\le n}\left\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\right\}e^{2\varepsilon\tau}\right]\left\|\varphi-u^*\right\|_{L^2}\\
&\quad+\frac{\|\bar\psi-u^*\|_{L^2}}{2}.
\end{aligned}\tag{16}$$

Since $V(0)\ge V(t)$, from (15) and (16), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{e^{2\varepsilon t}}{2}\left(\left\|u_i-u_i^*\right\|_{L^2}^2+\left\|v_i-v_i^*\right\|_{L^2}^2\right)\\
&\quad\le\left[\frac{1}{2}+\tau\sum_{j=1}^{n}\max_{1\le i\le n}\left\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\right\}e^{2\varepsilon\tau}\right]\left\|\varphi-u^*\right\|_{L^2}+\frac{\|\bar\psi-u^*\|_{L^2}}{2}.
\end{aligned}\tag{17}$$

By multiplying both sides of (17) with $2e^{-2\varepsilon t}$, we get

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\left\|u_i-u_i^*\right\|_{L^2}^2+\left\|v_i-v_i^*\right\|_{L^2}^2\right)\\
&\quad\le e^{-2\varepsilon t}\left\{\left[1+2\tau\sum_{j=1}^{n}\max_{1\le i\le n}\left\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\right\}e^{2\varepsilon\tau}\right]\left\|\varphi-u^*\right\|_{L^2}\right.\\
&\qquad\left.+\left\|\bar\psi-u^*\right\|_{L^2}\right\}\\
&\quad\le e^{-2\varepsilon t}\left\{1+2\tau\sum_{j=1}^{n}\max_{1\le i\le n}\left\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\right\}e^{2\varepsilon\tau}+\frac{\|\bar\psi-u^*\|_{L^2}}{\|\varphi-u^*\|_{L^2}}\right\}\\
&\qquad\times\left\|\varphi-u^*\right\|_{L^2},
\end{aligned}\tag{18}$$

for all $t\ge 0$.

Let $M=1+2\tau\sum_{j=1}^{n}\max_{1\le i\le n}\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\}e^{2\varepsilon\tau}+\frac{\|\bar\psi-u^*\|_{L^2}}{\|\varphi-u^*\|_{L^2}}>1$, we obtain from (18),

$$\sum_{i=1}^{n}\left\|u_i-u_i^*\right\|_{L^2}^2\le Me^{-2\varepsilon t}\left\|\varphi-u^*\right\|_{L^2},$$

for all $t\ge 0$. It implies that the equilibrium u^* of system (1) is globally exponentially stable. □

Theorem 2 *For system* (1), *under the hypothesis* (H), *system* (1) *there exist one periodic solution of system* (1) *and other solutions of* (1) *converge exponentially to it as* $t\to+\infty$ *if there exist constants* $0\le\beta_i\le 1$, $0\le\gamma_i\le 1$ ($i=1,2,\dots,n$) *such that*

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{|\alpha_{ji}|}{2}\left(l_i^{1-\beta_i}\right)^2+\sum_{j=1}^{n}\frac{|\beta_{ji}|}{2}\left(l_i^{1-\gamma_i}\right)^2+\frac{|b_i-e_i|}{2}-1<0,\\
&\sum_{j=1}^{n}\frac{|\alpha_{ij}|}{2}\left(l_j^{\beta_j}\right)^2+\sum_{j=1}^{n}\frac{|\beta_{ij}|}{2}\left(l_j^{\gamma_j}\right)^2+1+\frac{|b_i-e_i|}{2}-e_i<0,
\end{aligned}$$

for $i=1,2,\dots,n$.

Proof of Theorem 2 Let

$$\theta=\left\{\phi|\phi=(\varphi,\bar\psi)^T=(\varphi_1,\varphi_2,\dots,\varphi_n,\bar\psi_1,\bar\psi_2,\dots,\bar\psi_n)^T,\phi:(-\infty,0]\times\Omega\to R^{2n}\right\},$$

for any $\phi\in\theta$, we define

$$\|\phi\|_{L^2}=\left\|(\varphi,\bar\psi)^T\right\|_{L^2}=\sup_{-\infty<t\le 0}\sum_{i=1}^{n}\|\varphi_i\|_{L^2}^2+\sup_{-\infty<t\le 0}\sum_{i=1}^{n}\|\bar\psi_i\|_{L^2}^2,$$

then θ is the Banach space of continuous functions which maps $(-\infty,0]\times\Omega$ into R^{2n} with the topology of uniform convergence. For any $(\varphi,\bar\psi)^T,(\varphi^*,\bar\psi^*)^T\in\theta$, we denote the solutions of system (3) by

$$\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\varphi\\\bar{\psi}\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\varphi^*\\\bar{\psi}^*\end{pmatrix}\right),$$

as $u(t,\varphi,x) = (u_1(t,\varphi,x),u_2(t,\varphi,x),\dots,u_n(t,\varphi,x))^T$, $v(t,\bar{\psi},x) = (v_1(t,\bar{\psi},x),v_2(t,\bar{\psi},x),\dots,v_n(t,\bar{\psi},x)^T$, and $u(t,\varphi^*,x) = (u_1(t,\varphi^*,x),u_2(t,\varphi^*,x),\dots,u_n(t,\varphi^*,x))^T$, $v(t,\bar{\psi}^*,x) = (v_1(t,\bar{\psi}^*,x),v_2(t,\bar{\psi}^*,x),\dots,v_n(t,\bar{\psi}^*,x)^T$, respectively.

Defining $u_t(\varphi,x) = u(t+\delta,\varphi,x)$, $v_t(\bar{\psi},x) = v(t+\delta,\bar{\psi},x)$, $\delta\in(-\infty,0]$, $t\geq 0$, then $(u_t(\varphi,x),v_t(\bar{\psi},x))^T\in\theta$, for all $t\geq 0$.

Let

$$Z_{i,t,\varphi,\bar{\psi}}(t,x) = \begin{pmatrix}u_i(t,\varphi,x)-u_i(t,\varphi^*,x)\\v_i(t,\bar{\psi},x)-v_i(t,\bar{\psi}^*,x)\end{pmatrix},$$

thus from (3) we have

$$\begin{aligned}\frac{\partial Z_{i,t,\varphi,\bar{\psi}}(t,x)}{\partial t} &= P\begin{pmatrix}\sum_{k=1}^n\frac{\partial}{\partial x_k}(a_{ik}\frac{\partial(u_i(t,\varphi,x)-u_i(t,\varphi^*,x))}{\partial x_k})\\0\end{pmatrix} - B_iZ_{i,t,\varphi,\bar{\psi}}(t,x)\\&\quad + P\begin{pmatrix}\bigwedge_{j=1}^n\alpha_{ij}(f_j(u_j(t-\tau_{ij},\varphi,x))-f_j(u_j(t-\tau_{ij},\varphi^*,x)))\\0\end{pmatrix}\\&\quad + P\begin{pmatrix}\bigvee_{j=1}^n\beta_{ij}(f_j(u_j(t-\tau_{ij},\varphi,x))-f_j(u_j(t-\tau_{ij},\varphi^*,x)))\\0\end{pmatrix},\quad t\geq 0, x\in\Omega.\end{aligned}$$

We consider the Lyapunov functional

$$\begin{aligned}V(t) = \int_\Omega\sum_{i=1}^n\Biggl\{&\frac{\|Z_{i,t,\varphi,\bar{\psi}}(t,x)\|_{L^2}^2}{2}e^{2\varepsilon t} + \sum_{j=1}^n\frac{|\alpha_{ij}|(l_j^{1-\beta_j})^2+|\beta_{ij}|(l_j^{1-\gamma_j})^2}{2}\int_{t-\tau_{ij}}^t\bigl(u_j(s,\varphi,x)\\&-u_j\bigl(s,\varphi^*,x\bigr)\bigr)^2e^{2\varepsilon(s+\tau_{ij})}\,ds + \sum_{k=1}^n\frac{a_{ik}}{2}e^{2\varepsilon t}\left[\frac{\partial(u_i(t,\varphi,x)-u_i(t,\varphi^*,x))}{\partial x_k}\right]^2\Biggr\}\,dx, \quad (19)\end{aligned}$$

where $\varepsilon>0$ is sufficiently small.

By a minor modification of the proof of Theorem 1, we can derive easily

$$\begin{aligned}&\sum_{i=1}^n\bigl(\bigl\|u_i(t,\varphi,x)-u_i\bigl(t,\varphi^*,x\bigr)\bigr\|_{L^2}^2+\bigl\|v_i(t,\bar{\psi},x)-v_i\bigl(t,\bar{\psi}^*,x\bigr)\bigr\|_{L^2}^2\bigr)\\&\quad\leq e^{-2\varepsilon t}\Biggl\{\Biggl[1+2\tau\sum_{j=1}^n\max_{1\leq i\leq n}\Biggl\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\Biggr\}e^{2\varepsilon\tau}\Biggr]\bigl\|\varphi-\varphi^*\bigr\|_{L^2}\\&\qquad+\bigl\|\bar{\psi}-\bar{\psi}^*\bigr\|_{L^2}\Biggr\}\\&\quad\leq e^{-2\varepsilon t}\Biggl\{1+2\tau\sum_{j=1}^n\max_{1\leq i\leq n}\Biggl\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2+|\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\Biggr\}e^{2\varepsilon\tau}\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\|\bar{\psi} - \bar{\psi}^*\|_{L^2}}{\|\varphi - \varphi^*\|_{L^2}} \Bigg\} \|\varphi - \varphi^*\|_{L^2} \\
&\le e^{-2\varepsilon t} M \|\varphi - \varphi^*\|_{L^2},
\end{aligned} \tag{20}
$$

for all $t \ge 0$, where $M = 1 + 2\tau \sum_{j=1}^{n} \max_{1\le i\le n}\{\frac{|\alpha_{ji}|(l_i^{1-\beta_i})^2 + |\beta_{ji}|(l_i^{1-\gamma_i})^2}{2}\} e^{2\varepsilon\tau} + \frac{\|\bar{\psi}-\bar{\psi}^*\|_{L^2}}{\|\varphi-\varphi^*\|_{L^2}} > 1$.

We can choose a positive integer N and $\omega > 0$, such that $Me^{-2N\varepsilon\omega} \le \frac{1}{3}$. Now we define a Poincaré mapping $\theta \to \theta$ by

$$
F(\varphi, \bar{\psi})^T = \big(u_\omega(\varphi, x), v_\omega(\bar{\psi}, x)\big)^T,
$$

then

$$
F^N(\varphi, \bar{\psi})^T = \big(u_{N\omega}(\varphi, x), v_{N\omega}(\bar{\psi}, x)\big)^T.
$$

Let $t = N\omega$, then from (20) we have

$$
\left\| F^N \begin{pmatrix} \varphi \\ \bar{\psi} \end{pmatrix} - F^N \begin{pmatrix} \varphi^* \\ \bar{\psi}^* \end{pmatrix} \right\|_{L^2} \le \frac{1}{3} \left\| \begin{pmatrix} \varphi \\ \bar{\psi} \end{pmatrix} - \begin{pmatrix} \varphi^* \\ \bar{\psi}^* \end{pmatrix} \right\|_{L^2}.
$$

It implies that F^N is a contraction mapping, hence there exists a unique fixed point $(\varphi_*, \bar{\psi}_*)^T \in \theta$, such that $F^N(\varphi_*, \bar{\psi}_*)^T = (\varphi_*, \bar{\psi}_*)^T$. Since

$$
F^N \left(F \begin{pmatrix} \varphi_* \\ \bar{\psi}_* \end{pmatrix} \right) = F \left(F^N \begin{pmatrix} \varphi_* \\ \bar{\psi}_* \end{pmatrix} \right) = F \begin{pmatrix} \varphi_* \\ \bar{\psi}_* \end{pmatrix},
$$

$F(\varphi_*, \bar{\psi}_*)^T \in \theta$ is also a fixed point of F^N, and thus $F(\varphi_*, \bar{\psi}_*)^T = (\varphi_*, \bar{\psi}_*)^T$, i.e., $(u_\omega(\varphi_*), v_\omega(\bar{\psi}_*))^T = (\varphi_*, \bar{\psi}_*)^T$. Let $(u(t, \varphi_*, x), v(t, \bar{\psi}_*, x))^T$ be the solution of system (3) through

$$
\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_* \\ \bar{\psi}_* \end{pmatrix} \right),
$$

then $(u(t+\omega, \varphi_*, x), v(t+\omega, \bar{\psi}_*, x))^T$ is also a solution of system (3). Obviously we have

$$
\begin{pmatrix} u_{t+\omega}(\varphi_*, x) \\ v_{t+\omega}(\bar{\psi}_*, x) \end{pmatrix} = \begin{pmatrix} u_t(u_\omega(\varphi_*, x)) \\ v_t(v_\omega(\bar{\psi}_*, x)) \end{pmatrix} = \begin{pmatrix} u_t(\varphi_*, x) \\ v_t(\bar{\psi}_*, x) \end{pmatrix},
$$

for all $t \ge 0$. Hence

$$
\begin{pmatrix} u(t+\omega, \varphi_*, x) \\ v(t+\omega, \bar{\psi}_*, x) \end{pmatrix} = \begin{pmatrix} u(t, \varphi_*, x) \\ v(t, \bar{\psi}_*, x) \end{pmatrix},
$$

for all $t \ge 0$.

It shows that there is exactly one ω-periodic solution of system (3) and other solutions of system (3) converge exponentially to it as $t \to +\infty$, which implies that $u(t, \varphi_*, x)$ is exactly one ω-periodic solution of system (1) and other solutions of system (1) converge exponentially to is as $t \to +\infty$. □

Furthermore, as a consequence of the Theorem 2 we have the following corollary.

Corollary 1 *For system* (1), *under the hypothesis* (*H*), *there exists one periodic solution of system* (1), *and other solutions of system* (1) *converge exponentially to it as* $t \to +\infty$, *if one of the following conditions holds*:

$$\begin{cases} \sum_{j=1}^{n} \frac{|\alpha_{ji}|}{2} l_i + \sum_{j=1}^{n} \frac{|\beta_{ji}|}{2} l_i + \frac{|b_i - e_i|}{2} - 1 < 0, \\ \sum_{j=1}^{n} \frac{|\alpha_{ij}|}{2} l_j + \sum_{j=1}^{n} \frac{|\beta_{ij}|}{2} l_j + 1 + \frac{|b_i - e_i|}{2} - e_i < 0, \end{cases} \tag{21}$$

for $i = 1, 2, \ldots, n$,

$$\begin{cases} \sum_{j=1}^{n} \frac{|\alpha_{ji}|}{2} + \sum_{j=1}^{n} \frac{|\beta_{ji}|}{2} + \frac{|b_i - e_i|}{2} - 1 < 0, \\ \sum_{j=1}^{n} \frac{|\alpha_{ij}|}{2} l_j^2 + \sum_{j=1}^{n} \frac{|\beta_{ij}|}{2} l_j^2 + 1 + \frac{|b_i - e_i|}{2} - e_i < 0, \end{cases} \tag{22}$$

for $i = 1, 2, \ldots, n$, *and*

$$\begin{cases} \sum_{j=1}^{n} \frac{|\alpha_{ji}|}{2} l_i^2 + \sum_{j=1}^{n} \frac{|\beta_{ji}|}{2} l_i^2 + \frac{|b_i - e_i|}{2} - 1 < 0, \\ \sum_{j=1}^{n} \frac{|\alpha_{ij}|}{2} + \sum_{j=1}^{n} \frac{|\beta_{ij}|}{2} + 1 + \frac{|b_i - e_i|}{2} - e_i < 0, \end{cases} \tag{23}$$

In fact, the conditions (21)–(23) are special cases of Theorem 2 as $\beta_i = \gamma_i = \frac{1}{2}; \beta_i = \gamma_1 = 1; \beta_i = \gamma_1 = 0$, respectively. Therefore, by Theorem 2 we observe that Corollary 1 is true.

4 Numerical example

In this section, we give a numerical example to show the results of our method.

Example Consider the following neural networks with hyperbolic terms:

$$\begin{cases} \frac{\partial^2 u_i(t,x)}{\partial t^2} = \sum_{k=1}^{2} \frac{\partial}{\partial x_k} (a_{ik} \frac{\partial u_i(t,x)}{\partial x_k}) - b_i u_i(t,x) - e_i \frac{\partial u_i(t,x)}{\partial t} + \bigwedge_{j=1}^{2} \alpha_{ij} f_j(u_j(t - \tau_{ij}, x)) \\ \qquad + \bigvee_{j=1}^{2} \beta_{ij} f_j(u_j(t - \tau_{ij}, x) + \bigwedge_{j=1}^{2} T_{ij} \mu_j + \bigvee_{j=1}^{2} H_{ij} \mu_j, \quad t \geq 0, x \in \Omega, \end{cases} \tag{24}$$

for $i = 1, 2$, where $b_1 = 3$, $b_2 = 2$, $e_1 = 2.8$, $e_2 = 1.96$, $\alpha_{11} = 0.2$, $\alpha_{12} = 0.1$, $\alpha_{21} = -0.2$, $\alpha_{22} = -0.3$, $\beta_{11} = 0.25$, $\beta_{12} = 0.35$, $\beta_{21} = -0.15$, $\beta_{22} = -0.3$, $f_k(u) = \frac{1}{2}(|u+1| - |u-1|)$ $(k = 1, 2)$.

Obviously, $f_k(u)$ $(k = 1, 2)$ satisfies the condition (H) and $l_k = 1$ $(k = 1, 2)$.

By choosing $\beta_i = 1$, $\gamma_i = 1$ $(i = 1, 2)$, we have the following results after a simple calculation:

$$\sum_{j=1}^{2} \left(\frac{|\alpha_{j1}|}{2} + \frac{|\beta_{j1}|}{2} \right) + \frac{|b_1 - e_1|}{2} - 1 = -0.5 < 0,$$

$$\sum_{j=1}^{2} \left(\frac{|\alpha_{1j}|}{2} + \frac{|\beta_{1j}|}{2} \right) l_j^2 + 1 - e_1 + \frac{|b_1 - e_1|}{2} = -1.25 < 0,$$

$$\sum_{j=1}^{2} \left(\frac{|\alpha_{j2}|}{2} + \frac{|\beta_{j2}|}{2} \right) + \frac{|b_2 - e_2|}{2} - 1 = -0.455 < 0,$$

$$\sum_{j=1}^{2} \left(\frac{|\alpha_{2j}|}{2} + \frac{|\beta_{2j}|}{2} \right) l_j^2 + 1 - e_2 + \frac{|b_2 - e_2|}{2} = -1.305 < 0.$$

Hence, it follows from Theorem 1 that (24) has a unique equilibrium point which is globally exponentially stable. It also follows, from Theorem 2, that there exists exactly one periodic solution of (24) and all other solutions of (24) converge exponentially to it as $t \to +\infty$.

5 Conclusions

In this paper, some sufficient conditions have been derived for the globally exponential stability and existence of periodic solution of the fuzzy wave equations by constructing a suitable Lyapunov functional and using some analytical techniques. A numerical example is given to show the effectiveness of the results. The given algebra conditions are verifiable and useful in the theory and applications.

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Authors' contributions
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